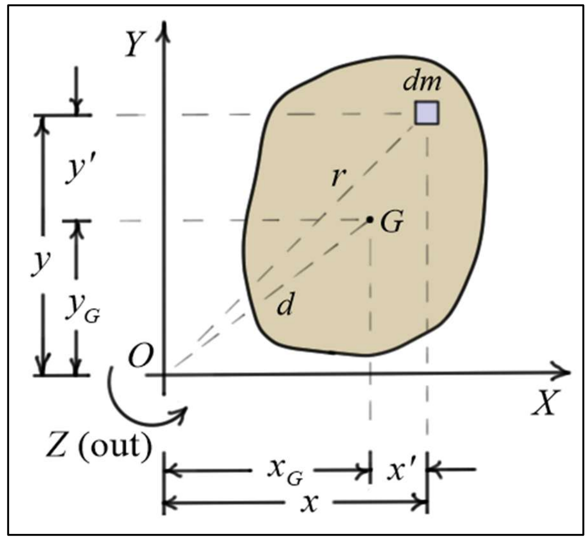
Elementary Dynamics
Mass Moments of Inertia in Two Dimensions

Definition of Mass Moment of Inertia



The figure depicts a rigid body in two dimensions. The moment of inertia of the body about the Z-axis passing through the point O is defined as

$$\boxed{(I_Z)_O = \int_B r^2 dm = \int_B \left(x^2 + y^2\right) dm}$$

The moment of inertia $(I_Z)_O$ measures the distribution of mass about the Z-axis.

The ***larger*** the inertia, the ***further*** the mass is spread away from the axis. The ***smaller*** the inertia, the ***closer*** the mass is located to the axis. The units of I_Z are ***slug-ft²*** or ***kg-m²***.

Parallel Axes Theorem

The moment of inertia of the body about the Z-axis can be related to the moment of inertia about an axis parallel to Z and passing through the mass center G as follows:

$$(I_Z)_O = \int_B \left(x^2 + y^2\right) dm = \int_B \left(\left(x_G + x'\right)^2 + \left(y_G + y'\right)^2\right) dm$$

$$= \int_B \left(x_G^2 + y_G^2\right) dm + \int_B 2x_G x' dm + \int_B 2y_G y' dm + \int_B \left(x'^2 + y'^2\right) dm$$

$$= \left(x_G^2 + y_G^2\right) \underbrace{\int_B dm}_{M} + 2x_G \underbrace{\left(\int_B x' dm\right)}_{\substack{\text{zero by} \\ \text{definition} \\ \text{of mass center}}} + 2y_G \underbrace{\left(\int_B y' dm\right)}_{\substack{\text{zero by} \\ \text{definition} \\ \text{of mass center}}} + (I_Z)_G$$

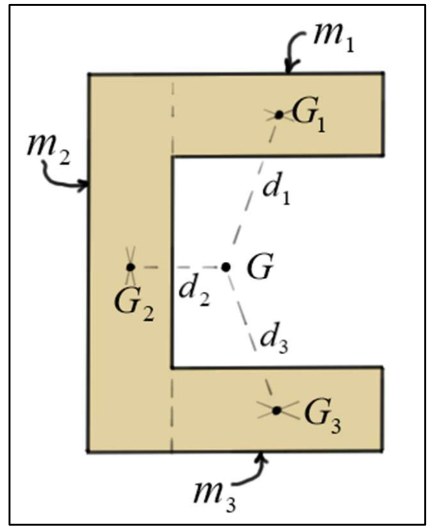
$$= M\,d^2 + (I_Z)_G$$

or

$$\boxed{(I_Z)_O = (I_Z)_G + M\,d^2} \quad \text{or} \quad \boxed{I_O = I_G + M\,d^2} \qquad \text{(Parallel Axes Theorem)}$$

Mass Moments of Inertia for Composite Shapes

The mass moment of inertia of a composite shape is the sum of the inertias of the individual component shapes. If G is the mass center of the composite shape, then



$$\boxed{I_G = \sum_i \left(I_G\right)_i}$$

Here, $\left(I_G\right)_i$ represents the moment of inertia of the i^{th} component shape about the composite mass center G and can be calculated as follows

$$\boxed{\left(I_G\right)_i = \left(I_{G_i}\right) + m_i d_i^2}.$$

Here, I_{G_i} represents the moment of inertia of the i^{th} component shape about its own mass center G_i and can be found in the tables for common geometric shapes.