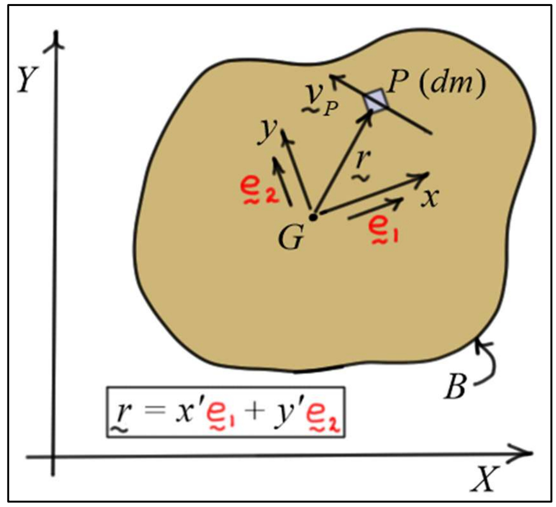
Elementary Dynamics
Angular Momentum of a Rigid Body in Two Dimensions



The figure depicts a rigid body B moving in two dimensions. The XY axes are fixed, the xy axes are fixed in and rotate with B. G is the mass center of B. The ***angular momentum*** of the body about G is defined as

$$\underset{\sim}{H}_G = \int_B \left(\underset{\sim}{r} \times \underset{\sim}{v}_P\right) dm$$

Using the ***relative velocity equation***, the angular momentum can be rewritten as follows.

$$\underset{\sim}{H}_G = \int_B \underset{\sim}{r} \times \left(\underset{\sim}{v}_G + \underset{\sim}{v}_{P/G}\right) dm = \int_B \left(\underset{\sim}{r} \times \underset{\sim}{v}_G\right) dm + \int_B \left(\underset{\sim}{r} \times \underset{\sim}{v}_{P/G}\right) dm = \underbrace{\left(\int_B \underset{\sim}{r}\, dm\right)}_{\substack{\text{zero by} \\ \text{definition of} \\ \text{mass center}}} \times \underset{\sim}{v}_G + \int_B \underset{\sim}{r} \times \left(\underset{\sim}{\omega} \times \underset{\sim}{r}\right) dm$$

$$\Rightarrow \boxed{\underset{\sim}{H}_G = \int_B \underset{\sim}{r} \times \left(\underset{\sim}{\omega} \times \underset{\sim}{r}\right) dm}$$

The triple vector product integrand can be written in component form as follows.

$$\underset{\sim}{\omega} \times \underset{\sim}{r} = \omega\, \underset{\sim}{k} \times \left(x' \underset{\sim}{e}_1 + y' \underset{\sim}{e}_2\right) = \omega\left(-y' \underset{\sim}{e}_1 + x' \underset{\sim}{e}_2\right)$$

$$\underset{\sim}{r} \times \left(\underset{\sim}{\omega} \times \underset{\sim}{r}\right) = \omega \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{k} \\ x' & y' & 0 \\ -y' & x' & 0 \end{vmatrix} = \omega\left(x'^2 + y'^2\right) \underset{\sim}{k}$$

Substituting into the boxed equation above gives

$$\underset{\sim}{H}_G = \int_B \omega\left(x'^2 + y'^2\right) \underset{\sim}{k}\, dm = \left(\int_B \left(x'^2 + y'^2\right) dm\right) \omega \underset{\sim}{k} = I_G \underset{\sim}{\omega}$$

So, the ***angular momentum*** of a rigid body about its ***mass center*** can be written as follows.

$$\boxed{\underset{\sim}{H}_G = I_G\, \underset{\sim}{\omega}}$$

I_G is the mass moment of inertia of the body about its mass center G.)

Note: Recall the ***linear momentum*** of a rigid body is $\boxed{\underset{\sim}{L} = m\, \underset{\sim}{v}_G}$.