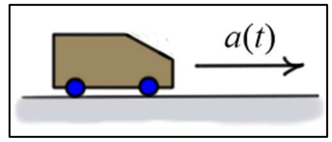
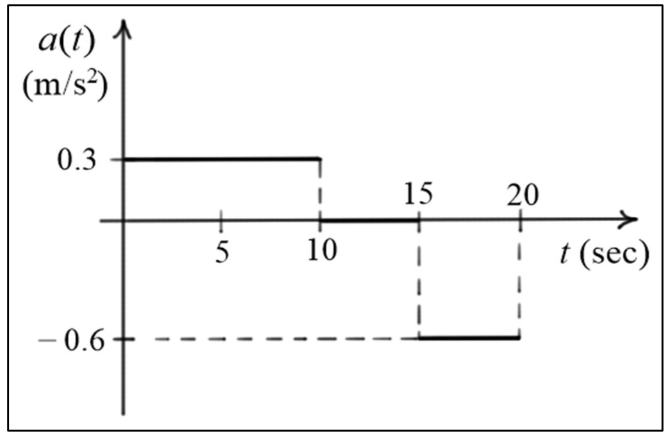
Elementary Engineering Mathematics
Applications of Integration in Elementary Dynamics

Example 1: Acceleration profiles



Given: A car has an acceleration profile as shown. It's initial position and velocity are zero. $\left(s(0)=v(0)=0\right)$

$$\boxed{v(t)=\int a(t)dt} \quad \text{and} \quad \boxed{s(t)=\int v(t)dt}$$



Find: (a) The velocity function $v(t)$; and (b) the displacement function $s(t)$ for the car for $0 \le t \le 20$ (sec).

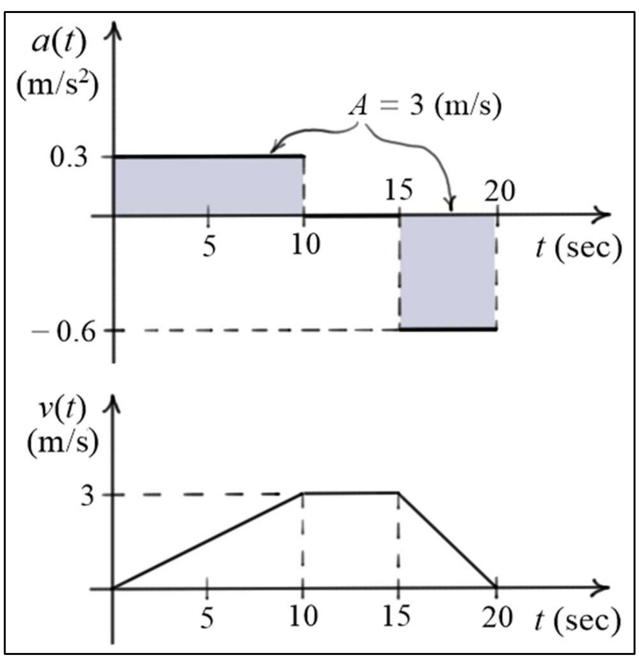
Solution:

(a) We can construct the velocity diagram easily from the acceleration diagram. When the acceleration is constant, the velocity varies linearly, and when the acceleration is zero, the velocity is constant.

$$\boxed{v(t)=\int 0.3dt = 0.3t + D = 0.3t} \quad \text{for } 0 \le t \le 10 \quad (\text{recall that } v(0)=0)$$

$$\boxed{v(t)=\int 0dt = D = v(10) = 3 \text{ (m/s)}} \quad \text{for } 10 \le t \le 15$$

$$\boxed{v(t)=\int -0.6dt = -0.6t + D = -0.6t + 12} \quad \text{for } 15 \le t \le 20 \quad \left(v(15)=3 \text{ (m/s)}\right)$$



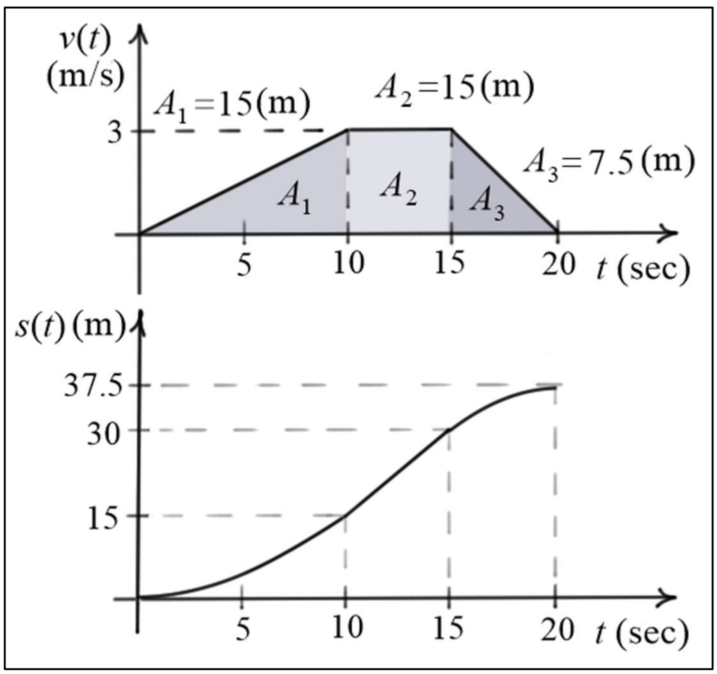
The ***areas*** under the acceleration profile give the ***changes*** in velocity over those periods of time

(b) The displacement function can now be derived from the velocity profile. When the velocity varies ***linearly***, the displacement will vary ***quadratically***, and when the velocity is ***constant***, the displacement will vary ***linearly***. The displacement changes are given by areas under the velocity function.

$$\boxed{s(t) = \int 0.3t\,dt = 0.15t^2 + D = 0.15t^2} \quad \text{for } 0 \le t \le 10 \quad (\text{recall that } s(0) = 0)$$

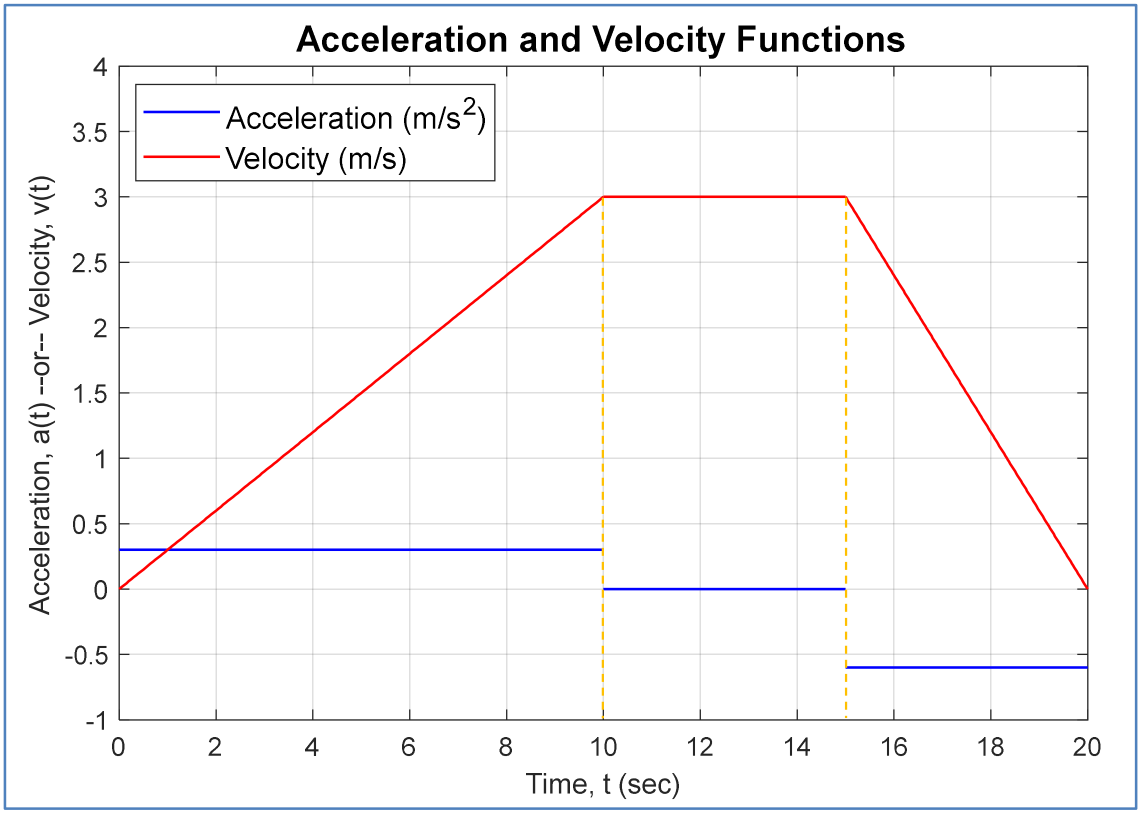
$$\boxed{s(t) = \int 3dt = 3t + D = 3t - 15} \quad \text{for } 10 \le t \le 15 \quad \big(s(10) = 15 \text{ (m)}\big)$$

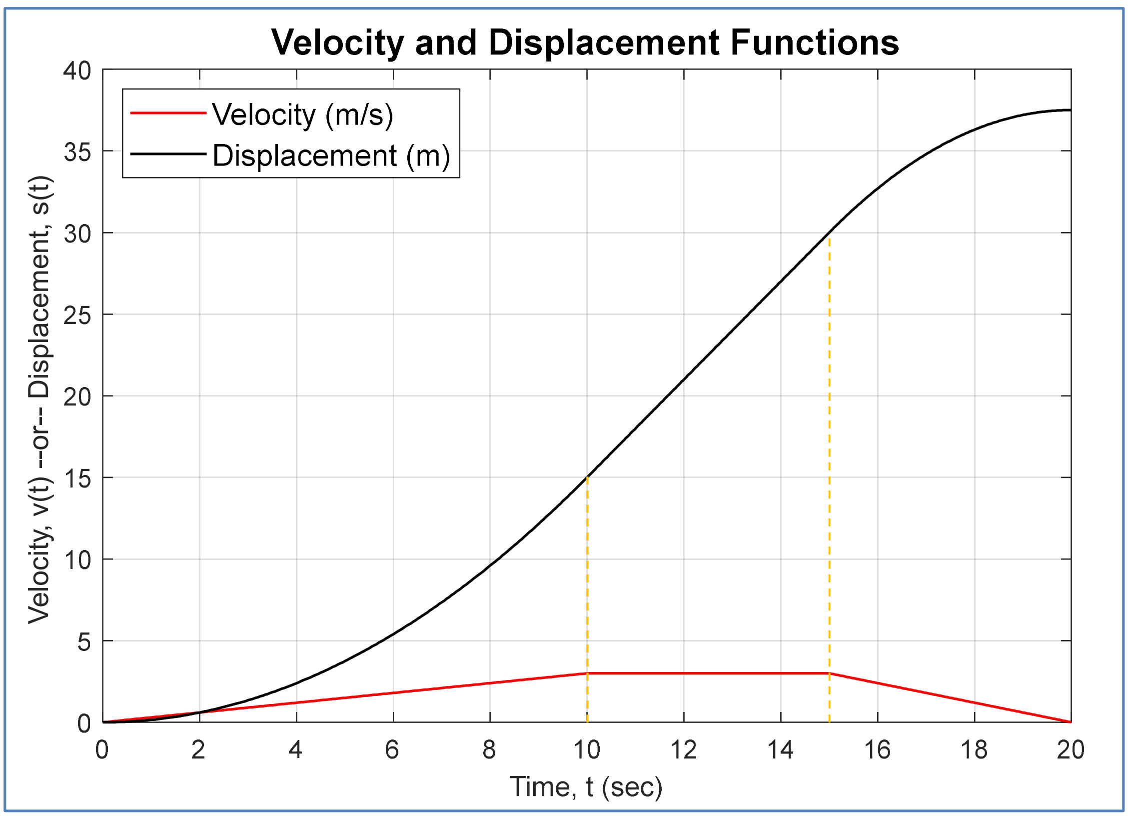
$$\boxed{\begin{aligned} s(t) &= \int -0.6t + 12\,dt = -0.3t^2 + 12t + D \\ &= -0.3t^2 + 12t - 82.5 \end{aligned}} \quad \text{for } 15 \le t \le 20 \quad \big(s(15) = 30 \text{ (m)}\big)$$



The ***areas*** under the velocity profile give the ***changes*** in displacement over those periods of time

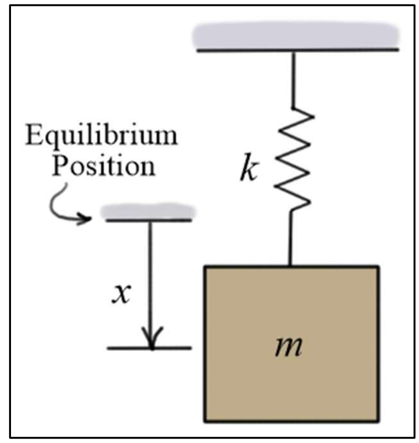
In this case, note that the acceleration profile is a ***discontinuous*** function, the velocity profile is a ***piece-wise continuous*** function, and the displacement profile is a ***continuous*** function. If we do not need the actual functions, we can still construct the velocity and displacement profiles by simply measuring areas under the acceleration and velocity profiles.


Acceleration and Velocity Functions
Acceleration (m/s²)
Velocity (m/s)
Acceleration, a(t) --or-- Velocity, v(t)
Time, t (sec)


Velocity and Displacement Functions
Velocity (m/s)
Displacement (m)
Velocity, v(t) --or-- Displacement, s(t)
Time, t (sec)

Example 2: Work done by a spring

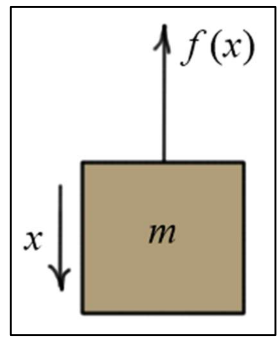
Given: The work done by the spring force $f(x)$ on the mass as it undergoes a downward displacement from A to B can be written as



$$W_{A\to B} = -\int_{x_A}^{x_B} f(x)dx$$

The work is ***negative***, because the force is opposite the displacement of the mass. If the mass has a positive downward velocity during this displacement, the spring will ***decrease*** its velocity.

Find: The work done by a spring as it is stretched from $x=1$ (in) to $x=4$ (in), assuming (a) a linear spring with $f(x)=6x$ (lb); and (b) a non-linear hardening spring with $f(x)=3x^2$. Assume x is measured in inches.



Solution:

(a) For the linear spring

$$W_{A\to B} = -\int_1^4 6x\,dx = -\left(3x^2\right)\Big|_1^4 = -3(16-1) = -45 \text{ (in-lb)}$$

(b) For the non-linear spring

$$W_{A\to B} = -\int_1^4 3x^2\,dx = -\left(x^3\right)\Big|_1^4 = -(64-1) = -63 \text{ (in-lb)}$$