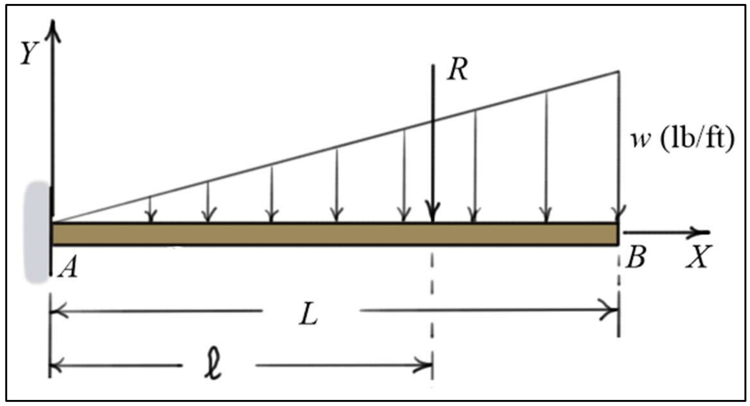
Elementary Engineering Mathematics
Applications of Integration in Statics and Mechanics of Materials

Example 1: Resultant of a distributed load

The diagram shows a cantilevered beam with a linearly varying load intensity. The maximum load intensity is w (lb/ft) at the right end of the beam. To calculate the support forces at A, the distributed load can be replaced by a single resultant load R acting at a distance ℓ from the left end.



The load R and distance ℓ are found by calculating the following integrals.

$$\boxed{R=\int_0^L w\,x\,dx} \quad \text{and} \quad \boxed{R\times\ell=\int_0^L (w\,x)\,x\,dx=\int_0^L w\,x^2\,dx} \qquad (1)$$

The first of Eqs. (1) equates the resultant ***force*** R with the ***summation*** (integral) of the ***load intensity***, and the second one equates the ***moment*** of R about A with the ***sum*** (integral) of the ***moments*** of the of the ***load intensity***.

Given: $L=10$ (ft), $w=100$ (lb/ft)

Find: (a) the resultant force R; and (b) the distance ℓ that it acts from the support.

Solution:

(a) The resultant load is

$$\boxed{R=\int_0^{10} 10\,x\,dx=\left(5\,x^2\right)\Big|_0^{10}=5\times 100=500 \text{ (lb)}} \quad \text{or} \quad \boxed{R=\int_0^{10} 10\,x\,dx=\underbrace{\tfrac{1}{2}\times 10\times 100}_{\text{area of the triangle}}=500 \text{ (lb)}}$$

(b) The distance ℓ is

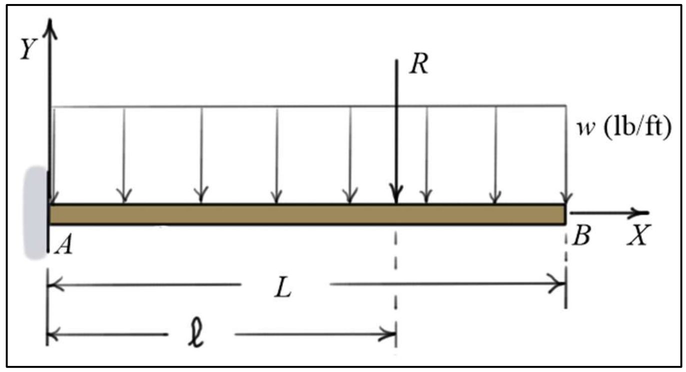
$$\boxed{500\,\ell=\int_0^{10} 10\,x^2\,dx=\left(\tfrac{10}{3}x^3\right)\Big|_0^{10} \quad\Rightarrow\quad \ell=\frac{10\times 10^3}{3\times 500}=\tfrac{2}{3}(10)=6.\overline{6} \text{ (ft)}}$$

For a linearly varying load (starting at zero), the resultant acts $\frac{2}{3}$ of the way along the distributed load. If the load is w (lb/ft) at the wall and zero at the end of the beam, the resultant would be located $\frac{1}{3}$ of the way along the load.

Example 2: Uniformly distributed load

Given: $L = 10$ (ft), $w = 100$ (lb/ft)

Find: (a) the resultant force R; and (b) the distance ℓ that it acts from the support.



Solution:

(a) The resultant load is

$$R = \int_0^{10} 100\,dx = 10 \times 100 = 1000 \text{ (lb)}$$

(b) The distance ℓ is

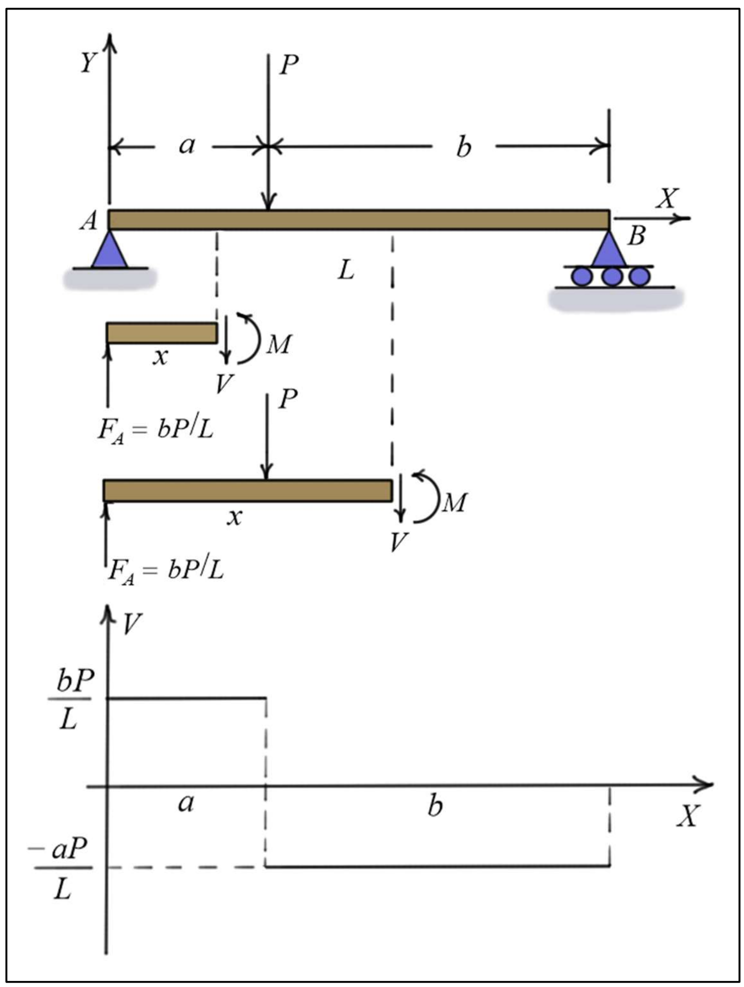
$$1000\,\ell = \int_0^{10} 100x\,dx = \tfrac{1}{2} \times 10 \times 1000 \quad \Rightarrow \quad \ell = \frac{5000}{1000} = \tfrac{1}{2}(10) = 5 \text{ (ft)}$$

So, for a uniformly distributed load, the resultant acts $\frac{1}{2}$ of the way along the load.

Example 3:

The diagram of the internal shearing force for the simply supported beam with a concentrated load is shown. The internal bending moment is related to the shearing force by the equation

$$M(x) = \int V(x)dx$$



Given: $P = 100$ (lbs), $L = 5$ (ft),

$a = 3.5$ (ft), $b = 1.5$ (ft),

and $M(0) = M(L) = 0$

Find: (a) moment diagram for the beam; and (b) $M_{\max}$ the maximum bending moment in the beam.

Solution:

(a) We can construct the moment diagram from the shear diagram. Where the shear is ***constant***, the moment varies ***linearly*** with x.

$\boxed{V_a = bP/L = 1.5\times 100/5 = 30 \text{ (lb)}}$ and $\boxed{V_b = -aP/L = -3.5\times 100/5 = -70 \text{ (lb)}}$

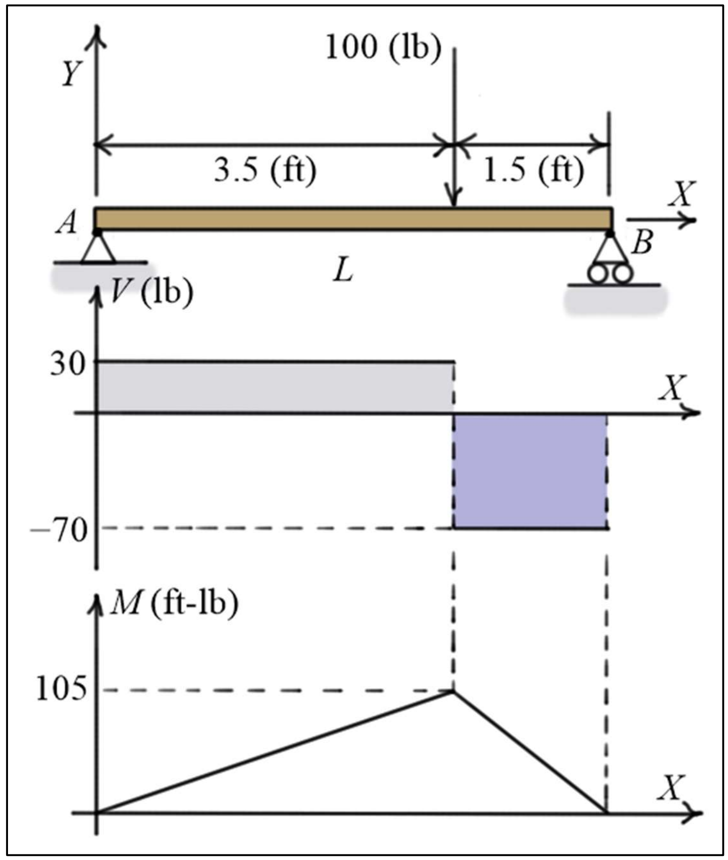
$\boxed{M(x) = \int 30\,dx = 30x + D = 30x}$ for $0 \le x \le 3.5$ (recall that $M(0)=0$)

$\boxed{M(x) = \int -70\,dx = -70x + D = -70x + 350}$ for $3.5 \le x \le 5$ $\left(M(3.5) = 105 \text{ (ft-lb)}\right)$

(b) The maximum bending moment occurs at the concentrated load. It is the area under the shear diagram from 0 to 3.5 feet.

$\boxed{M_{\max} = 30\times 3.5 = 105 \text{ (ft-lb)}}$

Note that the area under the shear diagram from 3.5 to 7 feet is the negative of this value, so the bending moment at both ends of the beam are zero.



Example 4:

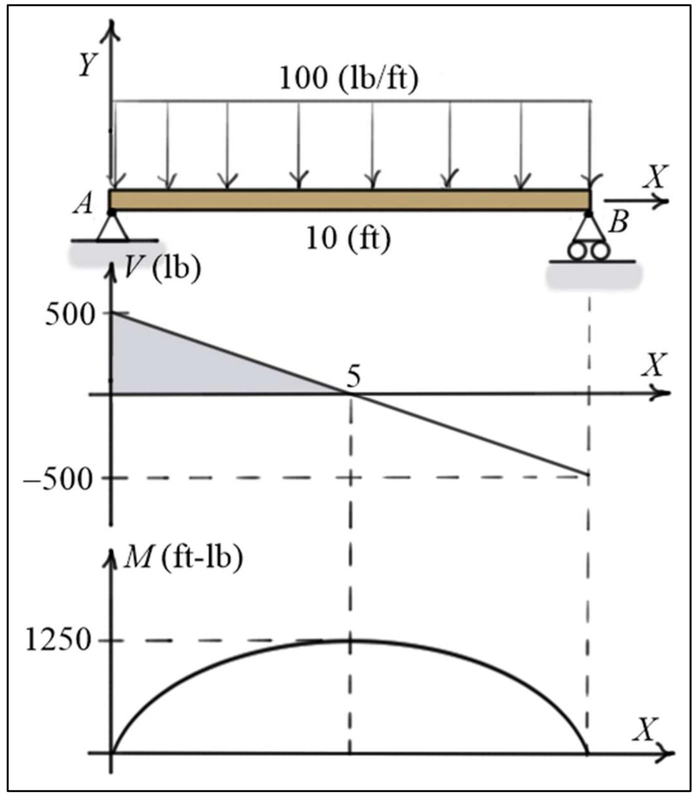
Given: $L = 10$ (ft), $w = 100$ (lb/ft),

$M(0) = M(L) = 0$, and

$\boxed{V(x) = 500 - 100x \text{ (lb)}}$ $\left(0 \le x \le L\right)$

Find: (a) bending moment diagram; and

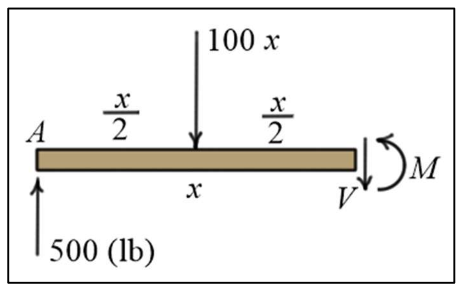
(b) the maximum bending moment.



Solution:

(a) Again, we can construct the moment diagram from the shear diagram. Where the shear varies ***linearly***, the moment varies ***quadratically*** with x.

$$\begin{aligned} M(x) &= \int \left(500 - 100x\right) dx \\ &= 500x - 50x^2 + D \\ &= 500x - 50x^2 \text{ (ft-lb)} \end{aligned} \qquad \left(M(0) = 0\right)$$



(b) The maximum bending moment occurs at the midpoint of the beam. It is the ***area under the shear diagram*** from 0 to $L/2$.

$$\boxed{M_{max} = \tfrac{1}{2} \times 5 \times 500 = 1250 \text{ (ft-lb)}}$$

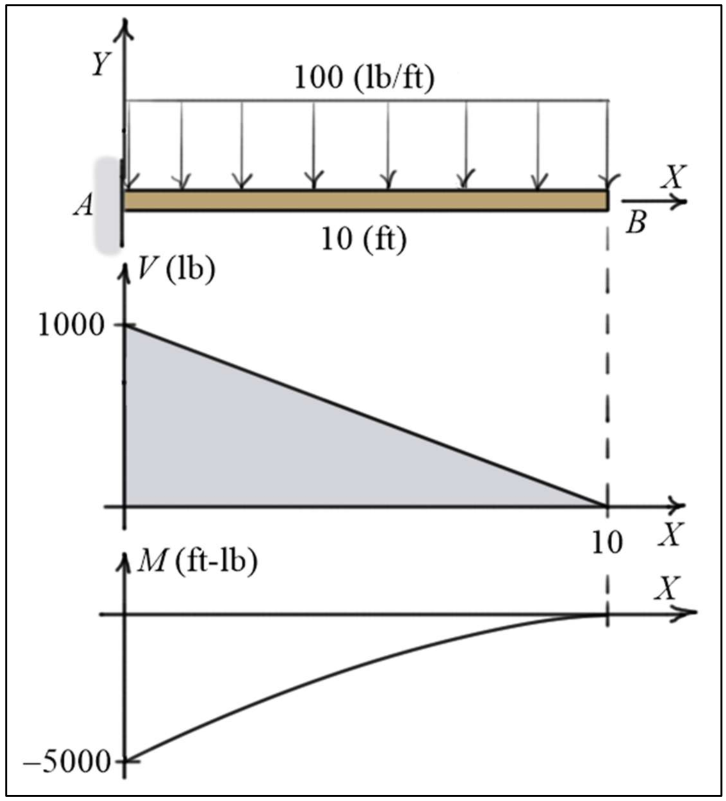
Example 5:

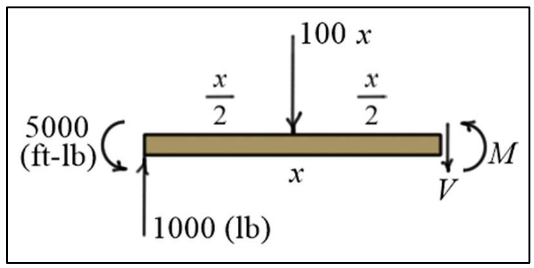
Given: $L = 10$ (ft), $w = 100$ (lb/ft), $M(L) = 0$,

$$\boxed{V(x) = 1000 - 100x \text{ (lb)}} \quad (0 \le x \le L)$$

Find: (a) bending moment diagram; and

(b) the maximum bending moment.





Solution:

(a) In this case, the shear varies ***linearly*** with x, so the moment will vary ***quadratically*** with x. Given $M(10) = 0$, we find

$$\begin{aligned} M(x) &= \int (1000 - 100x)\,dx \\ &= 1000x - 50x^2 + D \\ &= -5000 + 1000x - 50x^2 \text{ (ft-lb)} \end{aligned}$$

(b) The maximum moment occurs at the left end of the beam and is equal to the area under the shear diagram from 0 to L. Why?

$$\boxed{M_{\max} = \tfrac{1}{2} \times 10 \times 1000 = 5000 \text{ (ft-lb)}}$$