

Introductory Control Systems

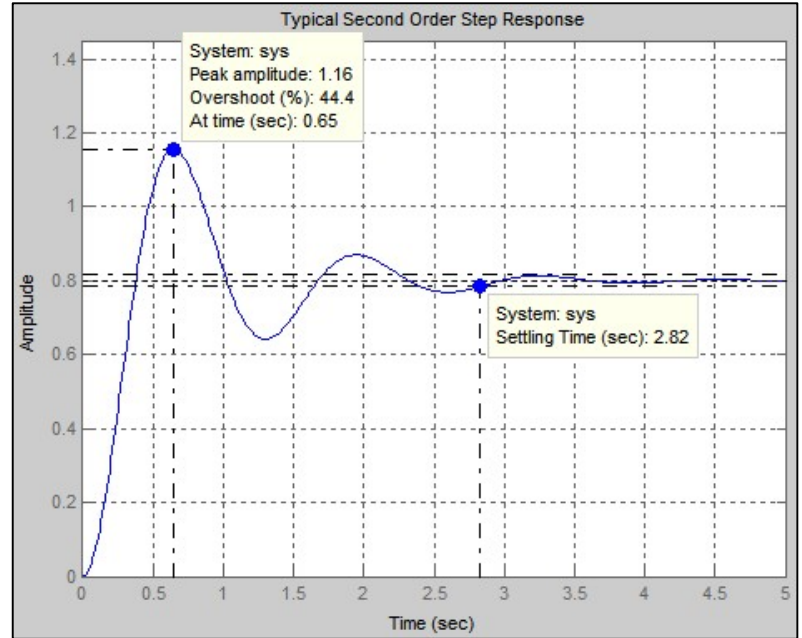
Characteristics of Under-damped, Second-Order System Step Response

Ref: Dorf & Bishop, Modern Control Systems, 12th edition, Prentice-Hall, Inc, 2010

The general form of the transfer function of an *under-damped, second-order system* with no zero (constant numerator) is

$$\frac{Y}{R}(s) = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For example, if $K = 20$, $\omega_n = 5$ (r/s), and $\zeta = 0.25$, the *unit step response* of the system is as shown in the figure. The MATLAB plot indicates there is approximately a 44% overshoot, a settling time $T_s \approx 2.8$ (sec), and a final value $f_v = 0.8$.



Using the table of Laplace transforms, it can be shown that the *response function* can be written as

$$y(t) = \frac{K}{\omega_n^2} \left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \left(\left(\omega_n \sqrt{1-\zeta^2} \right) t + \phi \right) \right)$$

From this result, relatively simple formulae can be derived for the basic *characteristics* of the response curve. The *percent overshoot* and the *peak time* (time it takes for the system to reach the top of the first peak) are given by the formulae

$$\%OS = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

Values of $\%OS$ and $\omega_n T_p$ can also be read from Fig. 5.8 from the Dorf & Bishop text. The settling time for the system is *approximately* $T_s \approx 4/\zeta\omega_n = 3.2$ (sec). Control system engineers often refer also to the *rise time* (T_r) of a system. This is often defined as the time it takes the

system to go from 10% to 90% of the final value for the first time. In the above example, $T_r \approx 0.25$ (sec).

For second order systems that ***have a zero***, the transient response can be affected by the ***location*** of the zero. If the zero is ***far left*** of the poles (along the real axis), then the above formula for percent overshoot can be used; however, if the zero is ***close to*** or ***to the right of*** the poles, then the above formula will under-predict the percent overshoot. The amount of error in using the equation can be large.