Elementary Engineering Mathematics
Introduction to Differential Equations in Mechanical Engineering

Differential equations are used to ***model*** (simulate) the response of physical systems. Their solutions allow us to ***predict*** the response of these systems without first building a prototype for testing. When used in this way, we refer to these equations as ***mathematical models***. These models can be used to ***simulate*** the ***response*** of very complex systems. The design of automobiles, airplanes, bridges, chemical plants, electronic circuits, power grids, biomedical devices, road systems (for improved traffic flow), and many others depend on them heavily.

Unfortunately, most differential equations do not have readily obtainable analytical solutions; however, they can be ***solved numerically*** using computer programs that employ a variety of numerical methods. As powerful as these methods are, it is often difficult to use them to draw general conclusions (and develop insight) about the many types of responses a system can exhibit. This is because we can only observe one solution at a time.

There are many classes of differential equations, however, for which we can find analytical solutions, and these solutions often provide reasonable predictions of the response of real systems. These solutions can be useful for developing insight into the nature of a system's response. In these notes, we consider the solutions of ***second-order, ordinary differential equations*** with ***constant coefficients***.

<u>Mass-Spring-Damper System</u>

m : mass of the block

k : spring stiffness $\left(f_{\text{sp}} = -k\,x\right)$

c : damping coefficient $\left(f_{\text{damper}} = -c\,\dfrac{dx}{dt} = -c\,\dot{x}\right)$

$f(t)$: applied external force (input)

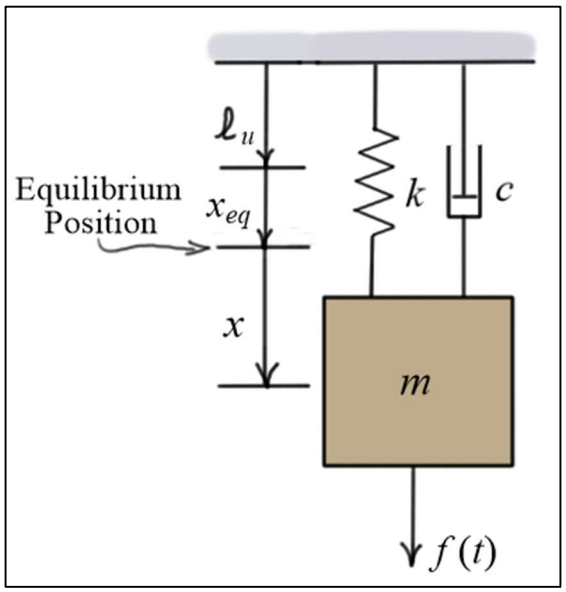
ℓ_u : unstretched (natural) length of spring

x_{eq} : static equilibrium position of mass

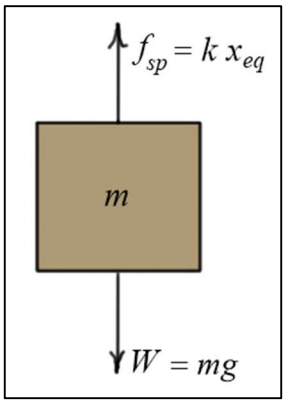
x : mass position ***relative to*** equilibrium position

$\dot{x}$: $v = \dfrac{dx}{dt}$, velocity of the mass

$\ddot{x}$: $a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$, acceleration of the mass



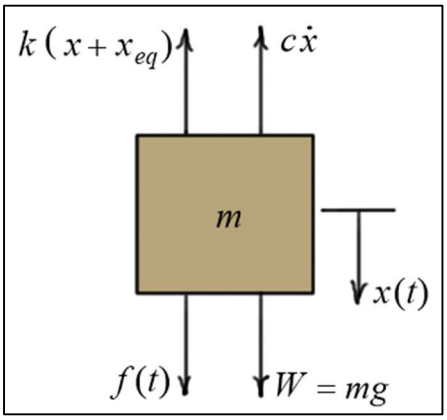
Static Equilibrium



For the mass to be in static equilibrium, the sum of the forces must be zero. We can use this fact to find the ***equilibrium position*** of the mass under its own weight.

$$+\downarrow \sum F = mg - k\,x_{eq} = 0 \quad \Rightarrow \quad x_{eq} = mg/k \qquad (1)$$

Differential Equation of Motion



Using Newton's second law, we can find the ***differential equation of motion*** of the mass. This equation can be used to ***predict*** the motion of the mass under a variety of conditions.

$$+\downarrow \sum F = f(t) + \cancel{mg} - k\left(x + \cancel{x_{eq}}\right) - c\dot{x} = ma = m\ddot{x}$$

$$\Rightarrow \boxed{m\,\ddot{x} + c\,\dot{x} + k\,x = f(t)} \qquad (2)$$

Note that ***static forces are not present*** in this equation.

The solution of this equation can be used to predict the ***forced response*** of the system ***relative*** to the ***equilibrium position***. The ***free response*** of the system is described by solving the equation with $f(t) \equiv 0$. In either case, to get a ***unique solution***, we must specify the ***initial conditions***, that is, we must specify $x(0)$ the ***initial position*** of the mass and $v(0) = \dot{x}(0)$ the ***initial velocity*** of the mass.

Eq. (2) is an example of a ***second-order, linear, constant coefficient ordinary differential equation***. In the following sections, we learn how to ***solve*** these types of equations.

Solving for Free Response $\left(f(t) \equiv 0\right)$

If we set $f(t)$ to ***zero***, Eq. (2) is called a ***homogeneous differential equation***, and its solution is called the ***homogeneous solution***. We can find this solution using the following steps.

1. Substitute $x(t) = Ae^{st}$ into the differential equation and find the conditions on A and s for this to be a solution to the equation.

$$x(t) = Ae^{st} \quad \Rightarrow \quad \dot{x}(t) = v(t) = \frac{dx}{dt} = \frac{d}{dt}\left(Ae^{st}\right) = Ase^{st}$$

$$\boxed{\dot{x}(t)=v(t)=Ase^{st} \quad \Rightarrow \quad \ddot{x}(t)=a(t)=\frac{dv}{dt}=\frac{d}{dt}\left(Ase^{st}\right)=As^2e^{st}}$$

Substituting these results into the differential equation gives

$$\boxed{\begin{aligned}0&=m\,\ddot{x}+c\,\dot{x}+k\,x=m\left(As^2e^{st}\right)+c\left(Ase^{st}\right)+k\left(Ae^{st}\right)\\&=\left(ms^2+cs+k\right)Ae^{st}\end{aligned}}$$

For a ***non-zero solution***, we require s to satisfy the equation $\boxed{ms^2+cs+k=0}$. This is called the ***characteristic equation*** of the system. There is ***no condition*** on A.

2. Find the roots of the characteristic equation and determine the ***type*** of ***response***.

$$\boxed{ms^2+cs+k=0 \quad \Rightarrow \quad s^2+\left(\tfrac{c}{m}\right)s+\tfrac{k}{m}=0}$$

Using the ***quadratic formula***,

$$\boxed{s_{1,2}=\frac{-\frac{c}{m}\pm\sqrt{\left(\frac{c}{m}\right)^2-4\left(\frac{k}{m}\right)}}{2}=-\frac{c}{2m}\pm\sqrt{\left(\frac{c}{2m}\right)^2-\left(\frac{k}{m}\right)}}$$

Case 1: ***real***, ***unequal roots***, $\boxed{\frac{c}{2m}>\sqrt{\frac{k}{m}}}$ (***over-damped***)

$\boxed{s_{1,2}=-\frac{c}{2m}\pm\sqrt{\left(\frac{c}{2m}\right)^2-\left(\frac{k}{m}\right)}}$ (two different real roots)

Case 2: ***real***, ***equal roots***, $\boxed{\frac{c}{2m}=\sqrt{\frac{k}{m}}}$ (***critical damping***)

$\boxed{s_1=s_2=-\frac{c}{2m}}$ (two identical real roots)

Case3: ***complex roots***, $\boxed{\frac{c}{2m}<\sqrt{\frac{k}{m}}}$ (***under-damped***)

$\boxed{s_{1,2}=-\frac{c}{2m}\pm i\sqrt{\left(\frac{k}{m}\right)-\left(\frac{c}{2m}\right)^2}=-\frac{c}{2m}\pm i\omega_d}$ (a pair of complex conjugate roots)

The frequency $\omega_d=\sqrt{\frac{k}{m}-\left(\frac{c}{2m}\right)^2}$ is the ***frequency*** in (rad/s) of the ***damped response***. If the damping is zero $\left(c=0\right)$, then the frequency of the response is $\omega_n=\sqrt{\frac{k}{m}}$ (rad/s). This is called the system's ***natural frequency***.

The ***types*** of ***solutions*** for each of these cases are summarized in the following table.

Case	Type of Roots	Type of Motion	Form of Solution
1	Real, unequal	Over-damped	$x(t) = Ae^{s_1 t} + Be^{s_2 t}$
2	Real, equal	Critically damped	$x(t) = Ae^{st} + Bte^{st}$
3	Complex conjugates	Under-damped	$x(t) = e^{-(c/2m)t}\left[A\sin(\omega_d t) + B\cos(\omega_d t)\right]$ $\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$

3. Finally, as we saw in earlier notes, the ***coefficients*** A and B are determined by applying the ***initial conditions***, $x(0) = x_0$ and $\dot{x}(0) = v_0$.

Example 1:

Given: $m = 0.5$ (slug), $c = 15$ (lb-s/ft), $k = 50$ (lb/ft), $x_0 = 0.25$ (ft), $v_0 = 5$ (ft/s)

Find: The displacement function $x(t)$

Solution:

Using the ***quadratic formula***, we find the ***roots*** of the ***characteristic equation***.

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)} = -\left(\frac{15}{2\times 0.5}\right) \pm \sqrt{\left(\frac{15}{2\times 0.5}\right)^2 - \frac{50}{0.5}} \approx \begin{Bmatrix} -3.82 \\ -26.18 \end{Bmatrix} \text{ (two different real roots)}$$

Given these ***real***, ***unequal roots***, the displacement function can be written as follows.

$$\boxed{x(t) = Ae^{-3.82t} + Be^{-26.18t}}$$

Now we can apply the ***initial conditions***:

$$\boxed{x(0) = \left(Ae^{-3.82t} + Be^{-26.18t}\right)\Big|_{t=0} = A + B = 0.25}$$

$$\boxed{\begin{aligned} \dot{x}(0) &= \left(\frac{d}{dt}\left[Ae^{-3.82t} + Be^{-26.18t}\right]\right)_{t=0} = \left(-3.82Ae^{-3.82t} - 26.18Be^{-26.18t}\right)_{t=0} \\ &= -3.82A - 26.18B \\ &= 5 \end{aligned}}$$

Or,

$$\begin{bmatrix} 1 & 1 \\ -3.82 & -26.18 \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0.25 \\ 5 \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} A \\ B \end{Bmatrix} \approx \begin{Bmatrix} 0.5163 \\ -0.2663 \end{Bmatrix}$$

Using the above results, the displacement function $x(t)$ is as follows.

$$x(t) = (0.5163)e^{-3.82t} - (0.2663)e^{-26.18t} \text{ (ft)}$$

Example 2:

Given: $m = 0.5$ (slug), $c = 5$ (lb-s/ft), $k = 50$ (lb/ft), $x_0 = 0.25$ (ft), $v_0 = 5$ (ft/s)

Find: The displacement function $x(t)$

Solution:

Using the quadratic formula, we find the roots of the characteristic equation.

$$s_{1,2} = -\tfrac{c}{2m} \pm \sqrt{\left(\tfrac{c}{2m}\right)^2 - \left(\tfrac{k}{m}\right)} = -\left(\tfrac{5}{2\times 0.5}\right) \pm \sqrt{\left(\tfrac{5}{2\times 0.5}\right)^2 - \tfrac{50}{0.5}} = -5 \pm \sqrt{75}\, i$$

Given these ***complex roots***, the solution is of the following form.

$$x(t) = e^{-(c/2m)t}\left[A\sin(\omega_d t) + B\cos(\omega_d t)\right] = e^{-5t}\left[A\sin(\sqrt{75}\, t) + B\cos(\sqrt{75}\, t)\right]$$

Now we can apply the ***initial conditions***:

$$x(0) = \left(e^{-(c/2m)t}\left[A\sin(\omega_d t) + B\cos(\omega_d t)\right]\right)_{t=0} = B = 0.25$$

And, using the ***product rule*** for ***differentiation***,

$$\begin{aligned} \dot{x}(0) &= \left\{\frac{d}{dt}\left(e^{-5t}\left[A\sin(\sqrt{75}\, t) + B\cos(\sqrt{75}\, t)\right]\right)\right\}_{t=0} \\ &= \left\{-5e^{-5t}\left[A\sin(\sqrt{75}\, t) + B\cos(\sqrt{75}\, t)\right]\right\}_{t=0} + \\ &\quad \left\{e^{-5t}\left[\sqrt{75}A\cos(\sqrt{75}\, t) - \sqrt{75}B\sin(\sqrt{75}\, t)\right]\right\}_{t=0} \\ &= -5B + \sqrt{75}A \\ &= 5 \end{aligned}$$

Solving gives, $A \approx 0.7217$ and $B = 0.25$. So, our ***under-damped solution*** is as follows.

$$x(t) \approx e^{-5t}\left[0.7217\sin(\sqrt{75}\, t) + 0.25\cos(\sqrt{75}\, t)\right] \text{ (ft)}$$

In previous notes, we found this could be written as a ***single***, ***phase-shifted sine function***. That is,

$$\boxed{x(t) = 0.7638\, e^{-5t} \sin\big((8.6603)t + 0.3335\big) \text{ (ft)}}$$

Solving for the Forced Response $\big(f(t) \neq 0\big)$

To find the forced response, we start by finding the "***particular solution***". The particular solution can sometimes be found using the ***method of undetermined coefficients*** provided the forcing function $f(t)$ has a simple form. The forced response is the sum of $x_P(t)$ the ***particular solution*** and $x_H(t)$ the solution of the homogenous equation (steps 1 and 2, only).

$$\boxed{x(t) = x_P(t) + x_H(t)} \qquad (3)$$

The coefficients of $x_H(t)$ are found by applying the initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$.

Forms of the particular solution for some common engineering forcing functions are given in the table below. In the method of undetermined coefficients, we substitute the ***general form*** of $x_P(t)$ back into the differential equation and ***solve*** for the ***unknown coefficients*** (referred to as B_0, B_1, and B_2 in the table below).

	$f(t)$	Form* of $x_P(t)$
constant	a_0	$B_0\, t^n$
linear	$a_1 t + a_0$	$\big(B_1 t + B_0\big)t^n$
quadratic	$a_2 t^2 + a_1 t + a_0$	$\big(B_2 t^2 + B_1 t + B_0\big)t^n$
exponential	$a\, e^{\beta t}$	$\big(B_1 e^{\beta t}\big)t^n$
sine or cosine	$a\sin(\omega t)$ ***or*** $a\cos(\omega t)$	$\big[B_1 \sin(\omega t) + B_2 \cos(\omega t)\big]t^n$
exponential, sine or cosine product	$a\, e^{\beta t}\sin(\omega t)$ ***or*** $a\, e^{\beta t}\cos(\omega t)$	$e^{\beta t}\big[B_1 \sin(\omega t) + B_2 \cos(\omega t)\big]t^n$

* The ***exponent*** n is the ***smallest, non-negative integer*** so that every term in $x_P(t)$ is ***different*** from every term in $x_H(t)$. That is, $n = 0$ unless the same type of term appears in $x_H(t)$.

Example 3:

Given: $m = 0.5$ (slug), $c = 15$ (lb-s/ft), $k = 50$ (lb/ft), $x_0 = 0.25$ (ft), $v_0 = 5$ (ft/s), $f(t) = 30\sin(5t)$

Find: (a) the form of the homogeneous solution $x_H(t)$; (b) the particular solution $x_P(t)$; and (c) the displacement function $x(t)$

Solution:

(a) Setting $f(t) = 0$, our differential equation is as it was in Example 1 above. The general form of that solution is the form or our ***homogeneous solution***.

$$\boxed{x_H(\mathrm{t}) = Ae^{-3.82t} + Be^{-26.18t}}$$ (over-damped response function)

(b) Given $f(t) = 3\sin(10t)$ is not of the form of any term in our homogeneous solution, we set $n = 0$, and the ***particular solution*** has the form

$$\boxed{x_P(t) = D\sin(5t) + E\cos(5t)}$$

We find values for the coefficients D and E by substituting the form of $x_P(t)$ into the differential equation and equating coefficients of like terms. First, we differentiate $x_P(t)$.

$$\boxed{\dot{x}_P(t) = \frac{d}{dt}\big(D\sin(5t) + E\cos(5t)\big) = 5D\cos(5t) - 5E\sin(5t)}$$

$$\boxed{\ddot{x}_P(t) = \frac{d}{dt}\big(5D\cos(5t) - 5E\sin(5t)\big) = -25D\sin(5t) - 25E\cos(5t)}$$

Substituting into the differential equation: $\boxed{m\ddot{x} + c\dot{x} + kx = f(t)}$

$$\begin{aligned} m\ddot{x}_P + c\dot{x}_P + kx_P &= m\big(-25D\sin(5t) - 25E\cos(5t)\big) + c\big(5D\cos(5t) - 5E\sin(5t)\big) \\ &\quad + k\big(D\sin(5t) + E\cos(5t)\big) \\ &= \big(-25mD - 5cE + kD\big)\sin(5t) + \big(-25mE + 5cD + kE\big)\cos(10t) \\ &= 30\sin(5t) \end{aligned}$$

Comparing coefficients of the sine and cosine functions on each side of the equation gives two algebraic equations that can be solved for D and E.

$$\begin{bmatrix} (k-25m) & -5c \\ 5c & (k-25m) \end{bmatrix} \begin{Bmatrix} D \\ E \end{Bmatrix} = \begin{bmatrix} 37.5 & -75 \\ 75 & 37.5 \end{bmatrix} \begin{Bmatrix} D \\ E \end{Bmatrix} = \begin{Bmatrix} 30 \\ 0 \end{Bmatrix} \quad \Rightarrow \boxed{\begin{Bmatrix} D \\ E \end{Bmatrix} = \begin{Bmatrix} 0.16 \\ -0.32 \end{Bmatrix}}$$

Given these results, the particular solution is as follows.

$$\boxed{x_P(t) = 0.16\sin(5t) - 0.32\cos(5t)}$$

(c) The form of the complete solution is the ***sum*** of the ***homogeneous*** and ***particular*** solutions.

$$\boxed{x(t) = x_H(t) + x_P(t) = Ae^{-3.82t} + Be^{-26.18t} + 0.16\sin(5t) - 0.32\cos(5t)}$$

To find the coefficients A and B, we ***apply*** the ***initial conditions***.

$$\boxed{x(0) = \left[Ae^{-3.82t} + Be^{-26.18t} + 0.16\sin(5t) - 0.32\cos(5t) \right]_{t=0} = A + B - 0.32 = 0.25}$$

$$\boxed{\begin{aligned} \dot{x}(0) &= \left(\frac{d}{dt}\left[Ae^{-3.82t} + Be^{-26.18t} + 0.16\sin(5t) - 0.32\cos(5t) \right] \right)_{t=0} \\ &= \left(-3.82Ae^{-3.82t} - 26.18Be^{-26.18t} + (5\times 0.16)\cos(5t) + (5\times 0.32)\sin(5t) \right)_{t=0} \\ &= -3.82A - 26.18B + 0.8 \\ &= 5 \end{aligned}}$$

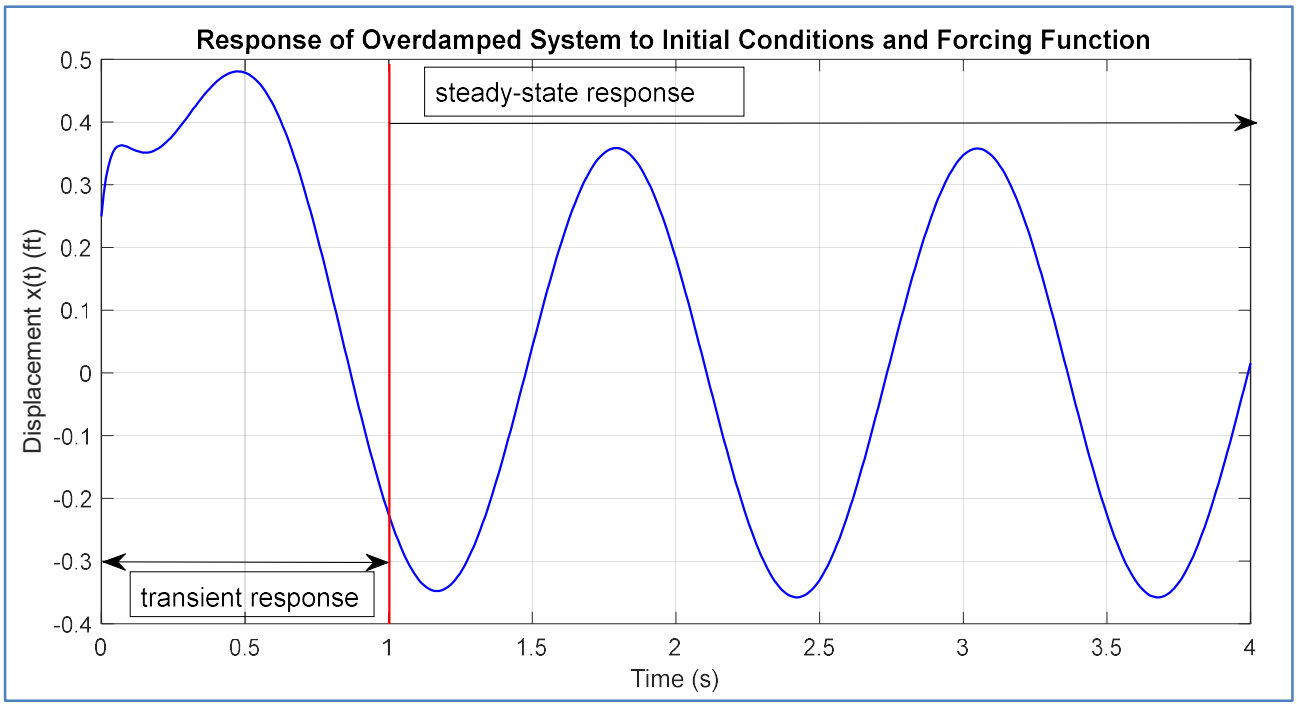
Or,

$$\begin{bmatrix} 1 & 1 \\ -3.82 & -26.18 \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0.57 \\ 4.2 \end{Bmatrix} \quad \Rightarrow \boxed{\begin{Bmatrix} A \\ B \end{Bmatrix} \approx \begin{Bmatrix} 0.855 \\ -0.2852 \end{Bmatrix}}$$

Finally, the complete solution is as follows.

$$\boxed{x(t) = x_H(t) + x_P(t) = \underbrace{0.855\,e^{-3.82t} - 0.2852\,e^{-26.18t}}_{\text{transient terms}} + \underbrace{0.16\sin(5t) - 0.32\cos(5t)}_{\text{steady-state terms}}}$$

This solution has both ***transient*** and ***steady-state*** terms. The transient terms go to ***zero*** as time progresses, but the steady-state terms do not. The solution is shown in the plot below. Approximately the first one second has contributions from the transient and steady-state terms. After that time, the transient terms are nearly zero, and the solution is dominated by the steady-state terms.


Response of Overdamped System to Initial Conditions and Forcing Function
steady-state response
transient response
Displacement x(t) (ft)
Time (s)
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0
0.5
1
1.5
2
2.5
3
3.5
4