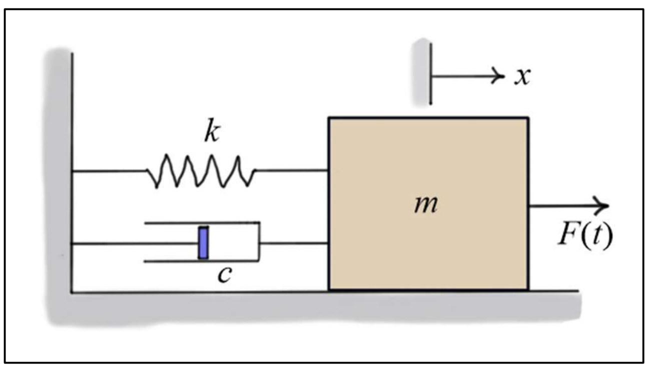
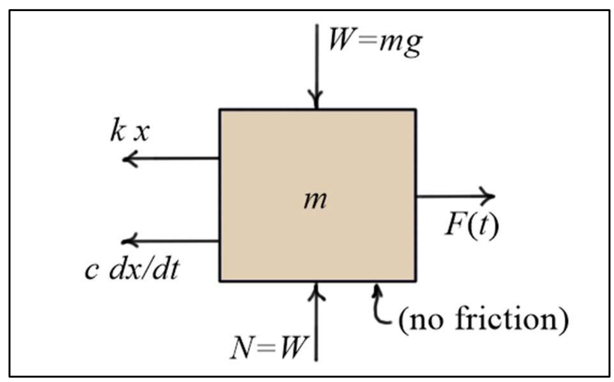
Elementary Dynamics
Introduction to Vibrations

Vibrations are a very common problem in mechanical and structural systems. Vibrations can lead to problems with wear, material fatigue, and noise that can affect the performance of a system. As an introduction to this important topic, consider the following simple spring-mass-damper (SMD) system shown below.

The mass m is excited by the external force $F(t)$ and the linear spring and damper that are connected to the wall. The displacement $x(t)$ is measured from the position where the spring is ***unstretched***. This is known as the equilibrium position. When the mass is away from its equilibrium position, the linear spring applies a force proportional to the displacement $x(t)$ and the linear damper applies a force proportional to the velocity $\dot{x}(t)$ as shown in the free body diagram. The spring coefficient k has units of force per unit length (lb/ft or N/m), and the damper coefficient has unit of force per unit velocity (lb-s/ft or N-s/m).





Free Body Diagram

Summing forces in the x-direction on the free body diagram gives

$$\sum F_x = F(t) - kx - c\frac{dx}{dt} = ma_x = m\frac{d^2x}{dt^2}$$

or

$$\boxed{\ddot{x} + \left(\frac{c}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = \frac{F(t)}{m}}$$

This equation is a ***linear***, ***second-order***, ***constant coefficient***, ***ordinary*** differential equation. It is the equation of motion of the mass and describes the motion of the mass for all time. Why?

Free, Undamped Response ($F(t) = 0,\ c = 0$)

In the case of free, undamped response the differential equation of motion becomes

$$\boxed{\ddot{x} + \left(\frac{k}{m}\right) x = 0}$$

The ***solution*** to this differential equation is $\boxed{x(t) = A\sin(\omega_n t + \phi)}$ where $\boxed{\omega_n = \sqrt{k/m}\ \text{(rad/s)}}$ is called the ***natural frequency*** of the system. So, if the mass is moved to the right and released from rest, it will oscillate about the equilibrium position with frequency ω_n. The values of A and ϕ depend on the initial conditions.

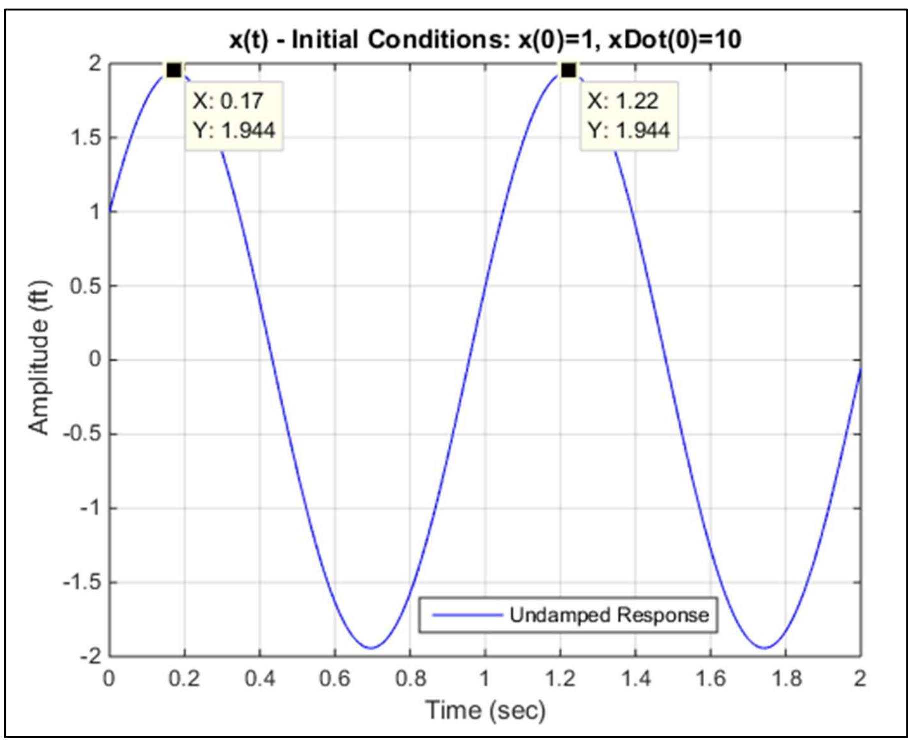
The ***natural frequency*** f in ***cycles/second*** or ***Hertz*** (Hz) is $\boxed{f = \omega_n / 2\pi\ \text{(Hz)}}$, and the ***period*** of ***oscillation*** $\boxed{T = 1/f}$ is the number of seconds required to complete one cycle. The plot below shows an example of free, undamped response for a system with

Physical parameters: $m = 1$ (slug), $k = 36$ (lb/ft)

Initial conditions: $x(0) = 1$ (ft), $\dot{x}(0) = 10$ (ft/s)

For this system, $\omega_n = \sqrt{k/m} = 6$ (rad/sec), $f = 6/2\pi \approx 0.9549$ (Hz), and $T = 1/f \approx 1.047$ (sec).

Note from the plot that $T \approx 1.22 - 0.17 = 1.05$ which is as expected.



Free, Damped Response ($F(t) = 0$)

For the case of free, damped response, the differential equation of motion becomes

$$\boxed{\ddot{x} + \left(\frac{c}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = 0}$$

To characterize the motion of this type of system, define the ***damping ratio*** $\zeta = \dfrac{c}{2\sqrt{mk}}$. This represents the ratio of the actual damping factor c to the "***critical***" damping factor $c_c \triangleq 2\sqrt{mk}$. In this way, the response can be characterized by the value of ζ as shown in the following table.

Value	Type of Response	Description of Response
$0 < \zeta < 1$	Under-damped	system oscillations decay with time
$\zeta = 1$	Critically Damped	system response decays with time, no oscillation
$\zeta > 1$	Over-damped	system response decays with time, no oscillation

Case1: Under-damped Response ($0 < \zeta < 1$)

For under-damped systems, the solution to the differential equation of motion is

$$\boxed{x(t) = B\, e^{(-c/2m)\,t} \sin(\omega_d\, t + \phi)}$$

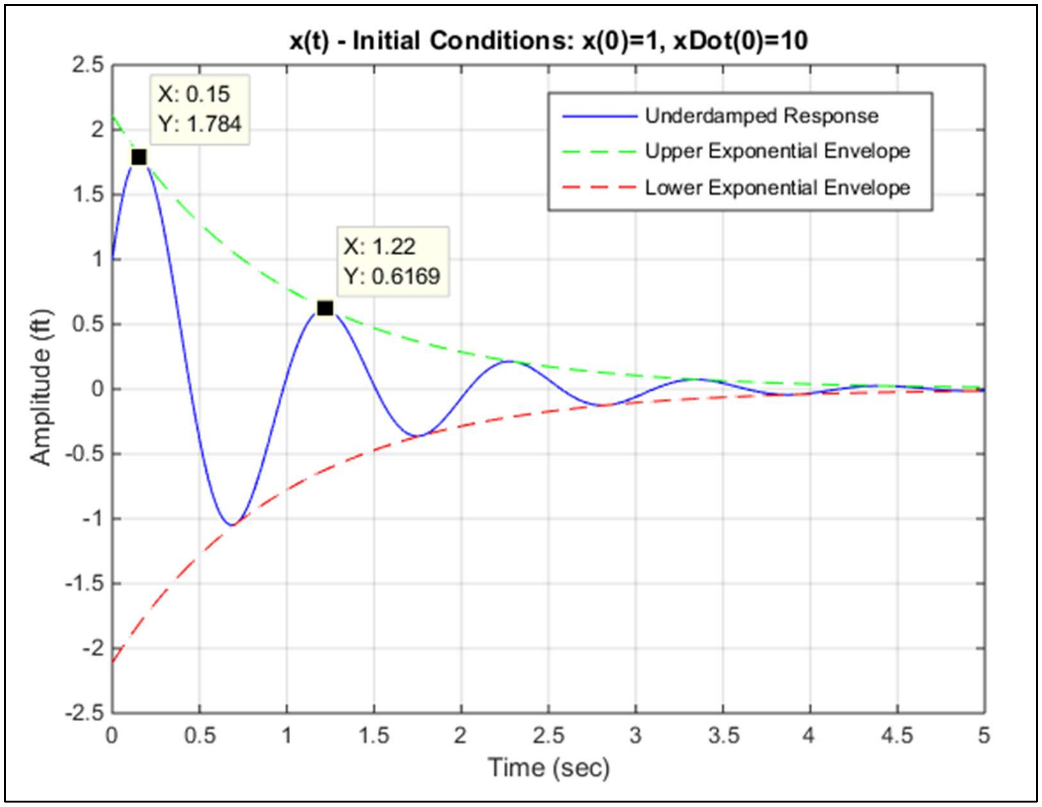
As before, the values of B and ϕ depend on the initial conditions. The damped frequency of oscillation is $\boxed{\omega_d = \sqrt{\omega_n^2 - (c/2m)^2}}$.

The plot below shows an example of free, damped response for a system with

Physical parameters: $m = 1$ (slug), $k = 36$ (lb/ft), $c = 2.0$ (lb-s/ft)

Initial conditions: $x(0) = 1$ (ft), $\dot{x}(0) = 10$ (ft/s)

For this system, $\omega_n = \sqrt{k/m} = 6$ (rad/sec) and $\omega_d \approx 5.916$ (rad/s). Other than the damping effect, these are the same parameters as for the free, undamped system above. Note from the plot the period of the damped oscillations is $T \approx 1.22 - 0.15 = 1.07$ (sec) which is ***slightly longer*** than the period of the undamped oscillations. Note also the amplitude of the first peak is less than that of the system without damping.



Case 2: Critically Damped Response ($\zeta = 1$)

For ***critically damped*** systems ($c = c_c$), the solution to the differential equation of motion can be written as

$$\boxed{x(t) = A\,e^{-at} + B\,t\,e^{-at}}$$

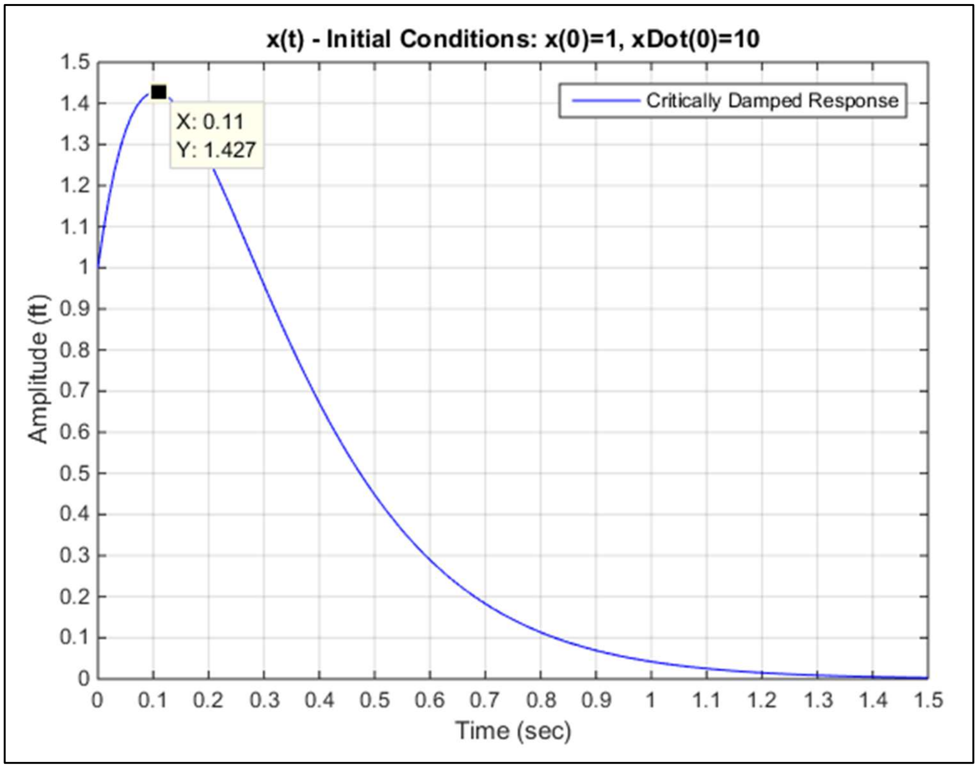
where the values of A and B depend on the ***initial conditions***. Note that in this case there is ***just enough*** damping to stop the oscillations from occurring.

The plot below shows an example of free, critically damped response for a system with

Physical parameters: $m = 1$ (slug), $k = 36$ (lb/ft), $c = c_c = 12.0$ (lb-s/ft)

Initial conditions: $x(0) = 1$ (ft), $\dot{x}(0) = 10$ (ft/s)

Other than the damping coefficient, these are the same parameters as for the underdamped system shown above. Note from the plot the system approaches the equilibrium position exponentially and does not oscillate about it. Note also the first peak of the response is less than that of the underdamped system.



Case 3: Over-damped Response ($\zeta > 1$)

For ***over-damped*** systems ($c > c_c$), the solution to the differential equation of motion is

$$\boxed{x(t) = A\, e^{-a\,t} + B\, e^{-b\,t}}$$

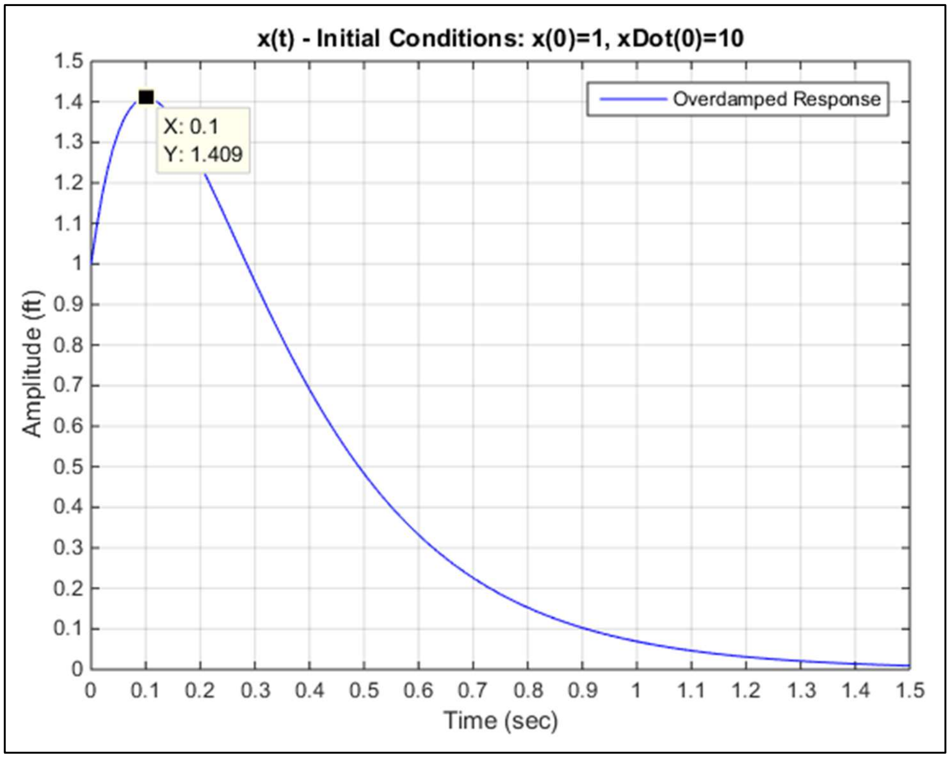
As before, the values of A and B depend on the ***initial conditions***. Note that in this case there is ***more than enough*** damping to stop the oscillations from occurring.

The plot below shows an example of free, overdamped response for a system with

Physical parameters: $m = 1$ (slug), $k = 36$ (lb/ft), $c = 13.0$ (lb-s/ft)

Initial conditions: $x(0) = 1$ (ft), $\dot{x}(0) = 10$ (ft/s)

Other than the damping coefficient, these are the same parameters as for the damped systems shown above. Note from the plot, as for the critically damped system, the overdamped system approaches the equilibrium position exponentially and does not oscillate about it. Note also the first peak of the response is less than that of the critically damped system.



Forced, Damped Response

In this case, the differential equation of motion is

$$\boxed{\ddot{x}+\left(\frac{c}{m}\right)\dot{x}+\left(\frac{k}{m}\right)x=\frac{F(t)}{m}}.$$

The ***solution*** of this differential equation is of the form

$$\boxed{x(t)=x_H(t)+x_P(t)}$$

Here, $x_H(t)$ represents the "***homogeneous***" solution (i.e. the free response when $F(t)=0$), and $x_P(t)$ represents the "***particular***" solution. The particular solution has the same form as the forcing function $F(t)$.

Definitions

Transient Response: The part of $x(t)$ that ***decays*** with time.

Steady-State Response: The part of $x(t)$ that remains after all decaying parts have vanished.

The plot below shows an example of forced, damped response for a system with

Physical parameters: $m = 1$ (slug), $k = 36$ (lb/ft), $c = 2.0$ (lb-s/ft)

Initial conditions: $x(0) = 1$ (ft), $\dot{x}(0) = 10$ (ft/s)

Forcing function: $F(t) = 5\sin(3t)$ (lb)

These system parameters and initial conditions are the same as for the example underdamped system shown above, but the system now also responds to a sinusoidal forcing function. The total response is a superposition (sum) of the responses due to the nonzero initial conditions and the forcing function. The response to initial conditions will decay to zero and is called the transient response. The response to the sinusoidal forcing function continues for as long as the force is applied and is called the steady-state response.

In the plot below, note the initial part of the response has a period of $T_{\text{initial}} \approx 1.19 - 0.16 = 1.03$ (sec) which is consistent with the response of the unforced system, and the later part of the response has a period of $T_{\text{later}} \approx 8.97 - 6.88 = 2.09$ (sec) which is consistent with the forcing function whose frequency is close to 0.5 (Hz).