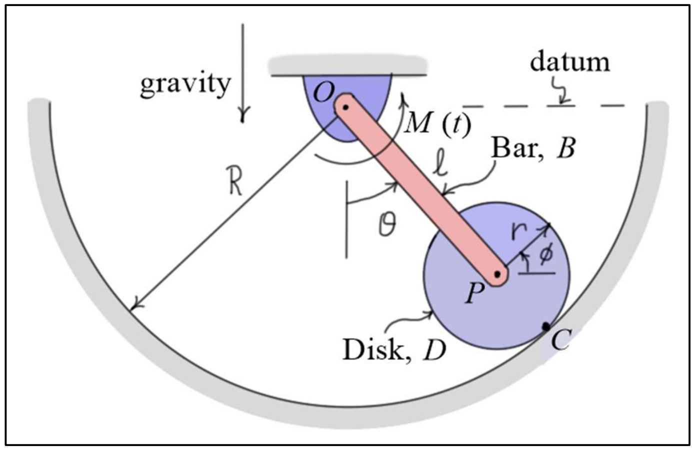
Intermediate Dynamics
Lagrange's Equations Examples

Example #1

The system at the right consists of ***two bodies***, a ***slender bar*** B and a ***disk*** D, moving together in a ***vertical plane***. As B rotates about O, D ***rolls without slipping*** on the fixed circular outer surface. The length of B is ℓ, the radius of D is r, and the radius of the outer surface is R. The mass of the bar and disk are both m. The external torque $M(t)$ drives the system.



Equation of Motion

Using θ as the single ***generalized coordinate***, the equation of motion of the system can be found from Lagrange's equation.

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = F_\theta} \tag{1}$$

Here, L is the Lagrangian of the system and is defined as follows.

$$\boxed{L = K - V = K_B + K_D - V_B - V_D}$$

The individual kinetic and potential energy terms are as follows.

$$K_D = \tfrac{1}{2}\underset{\sim}{\omega}_D \cdot \underset{\sim}{H}_C = \tfrac{1}{2} I_C \dot{\phi}^2 = \tfrac{1}{2}(\tfrac{1}{2}mr^2 + mr^2)\dot{\phi}^2 \quad \Rightarrow \boxed{K_D = \tfrac{3}{4}mr^2\dot{\phi}^2} \quad \text{(fixed axis rotation)}$$

$$K_B = \tfrac{1}{2}\underset{\sim}{\omega}_B \cdot \underset{\sim}{H}_O = \tfrac{1}{2} I_O \dot{\theta}^2 = \tfrac{1}{2}(\tfrac{1}{3}m\ell^2)\dot{\theta}^2 \quad \Rightarrow \boxed{K_B = \tfrac{1}{6}m\ell^2\dot{\theta}^2} \quad \text{(fixed axis rotation)}$$

$$V = V_D + V_B = -mg\ell C_\theta - \tfrac{1}{2}mg\ell C_\theta \quad \Rightarrow \boxed{V = -\tfrac{3}{2}mg\ell C_\theta}$$

The concept of ***instantaneous centers*** can now be used to express L in terms of θ and $\dot{\theta}$ only. Specifically, the velocity of point P can be calculated as a point at the end of B or at the center of D. That is, $\boxed{v_P = \ell\dot{\theta} = -r\dot{\phi}}$. Using this equation to remove $\dot{\phi}$ from the Lagrangian gives

$$\boxed{L = \tfrac{11}{12}m\ell^2\dot{\theta}^2 + \tfrac{3}{2}mg\ell C_\theta}$$

The ***generalized active force*** F_θ and the derivatives of the Lagrangian are as follows.

$$\boxed{F_\theta = M\,\underset{\sim}{k}\cdot\frac{\partial}{\partial\dot{\theta}}\left(\underset{\sim}{\omega}_B\right) = M\,\underset{\sim}{k}\cdot\underset{\sim}{k} = M(t)}$$

$$\frac{\partial L}{\partial\dot{\theta}} = \tfrac{11}{6}m\ell^2\dot{\theta} \qquad \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \tfrac{11}{6}m\ell^2\ddot{\theta}} \qquad \boxed{\frac{\partial L}{\partial\theta} = -\tfrac{3}{2}mg\ell S_\theta}$$

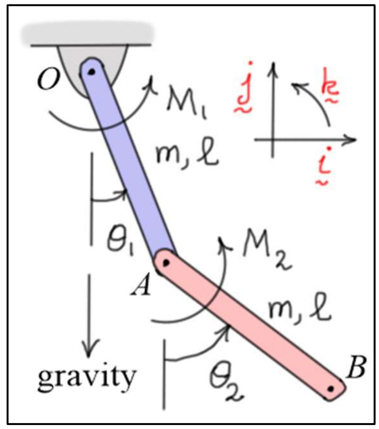
Substituting these results into Lagrange's Eq. (1) gives the ***equation of motion***.

$$\boxed{\tfrac{11}{6}m\ell^2\ddot{\theta} + \tfrac{3}{2}mg\ell S_\theta = M(t)} \tag{2}$$

Eq. (2) is a ***nonlinear***, ***second-order***, ***ordinary differential equation***.

Example #2 – Double Pendulum

The figure to the right shows a ***double pendulum*** in a vertical plane with ***driving torques*** M_1 and M_2 at the connecting joints. The two ***uniform slender links*** are assumed to be identical with mass m and length ℓ. The system has ***two degrees-of-freedom*** described by the ***generalized coordinate set*** (θ_1, θ_2).



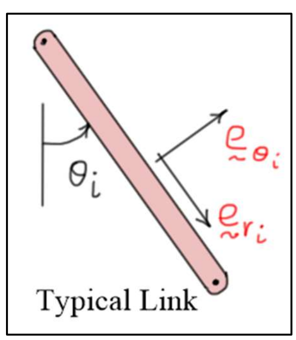
Equation of Motion

Using θ_1 and θ_2 as the two ***generalized coordinates***, the ***equations of motion*** of the system can be found using ***Lagrange's equations***.

$$\boxed{\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} &= F_{\theta_1} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) - \frac{\partial L}{\partial\theta_2} &= F_{\theta_2} \end{aligned}} \tag{3}$$

Kinematics

The velocities and squares of velocities of the mass centers of the two links can be found using the concept of ***relative velocity*** as follows. For the definition of the unit vectors, refer to the diagram.



Typical Link

$$\underset{\sim}{v}_{G_1} = \underset{\sim}{v}_O + \underset{\sim}{v}_{G_1/O} = \tfrac{1}{2}\ell\dot{\theta}_1\,\underset{\sim}{e}_{\theta_1}$$

$$\underset{\sim}{v}_{G_2} = \underset{\sim}{v}_A + \underset{\sim}{v}_{G_2/A} = \underset{\sim}{v}_O + \underset{\sim}{v}_{A/O} + \underset{\sim}{v}_{G_2/A} = \ell\dot{\theta}_1\,\underset{\sim}{e}_{\theta_1} + \tfrac{1}{2}\ell\dot{\theta}_2\,\underset{\sim}{e}_{\theta_2}$$

$$v_{G_1}^2 = \underset{\sim}{v}_{G_1} \cdot \underset{\sim}{v}_{G_1} = \tfrac{1}{4}\ell^2\dot{\theta}_1^2$$

$$v_{G_2}^2 = \underset{\sim}{v}_{G_2} \cdot \underset{\sim}{v}_{G_2} = \ell^2\dot{\theta}_1^2 + \tfrac{1}{4}\ell^2\dot{\theta}_2^2 + 2\left(\tfrac{1}{2}\ell^2\dot{\theta}_1\dot{\theta}_2\right)\left(\underset{\sim}{e}_{\theta_1} \cdot \underset{\sim}{e}_{\theta_2}\right) = \ell^2\dot{\theta}_1^2 + \tfrac{1}{4}\ell^2\dot{\theta}_2^2 + \ell^2\dot{\theta}_1\dot{\theta}_2 C_{2-1}$$

Here, C_{2-1} represents the cosine of $\theta_2 - \theta_1$.

Kinetic Energy

The ***kinetic energy*** of the system can be written as the sum of the kinetic energies of the two links.

$$\boxed{K = K_1 + K_2}$$

Here,

$$\boxed{K_1 = \tfrac{1}{2} I_O \dot{\theta}_1^2 = \tfrac{1}{2}\left(\tfrac{1}{3}m\ell^2\right)\dot{\theta}_1^2 = \tfrac{1}{6}m\ell^2\dot{\theta}_1^2} \qquad \text{(fixed axis rotation)}$$

$$K_2 = \tfrac{1}{2}mv_{G_2}^2 + \tfrac{1}{2}I_{G_2}\dot{\theta}_2^2 = \tfrac{1}{2}m\ell^2\dot{\theta}_1^2 + \tfrac{1}{8}m\ell^2\dot{\theta}_2^2 + \tfrac{1}{2}m\ell^2\dot{\theta}_1\dot{\theta}_2 C_{2-1} + \tfrac{1}{24}m\ell^2\dot{\theta}_2^2$$

$$\Rightarrow \boxed{K_2 = \tfrac{1}{2}m\ell^2\dot{\theta}_1^2 + \tfrac{1}{6}m\ell^2\dot{\theta}_2^2 + \tfrac{1}{2}m\ell^2\dot{\theta}_1\dot{\theta}_2 C_{2-1}} \qquad \text{(general plane motion)}$$

Potential Energy

Assuming the horizontal ***datum*** is level with the point O, the ***potential energy*** of the system can be written as follows.

$$\boxed{V = V_1 + V_2 = -\tfrac{1}{2}mg\ell C_1 - mg\left(\ell C_1 + \tfrac{1}{2}\ell C_2\right) = -\tfrac{3}{2}mg\ell C_1 - \tfrac{1}{2}mg\ell C_2}$$

Lagrangian $\boxed{L \triangleq K - V}$

Using the above results, the Lagrangian can be written as

$$\boxed{L = \tfrac{2}{3}m\ell^2\dot{\theta}_1^2 + \tfrac{1}{6}m\ell^2\dot{\theta}_2^2 + \tfrac{1}{2}m\ell^2\dot{\theta}_1\dot{\theta}_2 C_{2-1} + \tfrac{3}{2}mg\ell C_1 + \tfrac{1}{2}mg\ell C_2} \qquad (4)$$

Generalized Forces

The ***generalized active forces*** associated with the ***driving torques*** can be calculated as follows. Note that a torque of M_2 is applied to link 2 and a ***reaction torque*** of $-M_2$ is applied to link 1.

$$F_{\theta_1} = \left(M_1 \underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_1}{\partial \dot{\theta}_1}\right) + \left(-M_2 \underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_1}{\partial \dot{\theta}_1}\right) + \left(M_2 \underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_2}{\partial \dot{\theta}_1}\right) \quad \Rightarrow \boxed{F_{\theta_1} = M_1 - M_2}$$

$$F_{\theta_2} = \left(M_1 \underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_1}{\partial \dot{\theta}_2}\right) + \left(-M_2 \underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_1}{\partial \dot{\theta}_2}\right) + \left(M_2 \underset{\sim}{k} \cdot \frac{\partial \underset{\sim}{\omega}_2}{\partial \dot{\theta}_2}\right) \quad \Rightarrow \boxed{F_{\theta_2} = M_2}$$

Derivatives of Lagrangian

Using the expression given in Eq. (4), the derivatives of the Lagrangian can be calculated as follows.

$$\frac{\partial L}{\partial \dot{\theta}_1} = \tfrac{4}{3} m\ell^2 \dot{\theta}_1 + \tfrac{1}{2} m\ell^2 \dot{\theta}_2 C_{2-1}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = \tfrac{4}{3} m\ell^2 \ddot{\theta}_1 + \tfrac{1}{2} m\ell^2 C_{2-1} \ddot{\theta}_2 - \tfrac{1}{2} m\ell^2 \dot{\theta}_2 \left(\dot{\theta}_2 - \dot{\theta}_1\right) S_{2-1}}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \tfrac{1}{2} m\ell^2 C_{2-1} \dot{\theta}_1 + \tfrac{1}{3} m\ell^2 \dot{\theta}_2$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = \tfrac{1}{2} m\ell^2 C_{2-1} \ddot{\theta}_1 + \tfrac{1}{3} m\ell^2 \ddot{\theta}_2 - \tfrac{1}{2} m\ell^2 \dot{\theta}_1 \left(\dot{\theta}_2 - \dot{\theta}_1\right) S_{2-1}}$$

$$\boxed{\frac{\partial L}{\partial \theta_1} = \tfrac{1}{2} m\ell^2 \dot{\theta}_1 \dot{\theta}_2 S_{2-1} - \tfrac{3}{2} mg\ell S_1} \qquad \boxed{\frac{\partial L}{\partial \theta_2} = -\tfrac{1}{2} m\ell^2 \dot{\theta}_1 \dot{\theta}_2 S_{2-1} - \tfrac{1}{2} mg\ell S_2}$$

Substituting into Lagrange's equations of Eq. (3) gives the following ***equations of motion***.

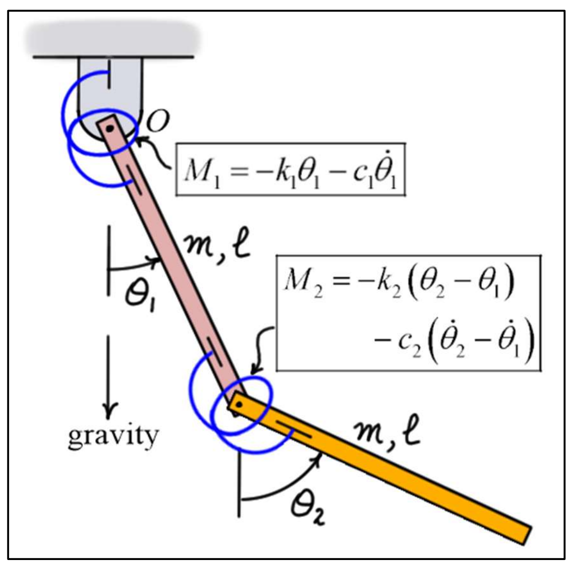
$$\boxed{\left(\tfrac{4}{3} m\ell^2\right)\ddot{\theta}_1 + \left(\tfrac{1}{2} m\ell^2 C_{2-1}\right)\ddot{\theta}_2 - \left(\tfrac{1}{2} m\ell^2 S_{2-1}\right)\dot{\theta}_2^2 + \tfrac{3}{2} mg\ell S_1 = M_1(t) - M_2(t)} \qquad (5)$$

$$\boxed{\left(\tfrac{1}{2} m\ell^2 C_{2-1}\right)\ddot{\theta}_1 + \left(\tfrac{1}{3} m\ell^2\right)\ddot{\theta}_2 + \left(\tfrac{1}{2} m\ell^2 S_{2-1}\right)\dot{\theta}_1^2 + \tfrac{1}{2} mg\ell S_2 = M_2(t)} \qquad (6)$$

Together, Eqs. (5) and (6) represent a ***coupled*** set of ***nonlinear***, ***second-order***, ***ordinary differential equations*** of motion for the ***double pendulum*** with ***driving torques***.

Example – Double Pendulum with Springs and Dampers

The figure at the right shows a ***double pendulum*** as in the above example with the ***driving torques*** replaced with a set of ***torsional springs*** and ***dampers***. The equations of motion of this system are easily found using the results from the previous example given that



$$\boxed{M_1 = -k_1\theta_1 - c_1\dot{\theta}_1}$$

$$\boxed{M_2 = -k_2\left(\theta_2 - \theta_1\right) - c_2\left(\dot{\theta}_2 - \dot{\theta}_1\right)}$$

Substituting these results into Eqs. (5) and (6) gives

$$\left(\tfrac{4}{3} m\ell^2\right)\ddot{\theta}_1 + \left(\tfrac{1}{2} m\ell^2 C_{2-1}\right)\ddot{\theta}_2 - \left(\tfrac{1}{2} m\ell^2 S_{2-1}\right)\dot{\theta}_2^2 + \tfrac{3}{2} mg\ell S_1 = -k_1\theta_1 - c_1\dot{\theta}_1 + k_2\left(\theta_2 - \theta_1\right) + c_2\left(\dot{\theta}_2 - \dot{\theta}_1\right)$$

$$\left(\tfrac{1}{2} m\ell^2 C_{2-1}\right)\ddot{\theta}_1 + \left(\tfrac{1}{3} m\ell^2\right)\ddot{\theta}_2 + \left(\tfrac{1}{2} m\ell^2 S_{2-1}\right)\dot{\theta}_1^2 + \tfrac{1}{2} mg\ell S_2 = -k_2\left(\theta_2 - \theta_1\right) - c_2\left(\dot{\theta}_2 - \dot{\theta}_1\right)$$

Or,

$$\left(\tfrac{4}{3}m\ell^2\right)\ddot{\theta}_1 + \left(\tfrac{1}{2}m\ell^2 C_{2-1}\right)\ddot{\theta}_2 - \left(\tfrac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_2^2 + \tfrac{3}{2}mg\ell S_1 + \left(c_1 + c_2\right)\dot{\theta}_1 - c_2\dot{\theta}_2 + \left(k_1 + k_2\right)\theta_1 - k_2\theta_2 = 0 \tag{7}$$

$$\left(\tfrac{1}{2}m\ell^2 C_{2-1}\right)\ddot{\theta}_1 + \left(\tfrac{1}{3}m\ell^2\right)\ddot{\theta}_2 + \left(\tfrac{1}{2}m\ell^2 S_{2-1}\right)\dot{\theta}_1^2 + \tfrac{1}{2}mg\ell S_2 + c_2\left(\dot{\theta}_2 - \dot{\theta}_1\right) + k_2\left(\theta_2 - \theta_1\right) = 0 \tag{8}$$

Eqs. (7) and (8) represent a set of ***two simultaneous, nonlinear***, ***second-order***, ***ordinary differential equations of motion*** of the double pendulum with springs and dampers at the connecting joints.