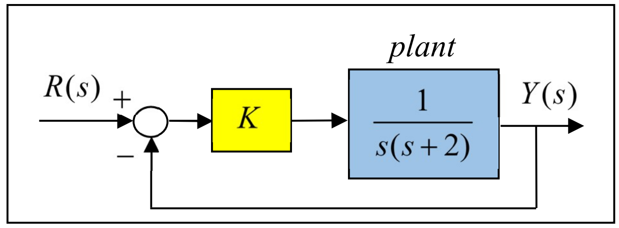
Introductory Control Systems
Root Locus (RL) Diagram Examples

1. Sketch the root locus diagram for the parameter K for the closed loop system shown in the diagram.



1) Characteristic Equation:

$$1+GH(s)=1+K\underbrace{\left[\frac{1}{s(s+2)}\right]}_{P(s)}=0$$

2) Zeros of $P(s)$: none $(n_z=0)$

Poles of $P(s)$: $s=0,-2 \quad (n_p=2) \quad \Rightarrow$ Number of branches = 2

Number of asymptotes: $n_A=n_p-n_z=2$

3) Poles on the real axis: $-2\le s\le 0$

Pole at $s=0$ moves to the ***left*** and pole at $s=-2$ moves to the ***right***, so there must be a break-away point in the range $-2\le s\le 0$.

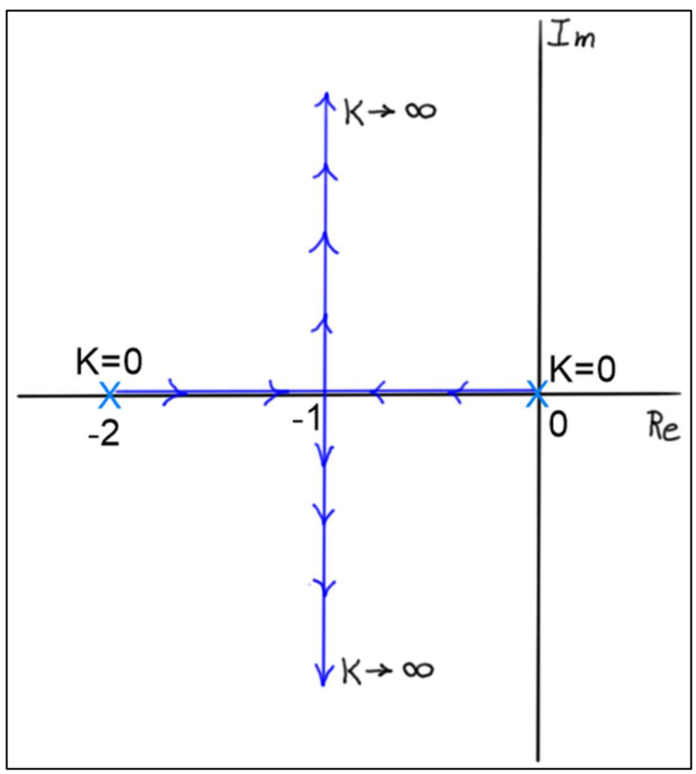
4) Asymptotes: $\phi_A=\left(\frac{2m+1}{n_A}\right)180=\begin{cases}90\text{ (deg)} & (m=0)\\ 270\text{ (deg)} & (m=1)\end{cases}$; $\sigma_A=\frac{0-2}{n_A}=\frac{-2}{2}=-1$

5) Break Points:

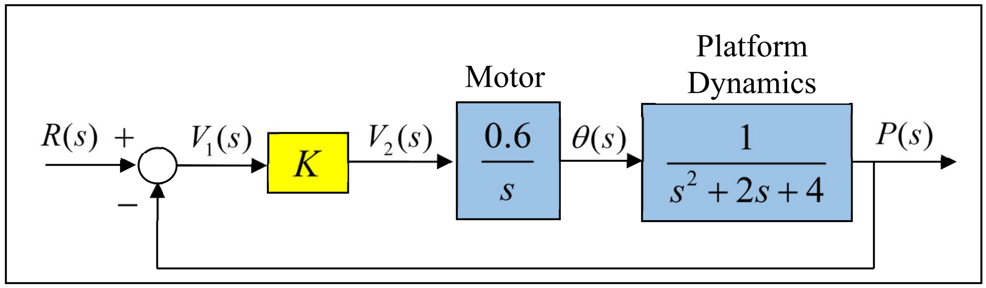
$$\frac{dP}{ds}=\frac{-1(2s+2)}{s^2(s+2)^2}=0$$

$\Rightarrow \boxed{2s+2=0}$

$\Rightarrow \boxed{s=-1}$



2. Sketch the root locus diagram for the parameter K for the closed loop system shown.



1) Characteristic Equation: $1 + GH(s) = 1 + K\underbrace{\left[\dfrac{0.6}{s(s^2+2s+4)}\right]}_{P(s)} = 0$

2) Zeros of $P(s)$: none $(n_z = 0)$

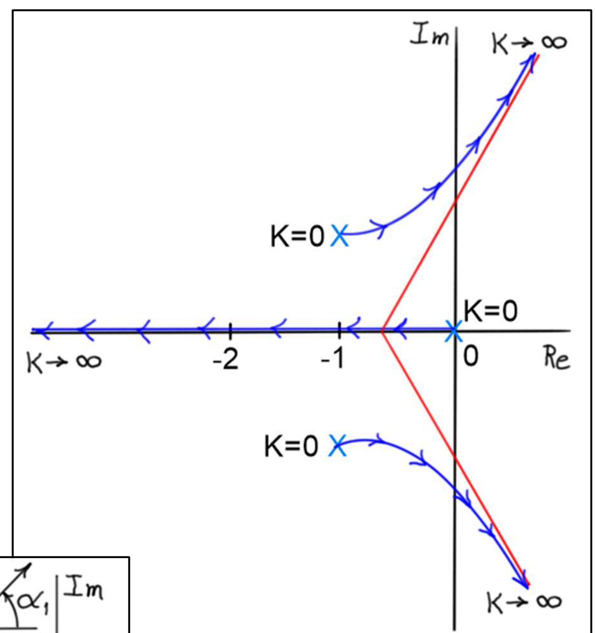
Poles of $P(s)$: $s = 0, -1 \pm 1.732j \quad (n_p = 3) \Rightarrow$ Number of branches = 3

Number of asymptotes: $n_A = n_p - n_z = 3$

3) Poles on the real axis: $-\infty < s \le 0$. Pole at $s = 0$ moves to the ***left***; poles move to infinity along the ***negative real axis*** (which is one of the asymptotes).

4) Asymptotes: $\phi_A = \left(\dfrac{2m+1}{n_A}\right)180 = 60, 180, 300 \text{ (deg)}$, $\sigma_A = \dfrac{0-1-1}{3} = -2/3$

Asymptotes at 60 and 300 deg. are shown in red.



5) Break Points: (break points must be on the real axis)

$$\frac{dP}{ds} = \frac{0.6\left(3s^2+4s+4\right)}{s^2\left(s^2+2s+4\right)^2} = 0 \quad \Rightarrow \boxed{3s^2+4s+4=0}$$

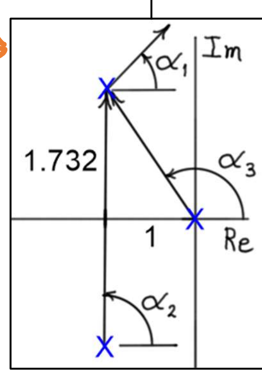
$$\Rightarrow \boxed{s = -0.667 \pm 0.94j} \quad \Rightarrow \text{ no break points}$$

6) Exit angle from the upper complex pole:

$$\alpha_1 + \alpha_2 + \alpha_3 = 180$$

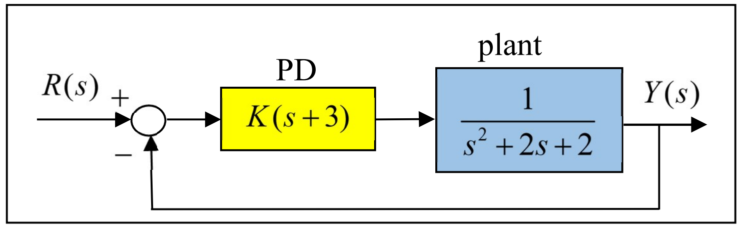
$$\alpha_1 + 90 + \left(180 - \tan^{-1}(1.732)\right) = 180$$

$$\Rightarrow \boxed{\alpha_1 = -30 \text{ (deg)}}$$



Exit angle from the lower complex pole is $+30$ (deg).

3. Sketch the root locus diagram for the parameter K for the closed loop system shown in the diagram.



1) Characteristic Equation: $$1+GH(s)=1+K\underbrace{\left[\frac{s+3}{s^2+2s+2}\right]}_{P(s)}=0$$

2) Zeros of $P(s)$: $s=-3 \quad (n_z=1)$

Poles of $P(s)$: $s=-1\pm 1j \quad (n_p=2) \quad \Rightarrow$ Number of branches = 2

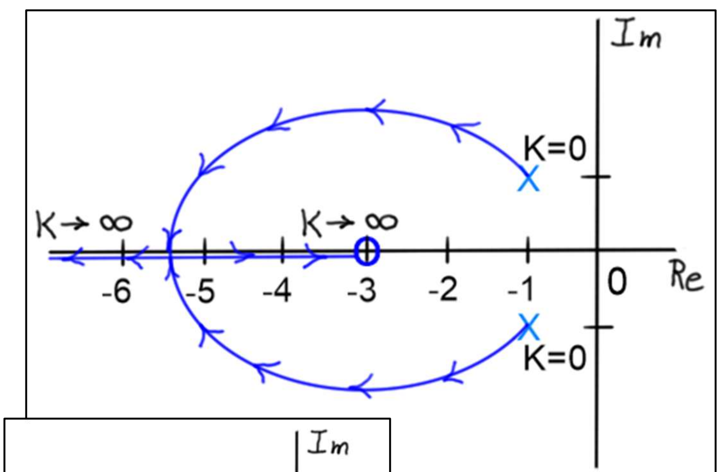
Number of asymptotes: $n_A=n_p-n_z=1$

3) Poles on the real axis: $-\infty<s\le -3$. Poles ***moving towards zero*** at $s=-3$, and poles moving ***towards*** ∞ ***along the negative real axis*** which is the only asymptote.

4) Asymptotes: No need for calculations. Lone asymptote is at 180 degrees.

5) Break Points: $$\frac{dP}{ds}=\frac{(1)(s^2+2s+2)-(s+3)(2s+2)}{(s^2+2s+2)^2}=0 \quad \Rightarrow \boxed{-s^2-6s-4=0}$$

$\Rightarrow \boxed{s=-0.764;\ -5.24}$. The break point must be in the range $-\infty<s\le -3$, so the break point for RL diagram is at $s=-5.24$.

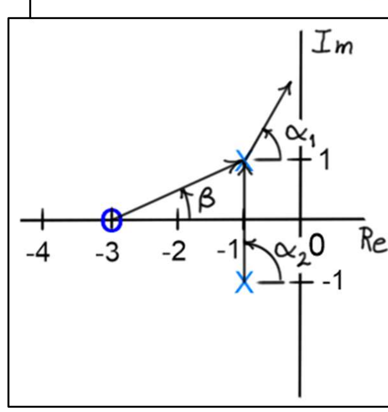


6) Exit angle from the upper complex pole:

$\alpha_1+\alpha_2-\beta=180$

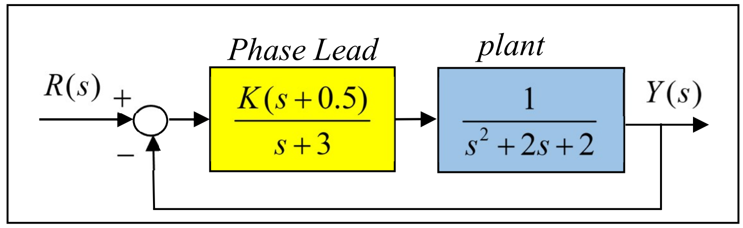
$\alpha_1+90-\tan^{-1}(1/2)=180$

$\Rightarrow \boxed{\alpha_1=116.6\ (\text{deg})}$



Exit angle from the lower complex pole is -116.6 (deg).

4. Sketch the root locus diagram for the parameter K for the closed loop system shown in the diagram.



1) Characteristic Equation: $$1+GH(s)=1+K\underbrace{\left[\frac{s+0.5}{(s+3)(s^2+2s+2)}\right]}_{P(s)}=0$$

2) Zeros of $P(s)$: $s=-0.5\;\;(n_z=1)$

   Poles of $P(s)$: $s=-3,\,-1\pm 1j\quad(n_p=3)\;\Rightarrow$ Number of branches = 3

   Number of asymptotes: $n_A=n_p-n_z=2$

3) Poles on the real axis: $-3\le s\le -0.5$

   Poles moving away from pole at $s=-3$, and poles moving towards the zero at $s=-0.5$.

4) Asymptotes:

$$\phi_A=\left(\frac{2m+1}{n_A}\right)180=\begin{cases}90\text{ (deg)} & (m=0)\\ 270\text{ (deg)} & (m=1)\end{cases},\quad \sigma_A=\frac{-3+2(-1)-(-0.5)}{n_A}=\frac{-4.5}{2}=-2.25$$

5) Break Points:

$$\frac{dP}{ds}=\frac{(1)(s+3)(s^2+2s+2)-(s+0.5)(3s^2+10s+8)}{(s+3)^2(s^2+2s+2)^2}=\frac{2s^3+6.5s^2+5s-2}{(s+3)^2(s^2+2s+2)^2}=0$$

$$\Rightarrow 2s^3+6.5s^2+5s-2=0$$

$$s=0.285;\;-1.77\pm 0.62j$$

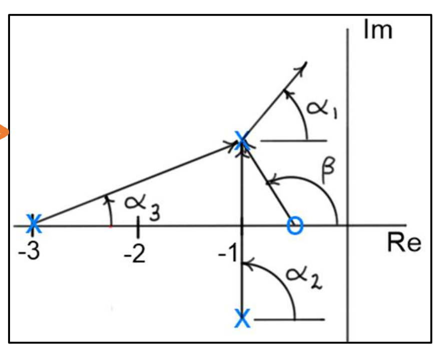
Break points on the RL diagram must be in the range $-3\le s\le -0.5$, so there are ***no break points***.

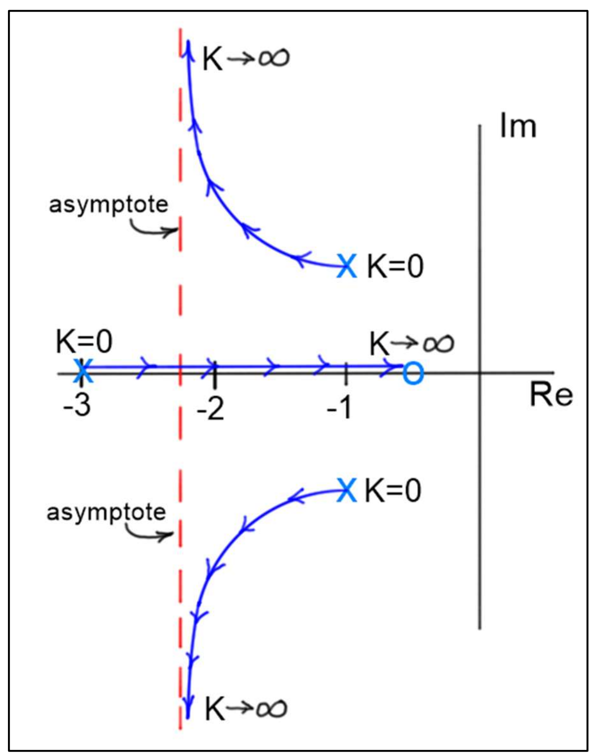
6) Exit angle from the upper complex pole:

$$\alpha_1+\alpha_2+\alpha_3-\beta=180$$

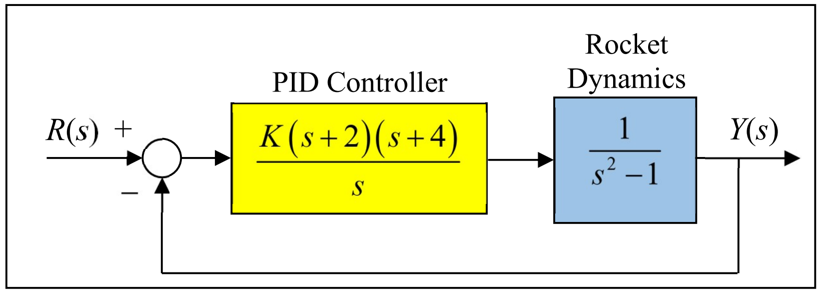
$$\alpha_1+90+\tan^{-1}(1/2)-\left(180-\tan^{-1}(1/0.5)\right)=180$$

$$\Rightarrow \alpha_1=180\text{ (deg)}$$





5. Sketch the root locus diagram for the parameter K for the closed loop system shown in the diagram.



1) Characteristic Equation: $$1+GH(s)=1+K\underbrace{\left[\frac{(s+2)(s+4)}{s(s^2-1)}\right]}_{P(s)}=0$$

2) Zeros of $P(s)$: $s=-2,-4$ $(n_z=2)$

   Poles of $P(s)$: $s=0,\pm 1$ $(n_p=3)$ $\Rightarrow$ Number of branches = 3

   Number of asymptotes: $n_A=n_p-n_z=1$

3) Poles on the real axis (3 segments): $-\infty<s<-4$, $-2<s<-1$, and $0<s<+1$

   Poles move to the ***right*** from $s=0$; poles move to the ***left*** from $s=+1$; poles move ***left*** from $s=-1$; poles move ***towards*** the zero at $s=-2$; poles move ***towards*** the zero at $s=-4$; poles move to infinity along the ***negative real axis*** which is the only asymptote.

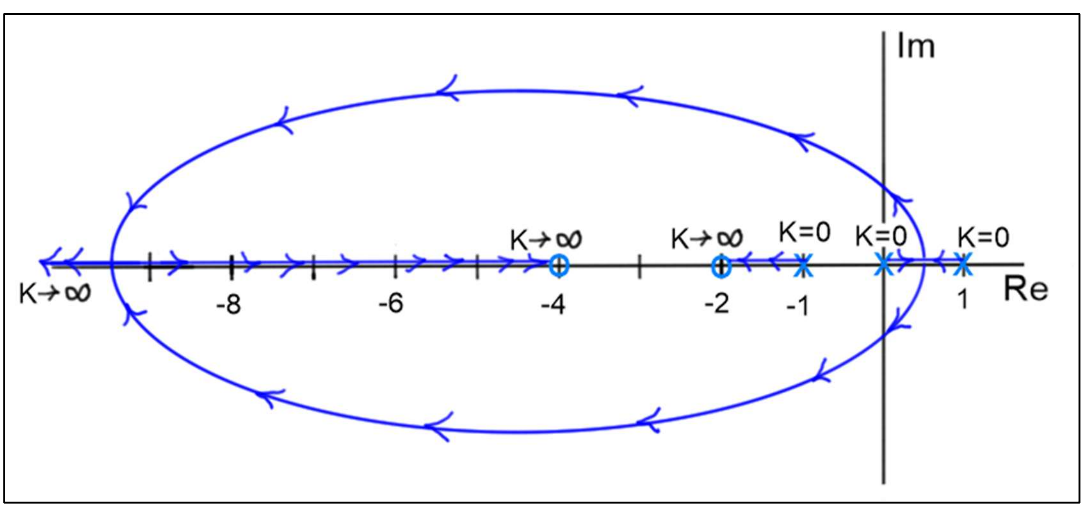
4) Asymptote: $\phi_A=180\,(\text{deg})$

5) Break Points:

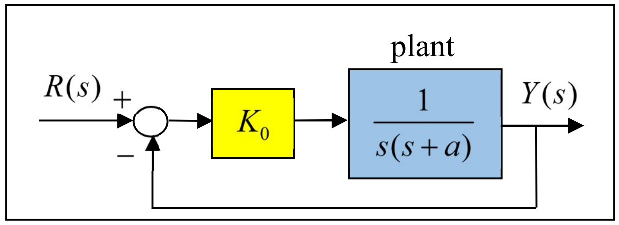
$$\frac{dP}{ds}=\frac{d}{ds}\left(\frac{(s+2)(s+4)}{s(s^2-1)}\right)=\frac{(2s+6)(s^3-s)-(s^2+6s+8)(3s^2-1)}{s^2(s^2-1)^2}=0$$

$\Rightarrow$ $\boxed{s^4+12s^3+25s^2-8=0}$ $\Rightarrow$ $\boxed{s=-9.33,\,-2.495,\,-0.68,\,0.5054}$

$\Rightarrow$ break points on RL diagram at $s=-9.33,\ 0.5054$



6. Sketch the root locus diagram for the parameter a for the closed loop system shown in the diagram.



1) Characteristic Equation: $1+GH(s)=1+\left[\frac{K_0}{s(s+a)}\right]=0 \;\Rightarrow\; 1+a\underbrace{\left[\frac{s}{s^2+K_0}\right]}_{P(s)}=0$

In this case, some ***algebraic manipulation*** is required to put the characteristic equation in standard form.

2) Zeros of $P(s)$: $s=0$ $(n_z=1)$

Poles of $P(s)$: $s=\pm\sqrt{K_0}\,j$ $(n_p=2)$ $\Rightarrow$ Number of branches = 2

Number of asymptotes: $n_A=n_p-n_z=1$

3) Poles on the real axis: $-\infty<s<0$

Poles move ***towards*** the zero at $s=0$, and poles move to infinity along the ***negative real axis*** which is the only asymptote.

4) Asymptotes: $\phi_A=180$ (deg)

5) Break Points: $\frac{dP}{ds}=\frac{d}{ds}\left(\frac{s}{s^2+K_0}\right)=\frac{(1)(s^2+K_0)-s(2s)}{\left(s^2+K_0\right)^2}=\frac{-s^2+K_0}{\left(s^2+K_0\right)^2}=0$

$\Rightarrow$ $s^2-K_0=0$ $\Rightarrow$ $s=\pm\sqrt{K_0}$

$\Rightarrow$ break point on the RL diagram at $s=-\sqrt{K_0}$