Intermediate Dynamics
Lagrange's Equations for Multi-Degree-of-Freedom Systems with Dependent Generalized Coordinates

Often it is more convenient to use a ***dependent set*** of ***generalized coordinates*** q_k $(k=1,\ldots,n)$ to define the configuration of a mechanical system. If the system possesses $N=n-m$ degrees of freedom (DOF), then there are "*m*" constraint equations that can be written in the following form.

$$\boxed{\sum_{k=1}^{n} a_{jk}\dot{q}_k + a_{j0} = 0} \quad (j=1,\ldots,m) \qquad \text{or} \qquad \boxed{\left[A\right]_{m\times n}\left\{\dot{q}\right\}_{n\times 1} + \left\{a_0\right\}_{m\times 1} = \left\{0\right\}_{m\times 1}} \tag{1}$$

The ***differential/algebraic*** equations of motion of the system can be derived using ***Lagrange's equations*** with ***Lagrange multipliers***.

$$\boxed{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_k}\right) - \frac{\partial K}{\partial q_k} = F_{q_k} + \sum_{j=1}^{m}\lambda_j a_{jk}} \qquad (k=1,\ldots,n) \tag{2}$$

Here, *K* is the ***kinetic energy*** of the system, F_{q_k} is the ***generalized force*** associated with the generalized coordinate q_k, λ_j is the ***Lagrange multiplier*** associated with the jth constraint equation, and the a_{jk} $(j=1,\ldots,m;\quad k=1,\ldots,n)$ are the ***coefficients*** from the constraint equations.

The "*n*" Lagrange's equations and the "*m*" constraint equations form a set of "$n+m$" ***differential/algebraic equations*** (DAE) for the "$n+m$" ***unknowns*** – the "*n*" ***generalized coordinates*** q_k $(k=1,\ldots,n)$ and the "*m*" ***Lagrange multipliers*** λ_j $(j=1,\ldots,m)$. The ***Lagrange multipliers*** are ***related*** to the ***forces*** and ***moments*** required to maintain the constraints as the system moves. It is ***important*** that *K*, F_{q_k}, and the a_{jk} be written ***only in terms*** of q_k, $\dot{q}_k$, and ***no other variables***.

If some of the forces and torques are ***conservative***, then their contributions to the equations of motion can be calculated in terms of ***potential energy functions***. In this case, the ***differential/algebraic*** EOM can be derived from the equations

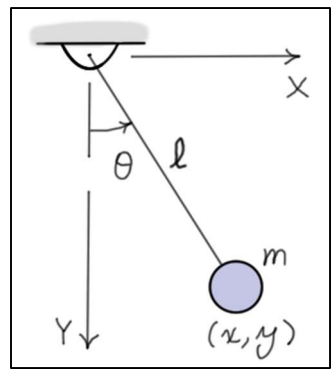
$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \left(F_{q_k}\right)_{nc} + \sum_{j=1}^{m}\lambda_j a_{jk}} \qquad (k=1,\ldots,n) \tag{3}$$

Here, $L = K - V$ is the ***Lagrangian*** of the system, V is the ***potential energy function*** for the ***conservative*** forces and torques, $\left(F_{q_k}\right)_{nc}$ is the ***generalized force*** associated with q_k for the ***non-conservative*** forces and torques, only, λ_j is the ***Lagrange multiplier*** associated with the jth constraint equation, and a_{jk} $(j = 1,\ldots m; k = 1,\ldots n)$ represents the ***coefficients*** from the ***constraint equations***.

As before, the "n" Lagrange's equations and the "m" constraint equations form a set of "$n+m$" ***differential/algebraic equations*** for the "$n+m$" unknowns – the "n" ***generalized coordinates*** q_k $(k = 1,\ldots,n)$ and the "m" ***Lagrange multipliers*** λ_j $(j = 1,\ldots,m)$.

<u>Example: Equations of Motion of the Simple Pendulum</u>

<u>Approach #1</u>: Using θ as the generalized coordinate



The ***simple pendulum*** is a single degree of freedom (SDOF) system. Using θ as the generalized coordinate, the Lagrangian of the system can be written as follows.

$$\boxed{L = \tfrac{1}{2} m \ell^2 \dot{\theta}^2 + mg\ell\cos(\theta)}$$

Using Lagrange's equations ***without constraints***, the equation of motion is found to be

$$\boxed{\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0} \tag{4}$$

<u>Approach #2</u>: Using (x, y) as the generalized coordinates

To use two generalized coordinates for a SDOF system, ***Lagrange's equations with a single constraint*** must be used. In this case, the configuration constraint is

$$\boxed{x^2 + y^2 = \ell^2}$$

Differentiating this equation, the constraint can be written in the form of Eq. (1).

$$x\dot{x} + y\dot{y} = 0 \qquad \text{or} \qquad \boxed{\begin{bmatrix} x & y \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = 0} \tag{5}$$

The ***coefficients*** a_{jk} $(j = 1; k = 0,1,2)$ are defined as follows.

$$\boxed{[A] = \begin{bmatrix} x & y \end{bmatrix}} \Rightarrow \boxed{a_{11} = x} \quad \boxed{a_{12} = y} \quad \text{and} \quad \boxed{a_{10} = 0} \tag{6}$$

Lagrange's equations ***with constraints*** can then be written in the form

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)-\frac{\partial L}{\partial q_k}=\lambda a_{1k}} \qquad k=1,2 \tag{7}$$

The ***Lagrangian*** can be written in terms of the generalized coordinates and their derivatives as

$$\boxed{L=\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2)+mgy}$$

Using Eq. (7) and the definitions of the constraint equation coefficients in Eq. (6), the following ***two coupled, second-order, differential/algebraic equations*** (DAE) can be found. The Lagrange multiplier λ appears as an algebraic unknown in the equations.

$$\boxed{\begin{aligned} m\ddot{x}-\lambda x&=0\\ m\ddot{y}-mg-\lambda y&=0\end{aligned}} \tag{8}$$

Differentiating the constraint Eq. (5) again, and combining it with Eqs. (8) gives the following ***complete set*** of ***three coupled, second-order, differential/algebraic equations*** in three unknowns (x,y,λ).

$$\boxed{\begin{aligned} m\ddot{x}-\lambda x&=0\\ m\ddot{y}-mg-\lambda y&=0\\ x\ddot{x}+y\ddot{y}+\dot{x}^2+\dot{y}^2&=0\end{aligned}} \tag{9}$$

Alternatively, the ***Lagrange multiplier*** λ can be eliminated from Eqs. (8) to form a single differential equation. For example, multiply the first equation by "y" and the second equation by "x" and subtract the two equations. Following this approach, the equations of motion are written as ***two coupled, second-order differential equations*** with no algebraic unknowns.

$$\boxed{\begin{aligned} y\ddot{x}-x\ddot{y}-gx&=0\\ x\ddot{x}+y\ddot{y}+\dot{x}^2+\dot{y}^2&=0\end{aligned}} \tag{10}$$