

Elementary Engineering Mathematics

Equations Sheet #4 – Spring-Mass-Damper Systems

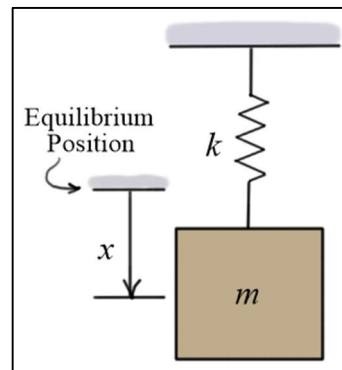
1. Spring-Mass System (no damping)

Initial Conditions: $x(0) = x_0$ $v(0) = v_0$

Natural Frequency: $\omega = \sqrt{\frac{k}{m}}$

Displacement: $x(t) = \frac{v_0}{\omega} \sin(\omega t) + x_0 \cos(\omega t)$

Velocity: $v(t) = v_0 \cos(\omega t) - x_0 \omega \sin(\omega t)$



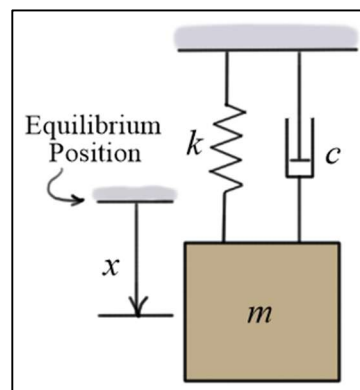
2. Spring-Mass-Damper System

Critical Damping Coefficient: $c_c = 2m\sqrt{k/m}$

Over-damped Motion: $c > c_c$

$$\begin{aligned} x(t) &= A e^{\lambda_1 t} + B e^{\lambda_2 t} \\ v(t) &= A \lambda_1 e^{\lambda_1 t} + B \lambda_2 e^{\lambda_2 t} \end{aligned}$$

$$\begin{cases} \lambda_1 \\ \lambda_2 \end{cases} = \begin{cases} -\left(\frac{c}{2m}\right) + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \\ -\left(\frac{c}{2m}\right) - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \end{cases}$$



Coefficients: (A, B) found by solving the simultaneous equations:

$$\begin{aligned} A + B &= x_0 \\ \lambda_1 A + \lambda_2 B &= v_0 \end{aligned}$$

Under-Damped Motion: $c < c_c$

Frequency: $\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$

Displacement: $x(t) = e^{-(c/2m)t} (A \sin(\omega_d t) + B \cos(\omega_d t))$

$$\begin{aligned} A &= \left[v_0 + \left(\frac{c}{2m}\right) x_0 \right] / \omega_d \\ B &= x_0 \end{aligned}$$

3. Decay/Growth Rate

$$\alpha = \frac{\ln(x_2/x_1)}{t_2 - t_1}$$

