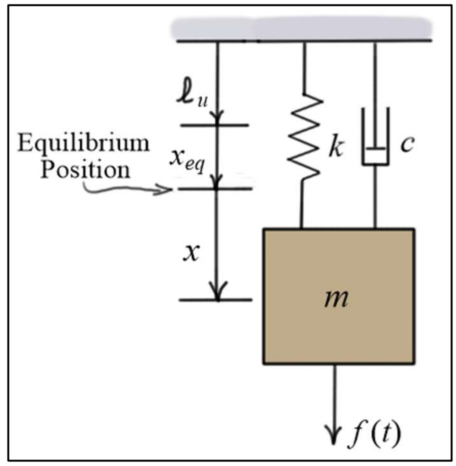
Elementary Engineering Mathematics
Equations Sheet #8 – Differential Equations for a Spring-Mass-Damper System



1. Differential Equation of Motion

$$\boxed{m\ddot{x} + c\dot{x} + kx = f(t)}$$

2. Free Response $\left(f(t) = 0\right)$

   (a) Characteristic Equation: $\boxed{ms^2 + cs + k = 0}$

   (b) Form of Solution depends on type of roots. Coefficients found by applying initial conditions.

Case	Type of Roots	Type of Motion	Form of Solution
1	Real, unequal	Over-damped	$x(t) = Ae^{s_1 t} + Be^{s_2 t}$
2	Real, equal	Critically damped	$x(t) = Ae^{st} + B\,t\,e^{st}$
3	Complex conjugates	Under-damped	$x(t) = e^{-\left(c/2m\right)t}\left[A\sin(\omega_d\, t) + B\cos(\omega_d\, t)\right]$ $\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$

3. Forced Response $\left(f(t) \neq 0\right)$

   (a) Solution is the sum of the homogeneous solution and a particular solution: $\boxed{x(t) = x_H(t) + x_P(t)}$.

   (b) Homogeneous solution has the ***form*** of the free response.

   (c) The ***forms*** of some particular solutions are shown in the table below.

   (d) Coefficients are found by applying initial conditions.

	$f(t)$	Form* of $x_P(t)$
constant	a_0	$B_0\, t^n$
linear	$a_1 t + a_0$	$\left(B_1 t + B_0\right)t^n$
quadratic	$a_2 t^2 + a_1 t + a_0$	$\left(B_2 t^2 + B_1 t + B_0\right)t^n$
exponential	$a\,e^{\beta t}$	$\left(B_1 e^{\beta t}\right)t^n$
sine **or** cosine	$a\sin\left(\omega t\right)$ **or** $a\cos\left(\omega t\right)$	$\left[B_1\sin\left(\omega t\right) + B_2\cos\left(\omega t\right)\right]t^n$
exponential-sine **or** exponential-cosine product	$a\,e^{\beta t}\sin\left(\omega t\right)$ **or** $a\,e^{\beta t}\cos\left(\omega t\right)$	$e^{\beta t}\left[B_1\sin\left(\omega t\right) + B_2\cos\left(\omega t\right)\right]t^n$

* The ***exponent*** n is the ***smallest, non-negative integer*** so every term in $x_P(t)$ is ***different*** from every term in $x_H(t)$. That is, $n = 0$ unless the same type of term appears in $x_H(t)$.