Introductory Control Systems
Exercises #16 – State-Space Equations and Transfer Functions

1. Speed control of a car in the presence of a disturbance force is described by the boxed equations. The desired speed is $r(t)$, the actual speed is $v(t)$, the speed error is $e(t)$, the driving force on the car is $f_a(t)$, the disturbance force on the car is $f_d(t)$, and the net force on the car is $f_{net}(t)$.

$$\frac{dv}{dt}+3v=f_{net}(t)$$
$$f_{net}(t)=f_a(t)-f_d(t)$$
$$f_a(t)=Ke(t)$$
$$e(t)=r(t)-v(t)$$

a) Express the equations in state-space form with output variables $v(t)$ and $e(t)$. b) Using the state-space equations, find the transfer functions $\frac{V}{R}(s)$, $\frac{E}{R}(s)$, $\frac{V}{F_d}(s)$, and $\frac{E}{F_d}(s)$. c) Using direct decomposition, find a set of state-space equations to represent $\frac{V}{R}(s)$.

Answers:

a) $\dot{x}_1=\left[-(K+3)\right]x_1+\begin{bmatrix}K & -1\end{bmatrix}\begin{Bmatrix}r\\f_d\end{Bmatrix}$ $\begin{Bmatrix}z_1\\z_2\end{Bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}x_1+\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\begin{Bmatrix}u_1\\u_2\end{Bmatrix}$

b) $\frac{V}{R}(s)=\frac{K}{s+(K+3)}$ $\frac{V}{F_d}(s)=\frac{-1}{s+(K+3)}$ $\frac{E}{R}(s)=\frac{s+3}{s+(K+3)}$ $\frac{E}{F_d}(s)=\frac{1}{s+(K+3)}$

c) $\dot{x}_1=\left[-(K+3)\right]x_1+[1]r(t)$ $v(t)=[K]x_1(t)$

2. The boxed equations describe **position control** of a spring-mass system using **proportional** (P) control. The **desired position** of the mass is $r(t)$, the **actual position** is $y(t)$, the **position error** is $e(t)$, and the **actuator force** applied to the mass is $f(t)$.

$$e(t)=r(t)-y(t)$$
$$f(t)=K\,e(t)$$
$$\ddot{y}+4y=f(t)$$

a) Express the equations in state-space form with output variables y, $\dot{y}$, and e. b) Using the state-space equations, find the transfer functions $\frac{Y}{R}(s)$ and $\frac{E}{R}(s)$. c) Using direct decomposition, find a set of state-space equations to represent $\frac{E}{R}(s)$.

Answers:

a) $\begin{Bmatrix}\dot{x}_1\\\dot{x}_2\end{Bmatrix}=\begin{bmatrix}0 & 1\\-(K+4) & 0\end{bmatrix}\begin{Bmatrix}x_1\\x_2\end{Bmatrix}+\begin{Bmatrix}0\\K\end{Bmatrix}u$ $\begin{Bmatrix}z_1\\z_2\\z_3\end{Bmatrix}=\begin{Bmatrix}x_1\\x_2\\e\end{Bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\\-1 & 0\end{bmatrix}\begin{Bmatrix}x_1\\x_2\end{Bmatrix}+\begin{Bmatrix}0\\0\\1\end{Bmatrix}u$

b) $\frac{Y}{R}(s)=\frac{K}{s^2+K+4}$ $\frac{E}{R}(s)=\frac{s^2+4}{s^2+K+4}$

c) $\begin{Bmatrix}\dot{x}_1\\\dot{x}_2\end{Bmatrix}=\begin{bmatrix}0 & 1\\-(K+4) & 0\end{bmatrix}\begin{Bmatrix}x_1\\x_2\end{Bmatrix}+\begin{Bmatrix}0\\1\end{Bmatrix}u$ $e=\begin{bmatrix}-K & 0\end{bmatrix}\begin{Bmatrix}x_1\\x_1\end{Bmatrix}+[1]u$

3. The boxed equations describe position control of a spring-mass-damper system using proportional-derivative control. The desired position of the mass is $r(t)$, the actual position is $x(t)$, the position error is $e(t)$, and the actuating force is $f(t)$.

$$\boxed{\begin{aligned} e(t) &= r(t) - x(t) \\ f(t) &= \dot{e}(t) + 5\,e(t) \\ \ddot{x} + 6\dot{x} + 20x &= f(t) \end{aligned}}$$

a) Express the equations in state-space form with output variables x and e. b) Using the state-space equations, find the transfer functions $\frac{X}{R}(s)$ and $\frac{E}{R}(s)$. c) Using direct decomposition, find a set of state-space equations to represent $\frac{X}{R}(s)$.

Answers:

a)
$$\boxed{\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -7 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 5 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}} \qquad \boxed{\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} x \\ e \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}$$

b)
$$\boxed{\frac{Z_1}{U_1}(s) = \frac{X}{R}(s) = \frac{s+5}{s^2 + 7s + 25}} \qquad \boxed{\frac{Z_2}{U_1}(s) = \frac{E}{R}(s) = \frac{s^2 + 6s + 20}{s^2 + 7s + 25}}$$

c)
$$\boxed{\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -7 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} r(t)} \qquad \boxed{x(t) = \begin{bmatrix} 5 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}}$$

4. Position control of a hydraulic actuator using proportional-integral control is described by the boxed equations. The desired position is $r(t)$, the actual position is $x(t)$, the position error is $e(t)$, and the actuating forces is $f(t)$.

$$\boxed{\begin{aligned} e(t) &= r(t) - x(t) \\ f(t) &= e(t) + 3\int_0^t e(t)\,dt \\ \ddot{x} + 5\dot{x} &= 2f(t) \end{aligned}}$$

a) Express the equations in state-space form with output variables x and e. b) Using the state-space equations, find the transfer functions $\frac{X}{R}(s)$ and $\frac{E}{R}(s)$. c) Using direct decomposition, find a set of state-space equations to represent $\frac{X}{R}(s)$.

Answers:

a)
$$\boxed{\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -2 & -5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 6 & 2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}} \qquad \boxed{\begin{Bmatrix} x \\ e \end{Bmatrix} = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}$$

b)
$$\boxed{\frac{X}{R}(s) = \frac{2(s+3)}{s^3 + 5s^2 + 2s + 6}} \qquad \boxed{\frac{E}{R}(s) = \frac{s^2(s+5)}{s^3 + 5s^2 + 2s + 6}}$$

c)
$$\boxed{\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -2 & -5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} r(t)} \qquad \boxed{x(t) = 6x_1 + 2x_2 = \begin{bmatrix} 6 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}}$$

5. The boxed equations describe speed control of a rotating mass system using proportional control. The desired rotational speed is $r(t)$, the actual rotational speed is $\omega(t)$, the rotational speed error is $e(t)$, the input voltage to the motor drive actuator is $v(t)$, and the actuator torque applied to the mass is $M(t)$.

$$\boxed{\begin{aligned} e(t) &= r(t) - \omega(t) \\ v(t) &= Ke(t) \\ \dot{M} + 8M &= v(t) \\ \dot{\omega} + 7\omega &= 3M(t) \end{aligned}}$$

a) Express the equations in state-space form with output variables ω and M. b) Using the state-space equations, find the transfer functions $\frac{\omega}{R}(s)$ and $\frac{M}{R}(s)$. c) Using direct decomposition, find a set of state-space equations to represent $\frac{M}{R}(s)$.

Answers:

a)
$$\left\{\begin{matrix} \dot{x}_1 \\ \dot{x}_2 \end{matrix}\right\} = \begin{bmatrix} -7 & 3 \\ -K & -8 \end{bmatrix}\left\{\begin{matrix} x_1 \\ x_2 \end{matrix}\right\} + \left\{\begin{matrix} 0 \\ K \end{matrix}\right\}u \qquad \left\{\begin{matrix} z_1 \\ z_2 \end{matrix}\right\} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\left\{\begin{matrix} x_1 \\ x_2 \end{matrix}\right\} + \left\{\begin{matrix} 0 \\ 0 \end{matrix}\right\}u$$

b)
$$\frac{\omega}{R}(s) = \frac{3K}{s^2 + 15s + (56 + 3K)} \qquad \frac{M}{R}(s) = \frac{K(s+7)}{s^2 + 15s + (56 + 3K)}$$

c)
$$\left\{\begin{matrix} \dot{x}_1 \\ \dot{x}_2 \end{matrix}\right\} = \begin{bmatrix} 0 & 1 \\ -(56+3K) & -15 \end{bmatrix}\left\{\begin{matrix} x_1 \\ x_2 \end{matrix}\right\} + \left\{\begin{matrix} 0 \\ 1 \end{matrix}\right\}r \qquad M = \begin{bmatrix} 7K & K \end{bmatrix}\left\{\begin{matrix} x_1 \\ x_2 \end{matrix}\right\}$$

6. The boxed equations describe **position control** of a spring-mass system using proportional control with a linear actuator. The **desired position** of the mass is $r(t)$, the **actual position** is $y(t)$, the **position error** is $e(t)$, and the **actuator force** applied to the mass is $f(t)$.

$$\boxed{\begin{aligned} e(t) &= r(t) - y(t) \\ v(t) &= 10\,e(t) \\ \dot{f} + 3f &= v(t) \\ \ddot{y} + 9y &= f(t) \end{aligned}}$$

a) Express the equations in state-space form with output variables y and v. b) Using the state-space equations, find the transfer functions $\frac{Y}{R}(s)$ and $\frac{V}{R}(s)$. c) Using direct decomposition, find a set of state-space equations to represent $\frac{Y}{R}(s)$.

Answers:

a)
$$\left\{\begin{matrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{matrix}\right\} = \begin{bmatrix} 0 & 0 & 1 \\ -10 & -3 & 0 \\ -9 & 1 & 0 \end{bmatrix}\left\{\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}\right\} + \left\{\begin{matrix} 0 \\ 10 \\ 0 \end{matrix}\right\}u \qquad \left\{\begin{matrix} z_1 \\ z_2 \end{matrix}\right\} = \begin{bmatrix} 1 & 0 & 0 \\ -10 & 0 & 0 \end{bmatrix}\left\{\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}\right\} + \left\{\begin{matrix} 0 \\ 10 \end{matrix}\right\}u$$

b)
$$\frac{Y}{R}(s) = \frac{10}{s^3 + 3s^2 + 9s + 37} \qquad \frac{V}{R}(s) = \frac{10(s^3 + 3s^2 + 9s + 27)}{s^3 + 3s^2 + 9s + 37}$$

c)
$$\left\{\begin{matrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{matrix}\right\} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -37 & -9 & -3 \end{bmatrix}\left\{\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}\right\} + \left\{\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right\}r \qquad y = \begin{bmatrix} 10 & 0 \end{bmatrix}\left\{\begin{matrix} x_1 \\ x_2 \end{matrix}\right\}$$