**Elementary Statics**
**Vector Components and Vector Addition in Three Dimensions**

**Cartesian Components of Vectors (3D)**

o  Given the *magnitude* of a vector and the *direction* of the vector relative to a set of three *reference axes*, the vector can be expressed in terms of its *components* along those axes.

o  For our convenience, it is beneficial to have the reference axes be a *right-handed*, *mutually perpendicular* set.



o  $V_x$, $V_y$, and $V_z$ represent the *components* of the vector $V$ along the *right-handed*, *mutually perpendicular* X, Y, and Z axes. By vector addition, $\boxed{V = V_x + V_y + V_z}$
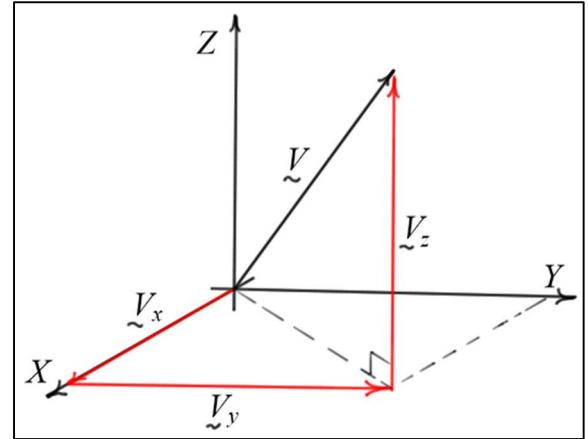
o  The triangles formed by $V$ and $V_x$, by $V$ and $V_y$, and by $V$ and $V_z$ are all *right triangles*.

o  If the *magnitude* of the vector $V$ is $|V| = V$, and if the *angles* that $V$ makes with the X, Y, and Z axes are $\theta_x$, $\theta_y$, and $\theta_z$, respectively, then



$$
\boxed{
\begin{aligned}
V &= V_x + V_y + V_z = V_x\,i + V_y\,j + V_z\,k \\
&= \left(V\cos(\theta_x)\right)i + \left(V\cos(\theta_y)\right)j + \left(V\cos(\theta_z)\right)k \\
&= V\left(\cos(\theta_x)\,i + \cos(\theta_y)\,j + \cos(\theta_z)\,k\right)
\end{aligned}
}
$$
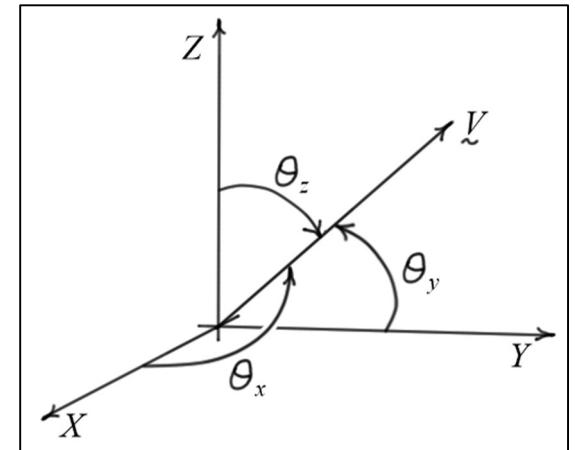
o  The *unit vectors* $i$, $j$, and $k$ indicate the positive X, Y, and Z coordinate directions.

o  The vector $\boxed{u_V = \cos(\theta_x)\,i + \cos(\theta_y)\,j + \cos(\theta_z)\,k}$ is a *unit vector* in the direction of $V$.

o  The angles $\theta_x$, $\theta_y$, and $\theta_z$ are *not independent*. It can be shown that,

$$\boxed{\cos^2(\theta_x) + \cos^2(\theta_y) + \cos^2(\theta_z) = 1}$$

o  The magnitude of $V$ is $\boxed{V = \sqrt{V_x^2 + V_y^2 + V_z^2}}$

**Cartesian Components – Polar and Elevation Angles**

o The Cartesian components of a vector can also be given in terms of *polar* and *elevation* angles. Unlike the three angles relative to the *X*, *Y*, and *Z* axes, these two angles are *independent*.

o In the diagram, angle $\theta$ measured in the *XY* plane is the *polar angle*, and the angle $\phi$ is the *elevation angle*.

o In this case, the Cartesian components are found using a two-step process. First, break $\underset{\sim}{V}$ into *two components*, one in the *XY* plane and one perpendicular to it (along the *Z* axis). Then, break the component in the *XY* plane into two components, one along the *X* axis and one along the *Y* axis.
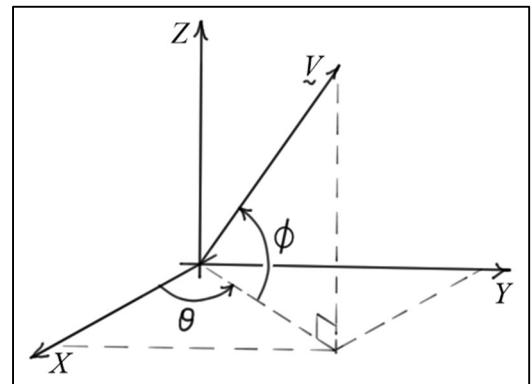
$$
\begin{aligned}
\underset{\sim}{V} &= V\cos(\phi)\big(\cos(\theta)\,\underset{\sim}{i} + \sin(\theta)\,\underset{\sim}{j}\big) + V\sin(\phi)\,\underset{\sim}{k} \\
&= V\cos(\phi)\cos(\theta)\,\underset{\sim}{i} + V\cos(\phi)\sin(\theta)\,\underset{\sim}{j} + V\sin(\phi)\,\underset{\sim}{k} \\
&= V\big(\cos(\phi)\cos(\theta)\,\underset{\sim}{i} + \cos(\phi)\sin(\theta)\,\underset{\sim}{j} + \sin(\phi)\,\underset{\sim}{k}\big)
\end{aligned}
$$

Example #1:

Given: Force $\underset{\sim}{F}$ has magnitude $\big|\underset{\sim}{F}\big| = 100$ (lb) and
angles $\theta_x = 40$ (deg) and $\theta_y = 70$ (deg).

Find: Express the force $\underset{\sim}{F}$ in terms of the unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$, and $\underset{\sim}{k}$.

Solution:

We know that: $\cos^2(\theta_x) + \cos^2(\theta_y) + \cos^2(\theta_z) = 1$. So,

$$
\theta_z = \cos^{-1}\left(\sqrt{1 - \cos^2(\theta_x) - \cos^2(\theta_y)}\right)
$$

$$
= \cos^{-1}\left(\sqrt{1 - \cos^2(40) - \cos^2(70)}\right) \approx 57.027 \approx 57.0 \text{ (deg)}
$$
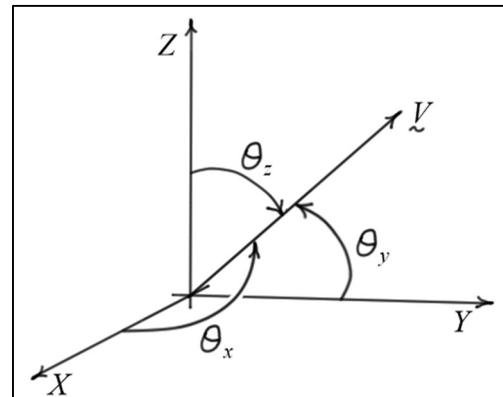
$$
\underset{\sim}{F} = 100\cos(40)\,\underset{\sim}{i} + 100\cos(70)\,\underset{\sim}{j} + 100\cos(57.027)\,\underset{\sim}{k} \approx 76.6\,\underset{\sim}{i} + 34.2\,\underset{\sim}{j} + 54.4\,\underset{\sim}{k} \text{ (lb)}
$$

Check: $\big|\underset{\sim}{F}\big| \approx \sqrt{76.6^2 + 34.2^2 + 54.4^2} \approx 99.9828 \approx 100$ (lb)

Example #2: (polar and elevation angles)

Given:   Force $\underset{\sim}{F}$ has magnitude $\left|\underset{\sim}{F}\right| = 100$ (lb) and angles
$\theta = 24$ (deg) and $\phi = 33$ (deg).

Find:    Express the force $\underset{\sim}{F}$ in terms of the unit vectors
$\underset{\sim}{i}$, $\underset{\sim}{j}$, and $\underset{\sim}{k}$.

Solution:
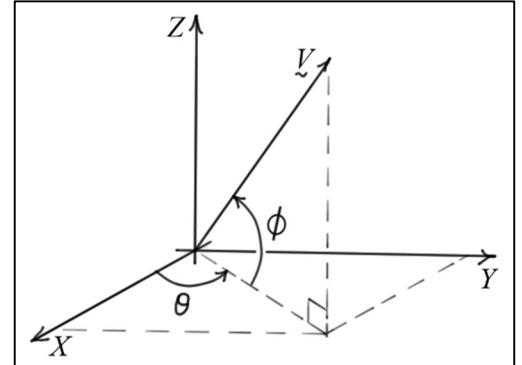
$$F_x = F\cos(\phi)\cos(\theta) = 100\cos(33)\cos(24) \approx 76.6164 \approx 76.6 \text{ (lbs)}$$

$$F_y = F\cos(\phi)\sin(\theta) = 100\cos(33)\sin(24) \approx 34.1118 \approx 34.1 \text{ (lbs)}$$

$$F_z = F\cos(\phi) = 100\sin(33) \approx 54.4639 \approx 54.5 \text{ (lbs)}$$

$$\Rightarrow \boxed{\underset{\sim}{F} \approx 76.6\,\underset{\sim}{i} + 34.1\,\underset{\sim}{j} + 54.5\,\underset{\sim}{k}}$$

Check: $\boxed{\left|\underset{\sim}{F}\right| \approx \sqrt{76.6^2 + 34.1^2 + 54.5^2} \approx 100.003 \approx 100 \text{ (lb)}}$

Example #3:

Given: A force $\underset{\sim}{F} = 70\,\underset{\sim}{i} + 50\,\underset{\sim}{j} - 80\,\underset{\sim}{k}$ (lb).

Find: The magnitude and direction of $\underset{\sim}{F}$ relative to the $X$, $Y$, and $Z$ axes.

Solution:
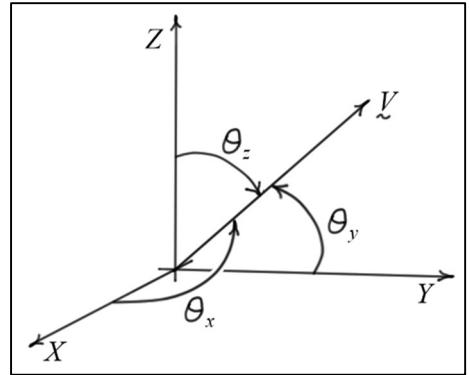
$$\boxed{\left|\underset{\sim}{F}\right| = F = \sqrt{70^2 + 50^2 + 80^2} = 117.473 \approx 117 \text{ (lb)}}$$

$$\boxed{\theta_x = \cos^{-1}\left(F_x/F\right) \approx \cos^{-1}\left(70/117.473\right) \approx 53.425 \approx 53.4\,(\text{deg})}$$

$$\boxed{\theta_y = \cos^{-1}\left(F_y/F\right) \approx \cos^{-1}\left(50/117.473\right) \approx 64.81 \approx 64.8\,(\text{deg})}$$

$$\boxed{\theta_z = \cos^{-1}\left(F_z/F\right) \approx \cos^{-1}\left(-80/117.473\right) \approx 132.92 \approx 133\,(\text{deg})}$$

Check: $\boxed{\cos^2(53.43) + \cos^2(64.8) + \cos^2(132.9) \approx 0.9997 \approx 1}$

# Vector Addition using Cartesian Components (3D)

To **add two** or **more vectors**, simply express them in terms of the same unit vectors, and then add **like components**. As before, we call the sum of the vectors the **resultant**.

Example #4:

Given:   $F_1 = 100\,i + 175\,j + 200\,k$ (lb)

   $F_2 = -75\,i + 25\,j - 100\,k$ (lb)

   $F_3 = -120\,i - 100\,j - 300\,k$ (lb)

Find:   The magnitude and the direction of the resultant force $F$.

Solution:

$$F = (100 - 75 - 120)\,i + (175 + 25 - 100)\,j + (200 - 100 - 300)\,k$$

$$\Rightarrow \boxed{F = -95\,i + 100\,j - 200\,k \text{ (lb)}}$$

$$\boxed{|F| = F = \sqrt{95^2 + 100^2 + 200^2} \approx 242.951 \approx 243 \text{ (lb)}}$$

$$\boxed{\theta_x = \cos^{-1}(F_x/F) \approx \cos^{-1}(-95/242.951) \approx 113.018 \approx 113 \,(\text{deg})}$$

$$\boxed{\theta_y = \cos^{-1}(F_y/F) \approx \cos^{-1}(100/242.951) \approx 65.6942 \approx 65.7 \,(\text{deg})}$$

$$\boxed{\theta_z = \cos^{-1}(F_z/F) \approx \cos^{-1}(-200/242.951) \approx 145.408 \approx 145 \,(\text{deg})}$$
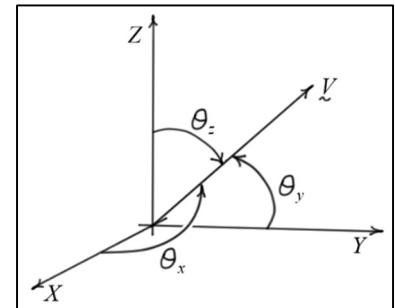


Check:   $\boxed{\cos^2(113.018) + \cos^2(65.6942) + \cos^2(145.408) \approx 1}$