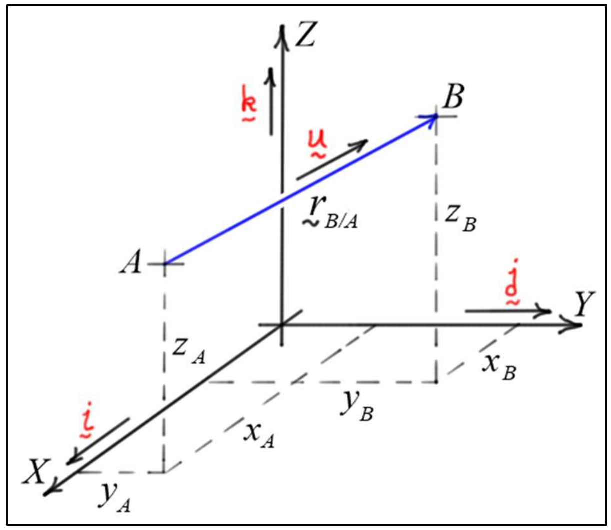
Elementary Statics
Position Vectors, Unit Vectors, and Forces Acting Along Lines

Position Vectors and Unit Vectors

- The diagram shows two ***points*** A and B and their ***coordinates*** relative to a known origin.



- The ***position vectors*** of the two points relative to the origin of the XYZ axes are

$$\underset{\sim}{r}_A = x_A\,\underset{\sim}{i} + y_A\,\underset{\sim}{j} + z_A\,\underset{\sim}{k}$$
$$\underset{\sim}{r}_B = x_B\,\underset{\sim}{i} + y_B\,\underset{\sim}{j} + z_B\,\underset{\sim}{k}$$

- Using the concept of ***vector addition***, we can write:

$$\underset{\sim}{r}_B = \underset{\sim}{r}_A + \underset{\sim}{r}_{B/A}$$

- Here, the vector $\underset{\sim}{r}_{B/A}$ represents the ***position vector*** of B ***relative*** to A.
- So, given the coordinates of A and B, we can find the vector $\underset{\sim}{r}_{B/A}$ as follows

$$\underset{\sim}{r}_{B/A} = \underset{\sim}{r}_B - \underset{\sim}{r}_A = \left(x_B - x_A\right)\underset{\sim}{i} + \left(y_B - y_A\right)\underset{\sim}{j} + \left(z_B - z_A\right)\underset{\sim}{k}$$

- The diagram also shows a ***unit vector*** $\underset{\sim}{u}$ which points in the ***direction*** of $\underset{\sim}{r}_{B/A}$. We can calculate $\underset{\sim}{u}$ by dividing $\underset{\sim}{r}_{B/A}$ by its magnitude

$$\underset{\sim}{u} = \frac{\underset{\sim}{r}_{B/A}}{|\underset{\sim}{r}_{B/A}|} = \frac{\left(x_B - x_A\right)}{|\underset{\sim}{r}_{B/A}|}\underset{\sim}{i} + \frac{\left(y_B - y_A\right)}{|\underset{\sim}{r}_{B/A}|}\underset{\sim}{j} + \frac{\left(z_B - z_A\right)}{|\underset{\sim}{r}_{B/A}|}\underset{\sim}{k}$$

- The magnitude of $\underset{\sim}{r}_{B/A}$ is $|\underset{\sim}{r}_{B/A}| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$
- The ***angles*** that $\underset{\sim}{u}$ makes with the X, Y, and Z axes are

$$\theta_x = \cos^{-1}\left(\frac{\left(x_B - x_A\right)}{|\underset{\sim}{r}_{B/A}|}\right) \qquad \theta_y = \cos^{-1}\left(\frac{\left(y_B - y_A\right)}{|\underset{\sim}{r}_{B/A}|}\right) \qquad \theta_z = \cos^{-1}\left(\frac{\left(z_B - z_A\right)}{|\underset{\sim}{r}_{B/A}|}\right)$$

Forces Acting Along Lines

- If a force $\underset{\sim}{F}$ acts along the line from A to B, we can write

$$\underset{\sim}{F} = |\underset{\sim}{F}|\,\underset{\sim}{u}$$

Example #1:

Given: Coordinates of A and B: $A(8,2,4)$ (ft) and $B(3,6,12)$ (ft)

Force $\underset{\sim}{F}$ of magnitude 200 (lb) acts along the line from A to B

Find: $\underset{\sim}{r}_{B/A}$ the position vector of B relative to A; $\underset{\sim}{u}$ the unit vector that points along the line AB, and the force $\underset{\sim}{F}$, and the angles that $\underset{\sim}{F}$ makes with the X, Y, and Z axes.

Solution:

$$\underset{\sim}{r}_{B/A} = (3-8)\underset{\sim}{i} + (6-2)\underset{\sim}{j} + (12-4)\underset{\sim}{k} = -5\underset{\sim}{i} + 4\underset{\sim}{j} + 8\underset{\sim}{k}$$

$$|\underset{\sim}{r}_{B/A}| = \sqrt{5^2+4^2+8^2} = 10.247 \text{ (ft)}$$

$$\underset{\sim}{u} = \left(\tfrac{-5}{10.247}\right)\underset{\sim}{i} + \left(\tfrac{4}{10.247}\right)\underset{\sim}{j} + \left(\tfrac{8}{10.247}\right)\underset{\sim}{k} = -0.4880\,\underset{\sim}{i} + 0.3904\,\underset{\sim}{j} + 0.7807\,\underset{\sim}{k}$$

$$\underset{\sim}{F} = 200\,\underset{\sim}{u} = -97.6\,\underset{\sim}{i} + 78.1\,\underset{\sim}{j} + 156\,\underset{\sim}{k} \text{ (lb)}$$

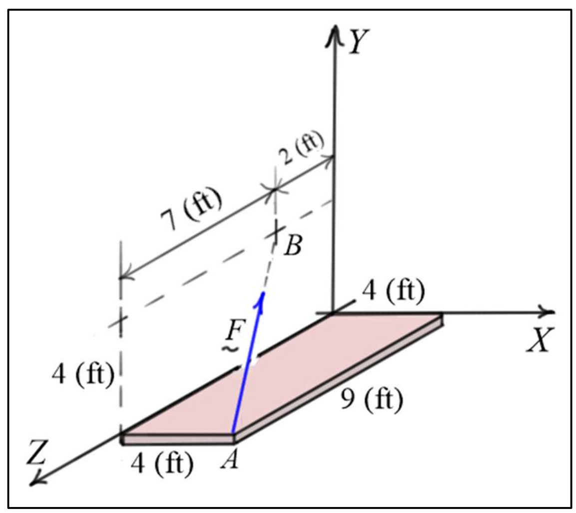
$$\theta_x = \cos^{-1}(-0.4880) \approx 119 \text{ (deg)}, \quad \theta_y = \cos^{-1}(0.3904) \approx 67 \text{ (deg)}$$

$$\theta_z = \cos^{-1}(0.7807) \approx 38.7 \text{ (deg)}$$

Example #2:

Given: Force $\underset{\sim}{F}$ of magnitude 450 (lb) acts on the rectangular door along the line from A to B.

Find: The XYZ components of $\underset{\sim}{F}$, and the angles that it makes with the X, Y, and Z axes.



Solution:

$$\underset{\sim}{r}_{B/A} = (0-4)\underset{\sim}{i} + (4-0)\underset{\sim}{j} + (2-9)\underset{\sim}{k} = -4\underset{\sim}{i} + 4\underset{\sim}{j} - 7\underset{\sim}{k}$$

$$|\underset{\sim}{r}_{B/A}| = \sqrt{4^2+4^2+7^2} = 9 \text{ (ft)}$$

$$\underset{\sim}{u} = -\left(\tfrac{4}{9}\right)\underset{\sim}{i} + \left(\tfrac{4}{9}\right)\underset{\sim}{j} - \left(\tfrac{7}{9}\right)\underset{\sim}{k}$$

$$\underset{\sim}{F} = 450\,\underset{\sim}{u} = 450\left[-\left(\tfrac{4}{9}\right)\underset{\sim}{i} + \left(\tfrac{4}{9}\right)\underset{\sim}{j} - \left(\tfrac{7}{9}\right)\underset{\sim}{k}\right] = -200\,\underset{\sim}{i} + 200\,\underset{\sim}{j} - 350\,\underset{\sim}{k} \text{ (lb)}$$

$$\theta_x = \cos^{-1}\left(\tfrac{-4}{9}\right) \approx 116.388 \approx 116 \text{ (deg)}, \quad \theta_y = \cos^{-1}\left(\tfrac{4}{9}\right) \approx 63.6122 \approx 63.6 \text{ (deg)},$$

$$\theta_z = \cos^{-1}\left(\tfrac{-7}{9}\right) \approx 141.058 \approx 141 \text{ (deg)}$$