

## Elementary Statics

### The Dot (or Scalar) Product of Two Vectors

#### Geometric Definition

The **dot** (or **scalar**) product of two vectors is defined as follows:

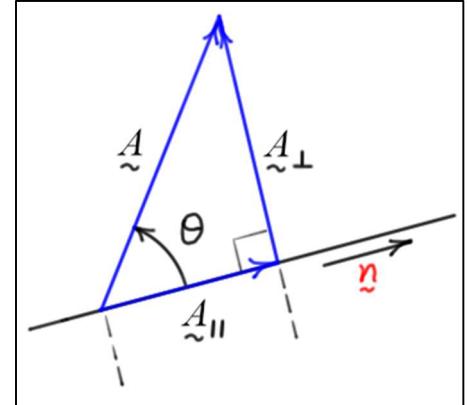
$$\underline{A} \cdot \underline{B} = |\underline{A}| |\underline{B}| \cos(\theta)$$

Here,  $\theta$  represents the **angle** between the two vectors. If one of the vectors is a **unit vector**, the dot product is the other vector's **projection** in the direction of the unit vector.

$$\underline{A} \cdot \underline{n} = |\underline{A}| |\underline{n}| \cos(\theta) = |\underline{A}| \cos(\theta)$$

The **components** of  $\underline{A}$  **parallel** and **perpendicular** to  $\underline{n}$  are

$$\underline{A}_{\parallel} = (\underline{A} \cdot \underline{n}) \underline{n} \quad \text{and} \quad \underline{A}_{\perp} = \underline{A} - \underline{A}_{\parallel}$$



The **dot product** of two vectors is **zero** if they are **perpendicular** to each other.

#### Calculation

Given two vectors  $\underline{A}$  and  $\underline{B}$  expressed in terms of a **mutually perpendicular set of unit vectors**  $\underline{i}$ ,  $\underline{j}$ , and  $\underline{k}$ , the **dot product** can be calculated as follows.

$$\underline{A} \cdot \underline{B} = (a_x \underline{i} + a_y \underline{j} + a_z \underline{k}) \cdot (b_x \underline{i} + b_y \underline{j} + b_z \underline{k}) = a_x b_x + a_y b_y + a_z b_z$$

#### Properties of the Dot Product

- Product is **commutative**:  $\underline{A} \cdot \underline{B} = \underline{B} \cdot \underline{A}$
- Product is **distributive** over addition:  $\underline{A} \cdot (\underline{B} + \underline{C}) = (\underline{A} \cdot \underline{B}) + (\underline{A} \cdot \underline{C})$
- Multiplication by a **scalar**  $\alpha$ :  $\alpha(\underline{A} \cdot \underline{B}) = (\alpha \underline{A}) \cdot \underline{B} = \underline{A} \cdot (\alpha \underline{B})$

### Example #1:

Given: Two vectors,  $\underline{A} = 10 \underline{i} + 2 \underline{j} + 8 \underline{k}$  and  $\underline{B} = 3 \underline{i} + 7 \underline{j} + 5 \underline{k}$

Find: The angle between the two vectors,  $\theta$ .

Solution:

The angle can be calculated using the **inverse cosine function** as follows.

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\underline{A} \cdot \underline{B}}{|\underline{A}||\underline{B}|}\right) = \cos^{-1}\left(\frac{30 + 14 + 40}{\sqrt{10^2 + 2^2 + 8^2} \sqrt{3^2 + 7^2 + 5^2}}\right) \approx \cos^{-1}\left(\frac{84}{118.085}\right) \\ &\approx 44.6548 \approx 44.7 \text{ (deg)}\end{aligned}$$

### Example #2:

Given: A vector,  $\underline{A} = 10 \underline{i} + 2 \underline{j} + 8 \underline{k}$ , and a **unit vector**  $\underline{n} = \left(\frac{2}{7}\right) \underline{i} + \left(\frac{6}{7}\right) \underline{j} + \left(\frac{3}{7}\right) \underline{k}$ .

Find: a)  $\theta$  the angle between the two vectors, b)  $\underline{A}_{\parallel}$  the component of  $\underline{A}$  **parallel** to  $\underline{n}$ , and c)  $\underline{A}_{\perp}$  the component of  $\underline{A}$  **perpendicular** to  $\underline{n}$ .

Solution:

a) The angle can be calculated using the **inverse cosine function** as before.

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\underline{A} \cdot \underline{n}}{|\underline{A}|}\right) = \cos^{-1}\left(\frac{(10 \cdot \frac{2}{7}) + (2 \cdot \frac{6}{7}) + (8 \cdot \frac{3}{7})}{\sqrt{10^2 + 2^2 + 8^2}}\right) \approx \cos^{-1}\left(\frac{8}{12.9615}\right) \\ &\approx 51.8871 \approx 51.9 \text{ (deg)}\end{aligned}$$

b)  $\underline{A}_{\parallel} = (\underline{A} \cdot \underline{n}) \underline{n} = 8 \left[ \left(\frac{2}{7}\right) \underline{i} + \left(\frac{6}{7}\right) \underline{j} + \left(\frac{3}{7}\right) \underline{k} \right] \approx 2.29 \underline{i} + 6.86 \underline{j} + 3.43 \underline{k}$

c)  $\underline{A}_{\perp} = \underline{A} - \underline{A}_{\parallel} \approx (10 \underline{i} + 2 \underline{j} + 8 \underline{k}) - (2.29 \underline{i} + 6.86 \underline{j} + 3.43 \underline{k}) \approx 7.71 \underline{i} - 4.86 \underline{j} + 4.57 \underline{k}$

Check:

$$\begin{aligned}\underline{A}_{\parallel} \cdot \underline{A}_{\perp} &\approx (2.29 \underline{i} + 6.86 \underline{j} + 3.43 \underline{k}) \cdot (7.71 \underline{i} - 4.86 \underline{j} + 4.57 \underline{k}) \\ &\approx 17.6327 - 33.3061 + 15.6735 \approx 0\end{aligned}$$

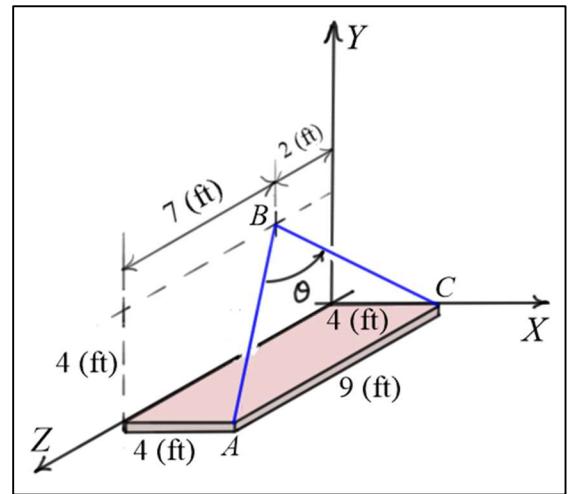
### Example #3:

Given: Two support cables  $AB$  and  $CB$  are attached to the rectangular plate as shown.

Find: The angle  $\theta$  between the two cables.

Solution:

The unit vectors pointing from  $A$  to  $B$  and  $C$  to  $B$  can be calculated using the dimensions shown in the figure as follows.



$$\underline{u}_{AB} = \left( -4\hat{i} + 4\hat{j} - 7\hat{k} \right) / \sqrt{4^2 + 4^2 + 7^2} = -\frac{4}{9}\hat{i} + \frac{4}{9}\hat{j} - \frac{7}{9}\hat{k}$$

$$\underline{u}_{CB} = \left( -4\hat{i} + 4\hat{j} + 2\hat{k} \right) / \sqrt{4^2 + 4^2 + 2^2} = -\frac{4}{6}\hat{i} + \frac{4}{6}\hat{j} + \frac{2}{6}\hat{k} = -\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}$$

Using the dot product, the angle  $\theta$  can be calculated as follows.

$$\begin{aligned} \theta &= \cos^{-1}(\underline{u}_{AB} \cdot \underline{u}_{CB}) = \cos^{-1} \left[ \left( -\frac{4}{9} \right) \left( -\frac{2}{3} \right) + \left( \frac{4}{9} \right) \left( \frac{2}{3} \right) + \left( -\frac{7}{9} \right) \left( \frac{1}{3} \right) \right] = \cos^{-1} \left( \frac{16-7}{27} \right) \\ &= \cos^{-1} \left( \frac{1}{3} \right) \\ \Rightarrow \theta &\approx 70.5 \text{ (deg)} \end{aligned}$$