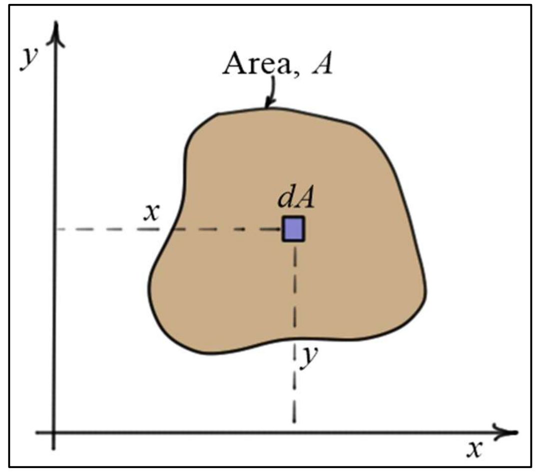
Elementary Statics
Moments of Inertia of Areas

Definition

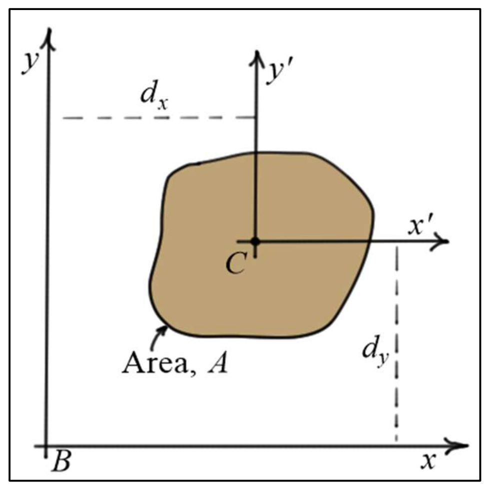


- The figure depicts an ***area***, A in the xy-plane. The ***distributions*** of this area relative to the x and y axes are measured by the ***moments of inertia of the area*** about these axes.
- The ***moments of inertia*** of A about the x and y axes are defined as

$$\boxed{I_x \triangleq \int_A y^2 dA} \qquad \boxed{I_y \triangleq \int_A x^2 dA}$$

- These ***inertias*** are always ***positive***. The ***units*** are those of L^4 (m^4, mm^4, ft^4, in^4, etc.).
- The ***larger*** the inertia, the ***farther*** the area is from the axis. The ***smaller*** the inertia, the ***closer*** it is to the axis.

Parallel Axes Theorem



- The ***moment of inertia*** of an area about ***any axis*** is related to the moment of inertia about an axis ***parallel*** to it and passing through the ***centroid*** C by the ***parallel axes theorem***.

$$\boxed{I_x = I_{x'}^C + Ad_y^2} \qquad \boxed{I_y = I_{y'}^C + Ad_x^2}$$

- It is clear from the parallel axes theorem that the ***minimum moments of inertia*** of an area occur about its ***centroidal axes***, because the quantity $Ad_a^2 > 0$.
- The ***moments of inertia about centroidal axes*** can often be found in ***inertia tables*** such as the ones in your textbook and other references.

Radius of Gyration

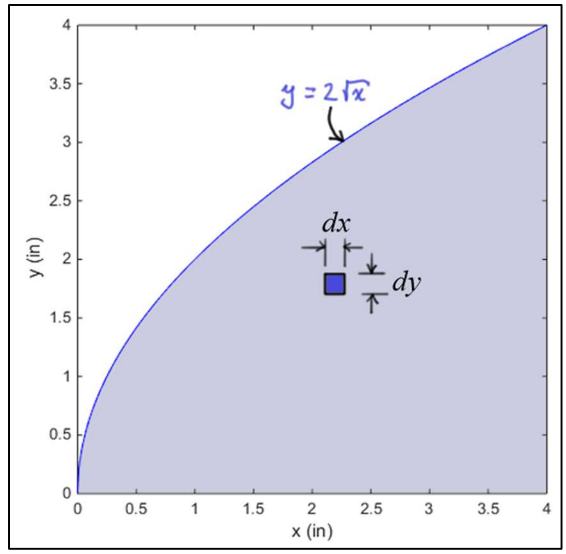
- The ***radius of gyration*** k_a of an area about ***axis*** a is defined as: $\boxed{k_a = \sqrt{\dfrac{I_a}{A}}}$.
- The ***units*** of k_a are those of ***length*** (m, mm, ft, in, etc.).

Example #1:

Given: Shaded area under the curve $y(x) = 2\sqrt{x}$ in the range $0 \le x \le 4$.

Find: I_x and I_y the moments of inertia of the shaded area about the x and y axes.

Solution using $dA = dx \times dy$:



$$I_x \triangleq \iint y^2 dA$$

$$= \int_0^4 \left(\int_0^{2\sqrt{x}} y^2 dy \right) dx = \int_0^4 \left(\tfrac{y^3}{3} \right)_0^{2\sqrt{x}} dx = \int_0^4 \tfrac{8x^{3/2}}{3} dx$$

$$= \tfrac{8}{3}\left(\tfrac{2}{5} x^{5/2} \right)_0^4 = \tfrac{16}{15}\left(4^{5/2}\right)$$

$$\Rightarrow \boxed{I_x \approx 34.1333 \approx 34.1 \text{ (in}^4\text{)}}$$

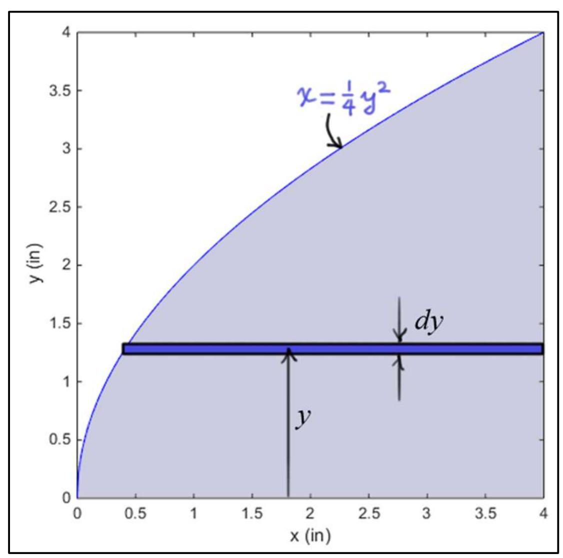
$$I_y \triangleq \iint x^2 dA$$

$$= \int_0^4 \left(\int_{y^2/4}^4 x^2 dx \right) dy = \int_0^4 \left(\tfrac{x^3}{3} \right)_{y^2/4}^4 dy = \int_0^4 \left(\tfrac{4^3}{3} - \frac{y^6}{3(4^3)} \right) dy = \left(\left(\tfrac{4^3}{3} \right) y - \frac{y^7}{3(4^3)7} \right)_0^4$$

$$= \left(\tfrac{4^4}{3} \right) - \frac{4^7}{21(4^3)} = \left(\tfrac{7}{21} - \tfrac{1}{21} \right) 4^4 = \left(\tfrac{6}{21} \right) 4^4$$

$$\Rightarrow \boxed{I_y \approx 73.1429 \approx 73.1 \text{ (in}^4\text{)}}$$

Solution for I_x using $dA = \left(4 - \frac{1}{4} y^2\right) \times dy$:



$$I_x \triangleq \iint y^2 dA = \int_0^4 y^2 \left(4 - \tfrac{1}{4} y^2\right) dy = \left(\tfrac{4}{3} y^3 - \tfrac{1}{20} y^5 \right)_0^4$$

$$= \tfrac{4^4}{3} - \tfrac{4^4}{5} = \left(\tfrac{5-3}{15} \right) 4^4 = \left(\tfrac{2}{15} \right) 4^4$$

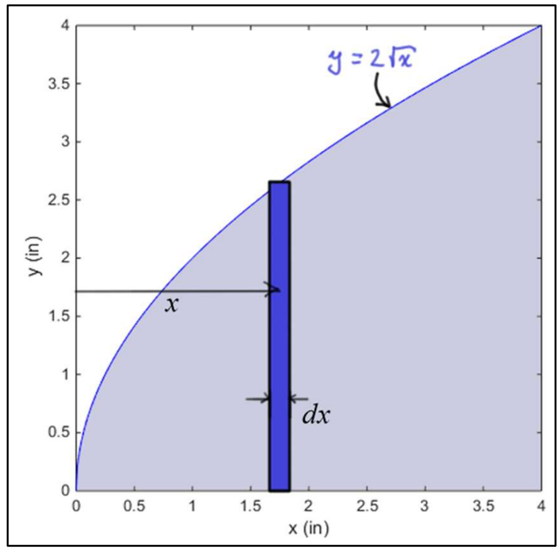
$$\Rightarrow \boxed{I_x \approx 34.1333 \approx 34.1 \text{ (in}^4\text{)}}$$

Solution for I_y using $dA = (2\sqrt{x}) \times dx$:

$$I_y \triangleq \iint x^2 dA = \int_0^4 x^2 \left(2\, x^{1/2}\right) dx = \int_0^4 2 x^{5/2}\, dx$$

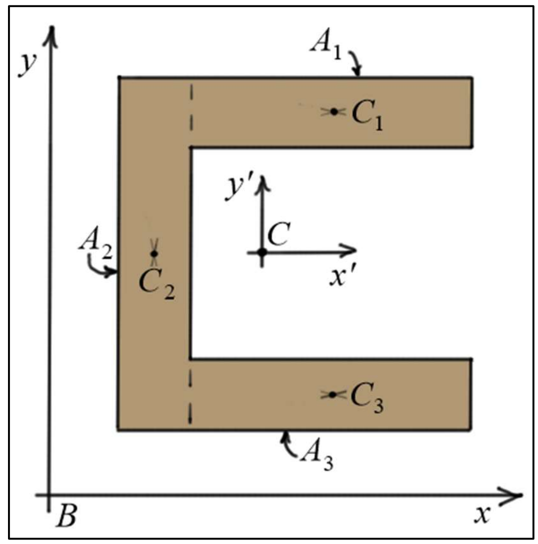
$$= \left(2\left(\tfrac{2}{7}\right) x^{7/2}\right)_0^4 = \left(\tfrac{4}{7}\right) 4^{7/2}$$

$$\Rightarrow \boxed{I_y \approx 73.1429 \approx 73.1 \text{ (in}^4\text{)}}$$



Composite Shapes

- The figure depicts a ***C-shaped area*** which has been ***divided*** into three ***rectangular areas*** A_1, A_2, and A_3 with centroids C_1, C_2, and C_3.



- The ***moment of inertia*** of the ***composite area*** about an axis is simply the ***sum of the inertias*** of the ***individual areas*** about that axis. So, for the inertia about the x-axis, we have

$$\boxed{I_x = \sum_{i=1}^{3} \left(I_x\right)_{A_i}}$$

- The ***inertia tables*** and the ***parallel axes theorem*** can be used to find the inertias of each of the individual areas about the specified axis.
- For example, $\boxed{\left(I_x\right)_{A_1} = \left(I_{x_1'}^{C_1}\right)_{A_1} + A_1 d_{y_1}^2}$. Here, $\left(I_{x_1'}^{C_1}\right)_{A_1}$ is the ***moment of inertia*** of area A_1 about an x-axis passing through ***its centroid*** C_1, and d_{y_1} is the ***distance*** between the x-axis passing through C_1 and the x-axis.
- The ***moment of inertia*** of the ***composite area*** about its ***centroidal axes*** can be related to the moments of inertia of the area about ***non-centroidal axes*** using the ***parallel axes theorem***.

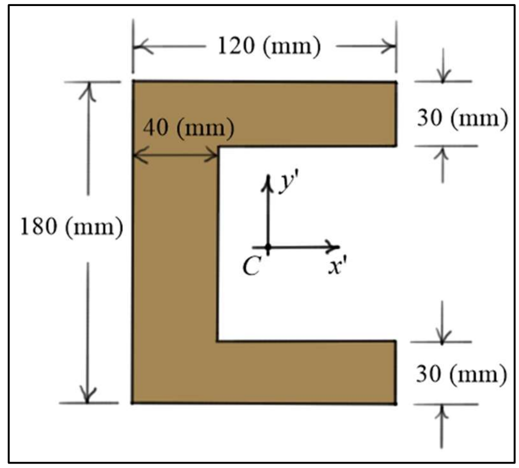
$$\boxed{I_x = I_{x'}^{C} + A d_y^2}$$

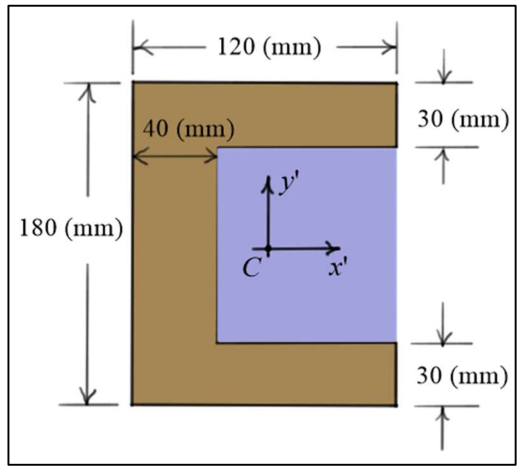
Example #2:

Given: Area shown

Find: a) Location of the centroid C

b) $I_{x'}$ and $I_{y'}$ the moments of inertia about the x' and y' axes





Solution:

In this solution, the shape shown in the first diagram will be thought of as the combination of two shapes. The 80×120 (mm) blue rectangular area in the second diagram will be removed from a 120×180 (mm) rectangular area.

a) Centroid: (relative to the lower left corner)

Let A_1 represent the 120×180 (mm) rectangular area and A_2 represent the 80×120 (mm) rectangular area.

$$\bar{x} = \frac{\sum_{i=1}^{2} \bar{x}_i A_i}{\sum_{i=1}^{2} A_i} = \frac{60(120 \times 180) - 80(80 \times 120)}{(120 \times 180) - (80 \times 120)} = \frac{528000}{12000}$$

$$\Rightarrow \boxed{\bar{x} = 44 \text{ (mm)}}$$

$$\bar{y} = \frac{\sum_{i=1}^{2} \bar{y}_i A_i}{\sum_{i=1}^{2} A_i} = \frac{90(120 \times 180) - 90(80 \times 120)}{(120 \times 180) - (80 \times 120)} = 90 \quad \Rightarrow \boxed{\bar{y} = 90 \text{ (mm)}}$$

The result for $\bar{y}$ can also be inferred by noting the shape is symmetrical about a plane cutting the shape half way up the vertical side, that is, at $y = 90$ (mm) .

b) Moments of Inertia:

The x' axis is a centroidal axis for both rectangular shapes, so there is no need for the parallel axes theorem. Using a table of inertias,

$$\left(I_{x'}\right)_{A_1} = \tfrac{1}{12} b h^3 = \tfrac{1}{12}(120)180^3 \approx 5.832 \times 10^7 \text{ (mm}^4\text{)}$$

$$\left(I_{x'}\right)_{A_2} = \tfrac{1}{12}bh^3 = \tfrac{1}{12}(80)120^3 \approx 1.152\times 10^7 \text{ (mm}^4\text{)}$$

$$I_{x'} = \left(I_{x'}\right)_{A_1} - \left(I_{x'}\right)_{A_2} = 5.832\times 10^7 - 1.152\times 10^7 \quad \Rightarrow \boxed{I_{x'} = 4.68\times 10^7 \text{ (mm}^4\text{)}}$$

The y' axis is not a centroidal axis for either of the two rectangular shapes, so the parallel axes theorem must be used.

$$\left(I_{y'}\right)_{A_1} = \left(I_{y'_1}^{C_1}\right)_{A_1} + A_1\left(d_{y_1}\right)^2 = \tfrac{1}{12}b^3h + A_1\left(d_{y_1}\right)^2 = \tfrac{1}{12}\left(120^3\right)180 + \left(120\times 180\right)\left(60-44\right)^2$$

$$= 2.592\times 10^7 + 5.5296\times 10^6 \quad \Rightarrow \boxed{\left(I_{y'}\right)_{A_1} = 3.14496\times 10^7 \text{ (mm}^4\text{)}}$$

$$\left(I_{y'}\right)_{A_2} = \left(I_{y'_2}^{C_2}\right)_{A_2} + A_2\left(d_{y_2}\right)^2 = \tfrac{1}{12}b^3h + A_2\left(d_{y_2}\right)^2 = \tfrac{1}{12}\left(80^3\right)120 + \left(80\times 120\right)\left(80-44\right)^2$$

$$= 5.12\times 10^6 + 1.24416\times 10^7 \quad \Rightarrow \boxed{\left(I_{y'}\right)_{A_1} = 1.75616\times 10^7 \text{ (mm}^4\text{)}}$$

$$I_{y'} = \left(I_{y'}\right)_{A_1} - \left(I_{y'}\right)_{A_2} = 3.14496\times 10^7 - 1.75616\times 10^7 \quad \Rightarrow \boxed{I_{y'} = 1.3888\times 10^7 \text{ (mm}^4\text{)}}$$