

Multibody Dynamics

Time Derivative of the (Coordinate) Transformation Matrices

Matrix Form of the Derivative of a Vector Fixed in a Rigid Body

Consider a body $B : (\underline{e}_1, \underline{e}_2, \underline{e}_3)$ moving in a fixed reference frame $R : (\underline{N}_1, \underline{N}_2, \underline{N}_3)$. If \underline{r} is a vector fixed in body B , then the derivative of \underline{r} may be written as

$$\boxed{\frac{{}^R d\underline{r}}{dt} = \dot{\underline{r}} = {}^R \underline{\omega}_B \times \underline{r}} \quad (1)$$

When performing the cross product, the individual vectors and the resulting cross product can be expressed in any reference frame. In the following paragraphs, “primes” indicate vector components in body $B : (\underline{e}_1, \underline{e}_2, \underline{e}_3)$, and “no primes” indicate vector components in $R : (\underline{N}_1, \underline{N}_2, \underline{N}_3)$. Also, let $[R]$ be the transformation matrix that relates the two sets of unit vectors as defined by the equation

$$\boxed{\{\underline{e}\} = [R] \{\underline{N}\}} \quad (2)$$

Case 1: $\dot{\underline{r}}$ expressed in $R : (\underline{N}_1, \underline{N}_2, \underline{N}_3)$, but ${}^R \underline{\omega}_B$ and \underline{r} expressed in $B : (\underline{e}_1, \underline{e}_2, \underline{e}_3)$

$$\text{In this case, write } \boxed{\dot{\underline{r}} = \sum_{i=1}^3 \dot{r}_i \underline{N}_i}, \quad \boxed{{}^R \underline{\omega}_B = \sum_{i=1}^3 \omega'_i \underline{e}_i}, \quad \text{and } \boxed{\underline{r} = \sum_{i=1}^3 r'_i \underline{e}_i}.$$

The three sets of vector components are related by the matrix form of Eq. (1).

$$\dot{\underline{r}} = {}^R \underline{\omega}_B \times \underline{r} \quad \rightarrow \quad \boxed{\{\dot{r}\} = [R]^T ([\tilde{\omega}'] \{r'\}) = ([R]^T [\tilde{\omega}']) \{r'\}} \quad (3)$$

Case 2: $\dot{\underline{r}}$ and ${}^R \underline{\omega}_B$ expressed in $R : (\underline{N}_1, \underline{N}_2, \underline{N}_3)$, but \underline{r} expressed in $B : (\underline{e}_1, \underline{e}_2, \underline{e}_3)$

$$\text{In this case, write } \boxed{\dot{\underline{r}} = \sum_{i=1}^3 \dot{r}_i \underline{N}_i}, \quad \boxed{{}^R \underline{\omega}_B = \sum_{i=1}^3 \omega_i \underline{N}_i}, \quad \text{and } \boxed{\underline{r} = \sum_{i=1}^3 r'_i \underline{e}_i}.$$

The three sets of vector components are again related by the matrix form of Eq. (1)

$$\dot{\underline{r}} = {}^R \underline{\omega}_B \times \underline{r} \quad \rightarrow \quad \boxed{\{\dot{r}\} = [\tilde{\omega}] ([R]^T \{r'\}) = ([\tilde{\omega}] [R]^T) \{r'\}} \quad (4)$$

Time Derivative of the Transformation Matrices

The above results can be used to determine *two different forms* of the *time derivative* of the transformation matrix $[R]$. To do this, first note that the components of position vector \underline{r} in the two different reference frames are related as follows

$$\{r\} = [R]^T \{r'\}$$

This matrix equation can be *differentiated* directly to give the following.

$$\boxed{\{\dot{r}\} = [\dot{R}]^T \{r'\} + [R]^T \{\dot{r}'\} = [\dot{R}]^T \{r'\}}$$

Here, advantage is taken of the fact that since r is **fixed** in the **body**, $\boxed{\{r'\} = \{0\}}$. Comparing this result with Eqs.

(3) and (4) gives the two forms of $[\dot{R}]$.

$$\boxed{[\dot{R}_K]^T = [R_K]^T [\tilde{\omega}'_K]} \quad \text{and} \quad \boxed{[\dot{R}_K]^T = [\tilde{\omega}_K][R_K]^T}$$

or

$$\boxed{[\dot{R}_K] = [\tilde{\omega}'_K]^T [R_K]} \quad \text{and} \quad \boxed{[\dot{R}_K] = [R_K][\tilde{\omega}_K]^T}$$