Multibody Dynamics
Angular Velocity and Orientation Angles

When using angle sequences to describe the orientation of a rigid body, the ***addition rule*** for angular velocities is used to find the angular velocity of the body. For example, the ***angular velocity*** when using a 1-2-3 rotation sequence can be written as

$$\boxed{{}^{R}\underset{\sim}{\omega}_B = {}^{R}\underset{\sim}{\omega}_{R'} + {}^{R'}\underset{\sim}{\omega}_{R''} + {}^{R''}\underset{\sim}{\omega}_B = \dot{\theta}_1 \underset{\sim}{N}_1 + \dot{\theta}_2 \underset{\sim}{N}_2' + \dot{\theta}_3 \underset{\sim}{N}_3'' = \dot{\theta}_1 \underset{\sim}{N}_1' + \dot{\theta}_2 \underset{\sim}{N}_2'' + \dot{\theta}_3 \underset{\sim}{n}_3}$$

Recall that the "***primed***" axes are intermediate axes that are introduced so the angular velocity components are all "***simple***". To make the form of ${}^{R}\underset{\sim}{\omega}_B$ most useful, it must be expressed in either the base reference frame $\left(R : \underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3\right)$ or in the body reference frame $\left(B : \underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3\right)$. For example, it is easy to show that

$$\boxed{{}^{R}\underset{\sim}{\omega}_B = \omega_1 \underset{\sim}{n}_1 + \omega_2 \underset{\sim}{n}_2 + \omega_3 \underset{\sim}{n}_3}$$

Here,

$$\boxed{\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{Bmatrix} \dot{\theta}_1 C_2 C_3 + \dot{\theta}_2 S_3 \\ -\dot{\theta}_1 C_2 S_3 + \dot{\theta}_2 C_3 \\ \dot{\theta}_1 S_2 + \dot{\theta}_3 \end{Bmatrix}}$$

These equations can be ***inverted*** to solve for $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ in terms of ω_1, ω_2, and ω_3.

$$\boxed{\begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} = \begin{Bmatrix} \left(\omega_1 C_3 - \omega_2 S_3\right)/C_2 \\ \omega_1 S_3 + \omega_2 C_3 \\ \omega_3 - S_2\left(\omega_1 C_3 - \omega_2 S_3\right)/C_2 \end{Bmatrix}}$$

<u>Singularity Positions</u>

The angular velocity equations shown above for the 1-2-3 orientation angle sequence are ***singular*** when $\cos(\theta_2)$ is ***zero***, that is, when the second orientation angle is 90 degrees. ***All*** orientation angle sequences display such a singularity at some position. This can cause problems for computer programs that use angle sequences to describe the orientation of rigid bodies.

Linearization of the 1-2-3 Orientation Angle Sequence about (0,0,0)

The above equations that relate the derivatives of the orientation angles to the angular velocity components were derived assuming the body is in an ***arbitrary position***. Now, it will be assumed that the body is exhibiting ***small angular motions*** about the orientation where all angles are zero, that is, around the ***equilibrium position*** $\theta_1 = \theta_2 = \theta_3 = \dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = 0$. In this case, we may be able to describe the angular velocity of the body using equations linearized about the equilibrium position. The linearization of the angular velocity expressions follows.

1. $m_1 = \left.\dfrac{\partial g}{\partial \theta_2}\right|_{eq} = \left(-\dot{\theta}_1 S_2 C_3\right)_{eq} = 0$

$$m_1 = \left.\frac{\partial f}{\partial \theta_2}\right|_{eq} = \left(-\dot{\theta}_1 S_2 C_3\right)_{eq} = 0 \qquad m_2 = \left.\frac{\partial f}{\partial \theta_3}\right|_{eq} = (-\dot{\theta}_1 C_2 S_3 + \dot{\theta}_2 C_3)_{eq} = 0$$

$$m_3 = \left.\frac{\partial f}{\partial \dot{\theta}_1}\right|_{eq} = \left(C_2 C_3\right)_{eq} = 1 \qquad m_4 = \left.\frac{\partial f}{\partial \dot{\theta}_2}\right|_{eq} = \left(S_3\right)_{eq} = 0$$

2. $g(\theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2) = -\dot{\theta}_1 C_2 S_3 + \dot{\theta}_2 C_3 \cong m_1\theta_2 + m_2\theta_3 + m_3\dot{\theta}_1 + m_4\dot{\theta}_2$

$$m_1 = \left.\frac{\partial g}{\partial \theta_2}\right|_{eq} = \left(\dot{\theta}_1 S_2 S_3\right)_{eq} = 0 \qquad m_2 = \left.\frac{\partial f}{\partial \theta_3}\right|_{eq} = \left(-\dot{\theta}_1 C_2 C_3 - \dot{\theta}_2 S_3\right)_{eq} = 0$$

$$m_3 = \left.\frac{\partial f}{\partial \dot{\theta}_1}\right|_{eq} = \left(-C_2 S_3\right)_{eq} = 0 \qquad m_4 = \left.\frac{\partial f}{\partial \dot{\theta}_2}\right|_{eq} = \left(C_3\right)_{eq} = 1$$

2. $h(\theta_2, \dot{\theta}_1, \dot{\theta}_3) = \dot{\theta}_1 S_2 + \dot{\theta}_3 \cong m_1\theta_2 + m_2\dot{\theta}_1 + m_3\dot{\theta}_3$

$$m_1 = \left.\frac{\partial g}{\partial \theta_2}\right|_{eq} = \left(\dot{\theta}_1 C_2\right)_{eq} = 0 \qquad m_2 = \left.\frac{\partial f}{\partial \dot{\theta}_1}\right|_{eq} = \left(S_2\right)_{eq} = 0$$

$$m_3 = \left.\frac{\partial f}{\partial \dot{\theta}_3}\right|_{eq} = 1$$

The ***conclusion*** of the linearization is that for small motions about the $(0,0,0)$ orientation, the angular velocity components are related very simply to the rates of the angles. That is,

$$\boxed{\omega_1 = \dot{\theta}_1} \quad \boxed{\omega_2 = \dot{\theta}_2} \quad \boxed{\omega_3 = \dot{\theta}_3}$$