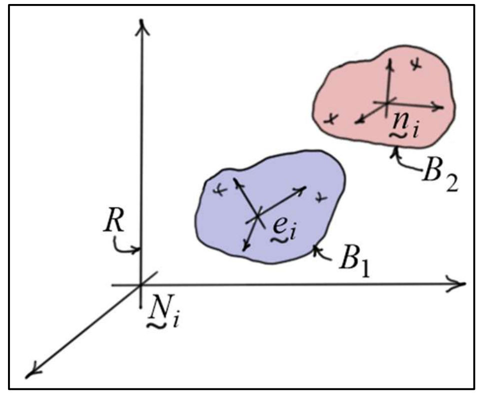
Multibody Dynamics
Angular Acceleration Using Relative Coordinates

Angle Derivatives as Generalized Speeds: 1-2-3 Rotation Sequence

Consider the two-body system shown. The fixed-frame components of the angular velocities of the bodies can be written as



$$\left\{\omega_{B_1}\right\} = \left[\omega_{B_1,\dot\theta_{B_1}}\right]\left\{\dot\theta_{B_1}\right\} + \left[\omega_{B_1,\dot\theta_{B_2}}\right]\left\{\dot\theta_{B_2}\right\}$$

$$\left\{\omega_{B_2}\right\} = \left[\omega_{B_2,\dot\theta_{B_1}}\right]\left\{\dot\theta_{B_1}\right\} + \left[\omega_{B_2,\dot\theta_{B_2}}\right]\left\{\dot\theta_{B_2}\right\}$$

Here,

$$\left[\omega_{B_1,\dot\theta_{B_1}}\right] = \begin{bmatrix} 1 & 0 & S_{12} \\ 0 & C_{11} & -S_{11}C_{12} \\ 0 & S_{11} & C_{11}C_{12} \end{bmatrix} \qquad \left[\omega_{B_1,\dot\theta_{B_2}}\right] = [\ 0\ ]_{3\times3}$$

$$\left[\omega_{B_2,\dot\theta_{B_1}}\right] = \left[\omega_{B_1,\dot\theta_{B_1}}\right] \qquad \left[\omega_{B_2,\dot\theta_{B_2}}\right] = \left[R_{B_1}\right]^T \begin{bmatrix} 1 & 0 & S_{22} \\ 0 & C_{21} & -S_{21}C_{22} \\ 0 & S_{21} & C_{21}C_{22} \end{bmatrix}$$

So, the fixed-frame components of the angular accelerations of the bodies can be written as

$$\left\{\alpha_{B_1}\right\} = \left\{\dot\omega_{B_1}\right\} = \left[\omega_{B_1,\dot\theta_{B_1}}\right]\left\{\ddot\theta_{B_1}\right\} + \left[\dot\omega_{B_1,\dot\theta_{B_1}}\right]\left\{\dot\theta_{B_1}\right\}$$

$$\left\{\alpha_{B_2}\right\} = \left\{\dot\omega_{B_2}\right\} = \left[\omega_{B_1,\dot\theta_{B_1}}\right]\left\{\ddot\theta_{B_1}\right\} + \left[\dot\omega_{B_1,\dot\theta_{B_1}}\right]\left\{\dot\theta_{B_1}\right\} + \left[\omega_{B_2,\dot\theta_{B_2}}\right]\left\{\ddot\theta_{B_2}\right\} + \left[\dot\omega_{B_2,\dot\theta_{B_2}}\right]\left\{\dot\theta_{B_2}\right\}$$

Here,

$$\left[\dot\omega_{B_1,\dot\theta_{B_1}}\right] = \frac{d}{dt}\begin{bmatrix} 1 & 0 & S_{12} \\ 0 & C_{11} & -S_{11}C_{12} \\ 0 & S_{11} & C_{11}C_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dot\theta_{12}C_{12} \\ 0 & -\dot\theta_{11}S_{11} & \left(\dot\theta_{12}S_{11}S_{12} - \dot\theta_{11}C_{11}C_{12}\right) \\ 0 & \dot\theta_{11}C_{11} & -\left(\dot\theta_{11}S_{11}C_{12} + \dot\theta_{12}C_{11}S_{12}\right) \end{bmatrix}$$

$$\left[\dot\omega_{B_2,\dot\theta_{B_2}}\right] = \frac{d}{dt}\left(\left[R_{B_1}\right]^T \begin{bmatrix} 1 & 0 & S_{22} \\ 0 & C_{21} & -S_{21}C_{22} \\ 0 & S_{21} & C_{21}C_{22} \end{bmatrix}\right)$$

$$= \left[\dot R_{B_1}\right]^T \begin{bmatrix} 1 & 0 & S_{22} \\ 0 & C_{21} & -S_{21}C_{22} \\ 0 & S_{21} & C_{21}C_{22} \end{bmatrix} + \left[R_{B_1}\right]^T \begin{bmatrix} 0 & 0 & \dot\theta_{22}C_{22} \\ 0 & -\dot\theta_{21}S_{21} & \left(\dot\theta_{22}S_{21}S_{22} - \dot\theta_{21}C_{21}C_{22}\right) \\ 0 & \dot\theta_{21}C_{21} & -\left(\dot\theta_{21}S_{21}C_{22} + \dot\theta_{22}C_{21}S_{22}\right) \end{bmatrix}$$

$$\Rightarrow \boxed{\left[\dot{\omega}_{B_2,\dot{\theta}_{B_2}}\right] = \left[\tilde{\omega}_{B_1}\right]\left[R_{B_1}\right]^T \begin{bmatrix} 1 & 0 & S_{22} \\ 0 & C_{21} & -S_{21}C_{22} \\ 0 & S_{21} & C_{21}C_{22} \end{bmatrix} + \left[R_{B_1}\right]^T \begin{bmatrix} 0 & 0 & \dot{\theta}_{22}C_{22} \\ 0 & -\dot{\theta}_{21}S_{21} & \left(\dot{\theta}_{22}S_{21}S_{22} - \dot{\theta}_{21}C_{21}C_{22}\right) \\ 0 & \dot{\theta}_{21}C_{21} & -\left(\dot{\theta}_{21}S_{21}C_{22} + \dot{\theta}_{22}C_{21}S_{22}\right) \end{bmatrix}}$$

Here, the ***time derivative*** of the ***transformation matrix*** $\left[R_{B_1}\right]^T$ is calculated using the ***skew symmetric matrix*** $\left[\tilde{\omega}_{B_1}\right]$ as discussed in previous notes.

Angular Velocity Components as Generalized Speeds

The ***fixed-frame components*** of ${}^R\underset{\sim}{\omega}_{B_2}$ the angular velocity of B_2 in the fixed frame R can be written as

$$\left\{\omega_{B_2}\right\} = \left\{\omega_{B_1}\right\} + \left[R_{B_1}\right]^T \left\{\hat{\omega}_{B_2}\right\} = \left[\omega_{B_2,\omega_{B_1}}\right]\left\{\omega_{B_1}\right\} + \left[\omega_{B_2,\hat{\omega}_{B_2}}\right]\left\{\hat{\omega}_{B_2}\right\}$$

Here,

$$\left[\omega_{B_2,\omega_{B_1}}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left[\omega_{B_2,\hat{\omega}_{B_2}}\right] = \left[R_{B_1}\right]^T$$

This equation can be differentiated to give the ***fixed-frame components*** of ${}^R\underset{\sim}{\alpha}_{B_2}$

$$\left\{\alpha_{B_2}\right\} = \left\{\dot{\omega}_{B_2}\right\} = \left\{\dot{\omega}_{B_1}\right\} + \left[R_{B_1}\right]^T \left\{\dot{\hat{\omega}}_{B_2}\right\} + \left[\dot{R}_{B_1}\right]^T \left\{\hat{\omega}_{B_2}\right\}$$

or

$$\boxed{\left\{\alpha_{B_2}\right\} = \left\{\dot{\omega}_{B_1}\right\} + \left[R_{B_1}\right]^T \left\{\dot{\hat{\omega}}_{B_2}\right\} + \left[\tilde{\omega}_{B_1}\right]\left[R_{B_1}\right]^T \left\{\hat{\omega}_{B_2}\right\}}$$

The ***body-frame components*** of ${}^R\underset{\sim}{\omega}_{B_2}$ the angular velocity of B_2 in the fixed frame R can be written as

$$\left\{\omega'_{B_2}\right\} = \left[{}^{B_1}R_{B_2}\right]\left\{\omega'_{B_1}\right\} + \left\{\hat{\omega}'_{B_2}\right\} = \left[\omega'_{B_2,\omega'_{B_1}}\right]\left\{\omega'_{B_1}\right\} + \left[\omega'_{B_2,\hat{\omega}'_{B_2}}\right]\left\{\hat{\omega}'_{B_2}\right\} \qquad \text{“}B_2 \text{ Components”}$$

Here,

$$\left[\omega'_{B_2,\omega'_{B_1}}\right] = \left[{}^{B_1}R_{B_2}\right] \qquad \left[\omega'_{B_2,\hat{\omega}'_{B_2}}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Differentiating this equation gives the B_2 components of ${}^R\underset{\sim}{\alpha}_{B_2}$

$$
\begin{aligned}
\left\{\alpha'_{B_2}\right\} &= \left\{\dot{\omega}'_{B_2}\right\} \\
&= \left[\omega'_{B_2,\omega'_{B_1}}\right]\left\{\dot{\omega}'_{B_1}\right\} + \left[\dot{\omega}'_{B_2,\omega'_{B_1}}\right]\left\{\omega'_{B_1}\right\} + \left[\omega'_{B_2,\hat{\omega}'_{B_2}}\right]\left\{\dot{\hat{\omega}}'_{B_2}\right\} + \left[\dot{\omega}'_{B_2,\hat{\omega}'_{B_2}}\right]\left\{\hat{\omega}'_{B_2}\right\} \\
&= \left[{}^{B_1}R_{B_2}\right]\left\{\dot{\omega}'_{B_1}\right\} + \left[{}^{B_1}\dot{R}_{B_2}\right]\left\{\omega'_{B_1}\right\} + \left\{\dot{\hat{\omega}}'_{B_2}\right\} \\
&= \left[{}^{B_1}R_{B_2}\right]\left\{\dot{\omega}'_{B_1}\right\} + \left[\tilde{\hat{\omega}}'_{B_2}\right]^T\left[{}^{B_1}R_{B_2}\right]\left\{\omega'_{B_1}\right\} + \left\{\dot{\hat{\omega}}'_{B_2}\right\}
\end{aligned}
$$

or

$$
\boxed{\left\{\alpha'_{B_2}\right\} = \left[{}^{B_1}R_{B_2}\right]\left\{\dot{\omega}'_{B_1}\right\} - \left[\tilde{\hat{\omega}}'_{B_2}\right]\left[{}^{B_1}R_{B_2}\right]\left\{\omega'_{B_1}\right\} + \left\{\dot{\hat{\omega}}'_{B_2}\right\}} \qquad \text{“}B_2\text{ Components”}
$$