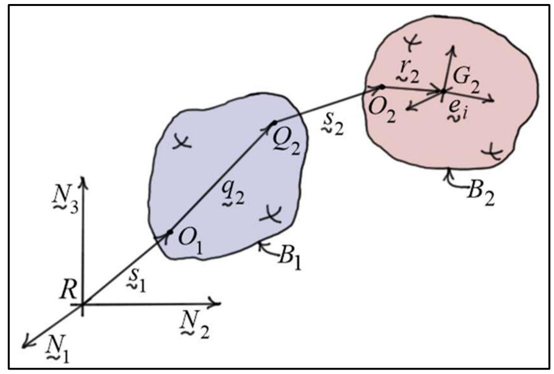
Multibody Dynamics
Velocities and Partial Velocities Using Relative Coordinates

Velocities

The motions of the mass centers of the bodies can also be determined by working ***through the other bodies*** of the system. For example, the fixed-frame components of the ***position vector*** of the mass center G_2 can be written as



$$\begin{aligned}\left\{p_{G_2}\right\} &= \left\{s_1\right\} + \left\{q_2\right\} + \left\{s_2\right\} + \left\{r_2\right\} \\ &= \left\{s_1\right\} + \left[R_{B_1}\right]^T \left\{q_2'\right\} + \left[R_{B_1}\right]^T \left\{s_2'\right\} + \left[R_{B_2}\right]^T \left\{r_2'\right\}\end{aligned}$$

Here it has been assumed that $\{s_1\}$ is given in the fixed frame, $\{q_2'\}$ and $\{s_2'\}$ are given in the B_1 frame, and $\{r_2'\}$ is given in the B_2 frame.

The ***velocity*** of G_2 can be found by ***differentiating*** the above equation.

$$\left\{v_{G_2}\right\} = \left\{\dot{s}_1\right\} + \left[\dot{R}_{B_1}\right]^T \left\{q_2'\right\} + \left[\dot{R}_{B_1}\right]^T \left\{s_2'\right\} + \left[R_{B_1}\right]^T \left\{\dot{s}_2'\right\} + \left[\dot{R}_{B_2}\right]^T \left\{r_2'\right\}$$

To expand this equation further, first decide whether to use ***fixed-frame*** or ***body-frame components*** for the ***angular velocity vectors***. The two results are as follows.

$$\boxed{\left\{v_{G_2}\right\} = \left\{\dot{s}_1\right\} + \left[\tilde{\omega}_{B_1}\right]\left[R_{B_1}\right]^T \left(\left\{q_2'\right\} + \left\{s_2'\right\}\right) + \left[R_{B_1}\right]^T \left\{\dot{s}_2'\right\} + \left[\tilde{\omega}_{B_2}\right]\left[R_{B_2}\right]^T \left\{r_2'\right\}} \tag{1}$$

$$\boxed{\left\{v_{G_2}\right\} = \left\{\dot{s}_1\right\} + \left[R_{B_1}\right]^T \left[\tilde{\omega}_{B_1}'\right]\left(\left\{q_2'\right\} + \left\{s_2'\right\}\right) + \left[R_{B_1}\right]^T \left\{\dot{s}_2'\right\} + \left[R_{B_2}\right]^T \left[\tilde{\omega}_{B_2}'\right]\left\{r_2'\right\}} \tag{2}$$

The velocities of ***other points*** of B_2 are found by simply replacing the coordinates $\{r_2'\}$ with the coordinates of that point.

To find the ***partial velocities*** of G_2, first ***rearrange*** Eqs. (1) and (2). As a first step in this process, note that ${}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r} = -\underset{\sim}{r} \times {}^R\underset{\sim}{\omega}_B$. This vector identity can be written in matrix form as $\left[\tilde{\omega}_B\right]\{r\} = -\left[\tilde{r}\right]\left\{\omega_B\right\}$. Using this result in Eqs. (1) and (2), write the following.

Using ***fixed-frame components*** for the ***angular velocity vectors***:

$$\begin{aligned}\left\{v_{G_2}\right\} &= \left\{\dot{s}_1\right\} + \left[\tilde{\omega}_{B_1}\right]\left[R_{B_1}\right]^T\left(\left\{q'_2\right\} + \left\{s'_2\right\}\right) + \left[R_{B_1}\right]^T\left\{\dot{s}'_2\right\} + \left[\tilde{\omega}_{B_2}\right]\left[R_{B_2}\right]^T\left\{r'_2\right\} \\ &= \left\{\dot{s}_1\right\} + \left[\tilde{\omega}_{B_1}\right]\left(\left\{q_2\right\} + \left\{s_2\right\}\right) + \left[R_{B_1}\right]^T\left\{\dot{s}'_2\right\} + \left[\tilde{\omega}_{B_2}\right]\left\{r_2\right\} \\ &= \left\{\dot{s}_1\right\} - \left(\left[\tilde{q}_2\right] + \left[\tilde{s}_2\right]\right)\left\{\omega_{B_1}\right\} + \left[R_{B_1}\right]^T\left\{\dot{s}'_2\right\} - \left[\tilde{r}_2\right]\left\{\omega_{B_2}\right\} \\ &= \left\{\dot{s}_1\right\} - \left(\left[\tilde{q}_2\right] + \left[\tilde{s}_2\right]\right)\left\{\omega_{B_1}\right\} + \left[R_{B_1}\right]^T\left\{\dot{s}'_2\right\} - \left[\tilde{r}_2\right]\left(\left\{\omega_{B_1}\right\} + \left[R_{B_1}\right]^T\left\{\hat{\omega}_{B_2}\right\}\right) \\ &= \left\{\dot{s}_1\right\} - \left(\left[\tilde{q}_2\right] + \left[\tilde{s}_2\right] + \left[\tilde{r}_2\right]\right)\left\{\omega_{B_1}\right\} + \left[R_{B_1}\right]^T\left\{\dot{s}'_2\right\} - \left[\tilde{r}_2\right]\left[R_{B_1}\right]^T\left\{\hat{\omega}_{B_2}\right\}\end{aligned} \tag{3}$$

Recall here that the entries of $\left\{\omega_{B_1}\right\}$ are expressed in the fixed frame, and the entries of $\left\{\hat{\omega}_{B_2}\right\}$ are expressed in the B_1 frame. In both cases, these are the frames in which the angular velocities are measured.

Using ***body-frame components*** for the ***angular velocity vectors***:

$$\begin{aligned}\left\{v_{G_2}\right\} &= \left\{\dot{s}_1\right\} + \left[R_{B_1}\right]^T\left[\tilde{\omega}'_{B_1}\right]\left(\left\{q'_2\right\} + \left\{s'_2\right\}\right) + \left[R_{B_1}\right]^T\left\{\dot{s}'_2\right\} + \left[R_{B_2}\right]^T\left[\tilde{\omega}'_{B_2}\right]\left\{r'_2\right\} \\ &= \left\{\dot{s}_1\right\} - \left[R_{B_1}\right]^T\left(\left[\tilde{q}'_2\right] + \left[\tilde{s}'_2\right]\right)\left\{\omega'_{B_1}\right\} + \left[R_{B_1}\right]^T\left\{\dot{s}'_2\right\} - \left[R_{B_2}\right]^T\left[\tilde{r}'_2\right]\left\{\omega'_{B_2}\right\} \\ &= \left\{\dot{s}_1\right\} - \left[R_{B_1}\right]^T\left(\left[\tilde{q}'_2\right] + \left[\tilde{s}'_2\right]\right)\left\{\omega'_{B_1}\right\} + \left[R_{B_1}\right]^T\left\{\dot{s}'_2\right\} \\ &\qquad - \left[R_{B_2}\right]^T\left[\tilde{r}'_2\right]\left(\left[{}^{B_2}R_{B_1}\right]^T\left\{\omega'_{B_1}\right\} + \left\{\hat{\omega}'_{B_2}\right\}\right) \\ &= \left\{\dot{s}_1\right\} - \left[R_{B_1}\right]^T\left(\left[\tilde{q}'_2\right] + \left[\tilde{s}'_2\right]\right)\left\{\omega'_{B_1}\right\} - \left[R_{B_2}\right]^T\left[\tilde{r}'_2\right]\left[{}^{B_2}R_{B_1}\right]^T\left\{\omega'_{B_1}\right\} \\ &\qquad + \left[R_{B_1}\right]^T\left\{\dot{s}'_2\right\} - \left[R_{B_2}\right]^T\left[\tilde{r}'_2\right]\left\{\hat{\omega}'_{B_2}\right\}\end{aligned} \tag{4}$$

<u>Partial Velocities</u>

Eqs. (3) and (4) can be used to find the ***partial velocities*** of the mass centers of the bodies.

<u>Case 1:</u>

If the ***generalized speeds*** are defined to be the components of $\left\{\dot{s}_1\right\}$, $\left\{\dot{s}'_2\right\}$, $\left\{\omega_{B_1}\right\}$, and $\left\{\hat{\omega}_{B_2}\right\}$, then

$$\boxed{\left\{v_{G_2}\right\} = \left[v_{G_2,\dot{s}_1}\right]\left\{\dot{s}_1\right\} + \left[v_{G_2,\omega_{B_1}}\right]\left\{\omega_{B_1}\right\} + \left[v_{G_2,\dot{s}'_2}\right]\left\{\dot{s}'_2\right\} + \left[v_{G_2,\hat{\omega}_{B_2}}\right]\left\{\hat{\omega}_{B_2}\right\}}$$

Here,

$\left[v_{G_2,\dot{s}_1}\right]$ is the 3×3 identity matrix

$\left[v_{G_2,\omega_{B_1}}\right] = -\left(\left[\tilde{q}_2\right] + \left[\tilde{s}_2\right] + \left[\tilde{r}_2\right]\right)$

$\left[v_{G_2,\dot{s}'_2}\right] = \left[R_{B_1}\right]^T$

$$\left[v_{G_2,\hat{\omega}_{B_2}}\right] = -\left[\tilde{r}_2\right]\left[R_{B_1}\right]^T$$

Case 2:

If the ***generalized speeds*** are defined to be the components of $\{\dot{s}_1\}$, $\{\dot{s}'_2\}$, $\{\omega'_{B_1}\}$, and $\{\hat{\omega}'_{B_2}\}$, then

$$\boxed{\left\{v_{G_2}\right\} = \left[v_{G_2,\dot{s}_1}\right]\left\{\dot{s}_1\right\} + \left[v_{G_2,\omega'_{B_1}}\right]\left\{\omega'_{B_1}\right\} + \left[v_{G_2,\dot{s}'_2}\right]\left\{\dot{s}'_2\right\} + \left[v_{G_2,\hat{\omega}'_{B_2}}\right]\left\{\hat{\omega}'_{B_2}\right\}}$$

Here,

$\left[v_{G_2,\dot{s}_1}\right]$ is the 3×3 identity matrix

$$\left[v_{G_2,\omega'_{B_1}}\right] = -\left[R_{B_1}\right]^T\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right) - \left[R_{B_2}\right]^T\left[\tilde{r}'_2\right]\left[{}^{B_2}R_{B_1}\right]^T$$

$$\left[v_{G_2,\dot{s}'_2}\right] = \left[R_{B_1}\right]^T$$

$$\left[v_{G_2,\hat{\omega}'_{B_2}}\right] = -\left[R_{B_2}\right]^T\left[\tilde{r}'_2\right]$$

Case 3:

If, instead, the generalized speeds are defined to be the components of $\{\dot{s}_1\}$, $\{\dot{s}_2\}$, and a set of ***orientation angle derivatives***, then we can write $\{\omega_{B_1}\}$ and $\{\hat{\omega}_{B_2}\}$ (or $\{\omega'_{B_1}\}$ and $\{\hat{\omega}'_{B_2}\}$) in terms of the orientation angles and then identify the partial velocities. Returning to Eq. (3), for example, write

$$\begin{aligned}
\left\{v_{G_2}\right\} &= \left\{\dot{s}_1\right\} - \left(\left[\tilde{q}_2\right]+\left[\tilde{s}_2\right]+\left[\tilde{r}_2\right]\right)\left\{\omega_{B_1}\right\} + \left[R_{B_1}\right]^T\left\{\dot{s}'_2\right\} - \left[\tilde{r}_2\right]\left[R_{B_1}\right]^T\left\{\hat{\omega}_{B_2}\right\} \\
&= \left\{\dot{s}_1\right\} - \left(\left[\tilde{q}_2\right]+\left[\tilde{s}_2\right]+\left[\tilde{r}_2\right]\right)\left[\omega_{B_1,\dot{\theta}_{B_1}}\right]\left\{\dot{\theta}_{B_1}\right\} + \left[R_{B_1}\right]^T\left\{\dot{s}'_2\right\} \\
&\qquad - \left[\tilde{r}_2\right]\left[R_{B_1}\right]^T\left[\hat{\omega}_{B_2,\dot{\theta}_{B_2}}\right]\left\{\dot{\theta}_{B_2}\right\} \\
&= \left[v_{G_2,\dot{s}_1}\right]\left\{\dot{s}_1\right\} + \left[v_{G_2,\dot{\theta}_{B_1}}\right]\left\{\dot{\theta}_{B_1}\right\} + \left[v_{G_2,\dot{s}'_2}\right]\left\{\dot{s}'_2\right\} + \left[v_{G_2,\dot{\theta}_{B_2}}\right]\left\{\dot{\theta}_{B_2}\right\}
\end{aligned} \tag{5}$$

Here,

$\left[v_{G_2,\dot{s}_1}\right]$ is the 3×3 identity matrix

$$\left[v_{G_2,\dot{\theta}_{B_1}}\right] = -\left(\left[\tilde{q}_2\right]+\left[\tilde{s}_2\right]+\left[\tilde{r}_2\right]\right)\left[\omega_{B_1,\dot{\theta}_{B_1}}\right]$$

$$\left[v_{G_2,\dot{s}'_2}\right] = \left[R_{B_1}\right]^T$$

$$\left[v_{G_2,\dot{\theta}_{B_2}}\right] = -\left[\tilde{r}_2\right]\left[R_{B_1}\right]^T\left[\hat{\omega}_{B_2,\dot{\theta}_{B_2}}\right]$$

The partial angular velocity matrices are defined in previous notes for a 1-2-3 rotation sequence.