Introductory Motion and Control
Compensator Design for ITAE Optimal Response with Pre-filters

(Reference: Dorf & Bishop, Modern Control Systems, Prentice-Hall, 2001)

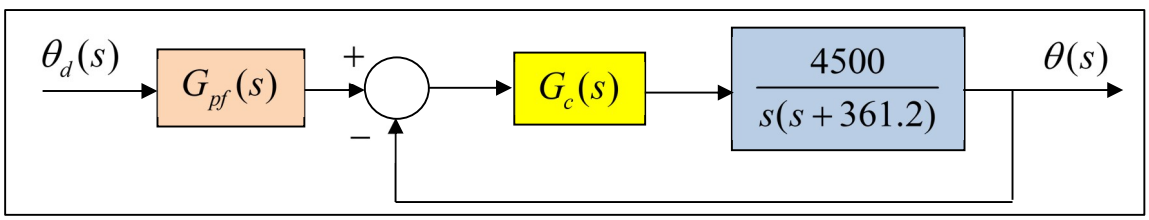
The forms of ***ITAE optimal*** transfer functions for ***step*** and ***ramp*** inputs are given in Tables 5.6 and 5.7 of Dorf & Bishop's Modern Control Systems. To guarantee ITAE optimal behavior, a compensator can be designed to force the closed-loop transfer function to be of ***ITAE optimal form***. The use of ***pre-filters*** may be required to obtain optimal form.

Design Strategy

1. ***Step Input***: Assume $\zeta = 0.7$; ***Ramp Input***: Assume $\zeta = 1.6$
2. Then ***select*** ω_n to satisfy the ***settling time*** specification.
3. Find the compensator parameters to give a ***characteristic equation*** of the form given in Table 5.6 (step input) or Table 5.7 (ramp input).
4. Design a ***pre-filter*** to "cancel" the unwanted terms in the numerator of the closed-loop transfer function.

Example

Design a PID compensator that will produce ***ITAE optimal, closed-loop step response*** with a settling time $T_s = 0.005$ (sec) for the aircraft attitude control system shown in the block diagram. Use a pre-filter as necessary.



Solution

1. Find ω_n by setting $T_s = 4/\zeta\omega_n = 0.005$ (sec). Assuming $\zeta = 0.7$ (for a step input), $\omega_n = 1143$ (rad/sec).
2. Assuming a PID controller of the form $G_c(s) = \dfrac{K(s^2 + as + b)}{s}$, the closed loop transfer function (without the pre-filter) is found to be

$$\boxed{T(s) = \frac{4500K(s^2 + as + b)}{s^2(s + 361.2) + 4500K(s^2 + as + b)}}$$

With ***PID*** control and ***no pre-filter***, this system is ***type-2***, so it can be ***optimized*** for ***ramp input***. However, in this example, the ***optimization*** is for ***step response***.

3. Equate the ***characteristic equation*** with that provided by Table 5.6 with $\omega_n = 1143$.

$$\boxed{\begin{aligned} &s^3 + (361.2 + 4500K)s^2 + (4500aK)s + (4500bK) \\ &\quad = s^3 + 1.75\omega_n s^2 + 2.15\omega_n^2 s + \omega_n^3 \end{aligned}}$$

Equating the polynomial coefficients and solving for K, a, and b gives

$$K = 0.36423 \qquad a = 1713.73 \qquad b = 9.110673\times 10^5$$

or

$$K_p = 624.2 \qquad K_I = 331{,}838 \qquad K_D = 0.36423$$

4. ***Pre-filter***: To provide ITAE optimal response, choose the pre-filter

$$\boxed{G_{pf}(s) = \frac{(1143)^3}{1639(s^2 + as + b)}}$$

So, the final transfer function has ***ITAE optimal form***

$$\frac{\theta_d(s)}{\theta(s)} = \left(\frac{(1143)^3}{1639(s^2 + as + b)}\right)\left(\frac{1639(s^2 + as + b)}{s^3 + 2000s^2 + (2.8089\times 10^6)s + (1143)^3}\right)$$

$$\Rightarrow \quad \boxed{\frac{\theta_d(s)}{\theta(s)} = \frac{(1143)^3}{s^3 + 2000s^2 + (2.8089\times 10^6)s + (1143)^3}}$$

5. Closed-loop step response ***with*** and ***without*** pre-filter: From the plots below, it is clear the closed loop system ***without a pre-filter*** has ***large overshoot*** due to the zeros in the PID compensator. This problem is removed by including a pre-filter. Both systems have ***settling times*** slightly larger than 0.005 (sec). Further increases on the choice of ω_n will resolve this issue as well.

6. Error analysis ***with a pre-filter***: The system error can be calculated as follows

$$E(s) = \theta_d(s) - \theta(s) = \theta_d(s) - \left(G_{pf}(s)T(s)\right)\theta_d(s) = \left(1 - G_{pf}(s)T(s)\right)\theta_d(s)$$

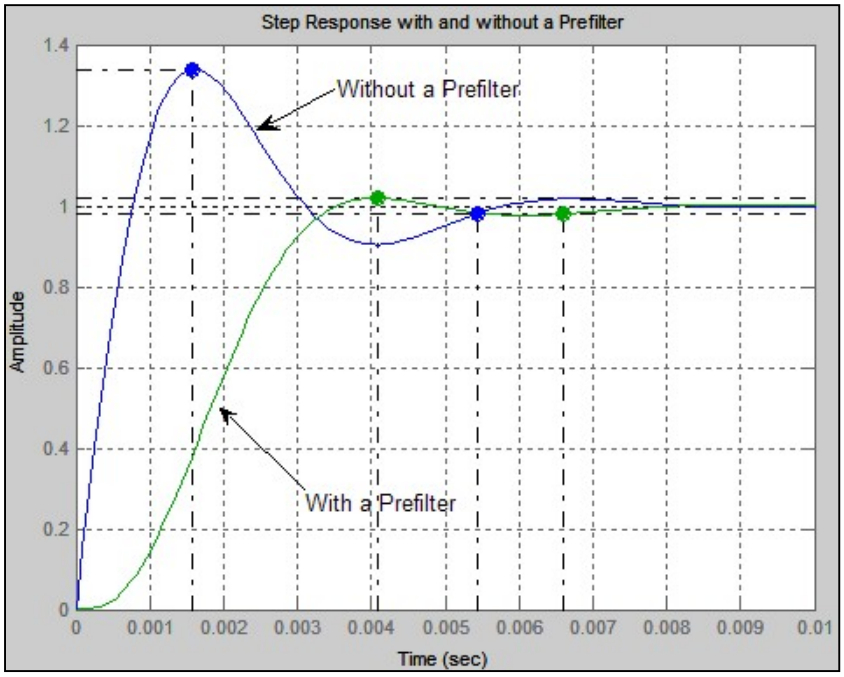
$$= \left(1 - \frac{(1143)^3}{s^3 + 2000s^2 + (2.8089E6)s + (1143)^3}\right)\theta_d(s)$$

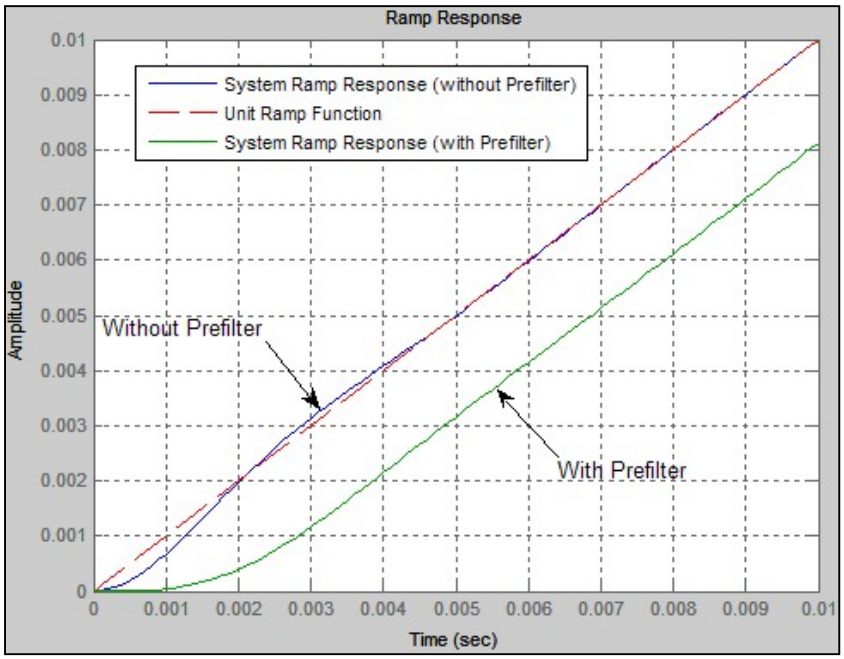
So, the error transfer function is

$$\boxed{\frac{E(s)}{\theta_d(s)} = \frac{s(s^2 + 2000s + 2.8089E6)}{s^3 + 2000s^2 + (2.8089E6)s + (1143)^3}}$$

Note the system ***with the pre-filter*** has a ***zero*** steady-state error to a ***step input*** and a ***finite*** steady-state error to a ***ramp input*** (typical of a ***type-1*** system).

7. Error analysis ***with no pre-filter***: The system with no pre-filter is a ***type-2*** system, so it has a ***zero*** steady-state error to both ***step*** and ***ramp*** inputs. Clearly, ***some error control is lost when using the pre-filter***.


Step Response with and without a Prefilter
Without a Prefilter
With a Prefilter
Amplitude
Time (sec)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0.001
0.002
0.003
0.004
0.005
0.006
0.007
0.008
0.009
0.01


Ramp Response
System Ramp Response (without Prefilter)
Unit Ramp Function
System Ramp Response (with Prefilter)
Without Prefilter
With Prefilter
Amplitude
Time (sec)
0
0.001
0.002
0.003
0.004
0.005
0.006
0.007
0.008
0.009
0.01