Multibody Dynamics
Constraint Relaxation Method: Meaning of Lagrange Multipliers

- Previously, it was noted that if a dynamic system is described using "n" ***generalized coordinates*** $q_k\ (k=1,\ldots,n)$, and if the system is subjected to "m" ***independent holonomic*** and/or ***nonholonomic*** constraint equations of the form

$$\boxed{\sum_{k=1}^{n} a_{jk}\dot{q}_k + a_{j0} = 0} \qquad (j=1,\ldots,m) \tag{1}$$

tthe equations of motion of the system can be found by using one of the following two forms of Lagrange's equations with Lagrange multipliers.

$$\boxed{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_k}\right) - \frac{\partial K}{\partial q_k} = F_{q_k} + \sum_{j=1}^{m}\lambda_j a_{jk}} \qquad (k=1,\ldots,n) \tag{2}$$

or

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \left(F_{q_k}\right)_{nc} + \sum_{j=1}^{m}\lambda_j a_{jk}} \qquad (k=1,\ldots,n) \tag{3}$$

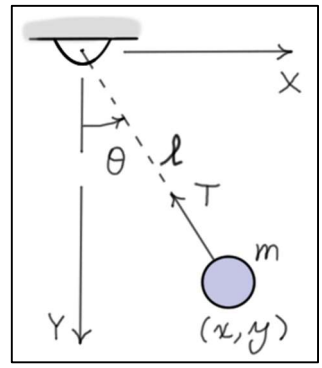
Here,

- K : ***kinetic energy*** of the system
- F_{q_k} : ***generalized force*** associated with the generalized coordinate q_k
- L : ***Lagrangian*** of the system
- V : ***potential energy*** function for the ***conservative*** forces and torques
- $\left(F_{q_k}\right)_{nc}$: ***generalized force*** associated with q_k for ***only*** the ***nonconservative*** forces/torques
- λ_j : Lagrange multiplier associated with the jth constraint equation
- a_{jk} : coefficients from the constraint equations $\left(j=1,\ldots,m;\ k=1,\ldots,n\right)$

- Eqs. (1) and (2) or Eqs. (1) and (3) form a set of "$n+m$" ***differential/algebraic equations*** for the "n" ***generalized coordinates*** and the "m" ***Lagrange multipliers***.
- Alternatively, some or all the constraints can be ***relaxed*** (or ***removed***) and replaced with ***force*** and/or ***torque*** components that are required to ***maintain*** the ***constraints***. Then, formulate the "n" Lagrange's equations in terms of the "n" generalized coordinates and the "m" constraint force (or torque) components.
- ***Together*** with the ***constraint equations***, this forms a set of "$n+m$" ***differential/ algebraic equations*** for the "n" ***generalized coordinates*** and the "m" ***constraint force*** and/or ***torque components***. If ***all*** the constraints are ***relaxed***, then Eqs. (3) can be written as

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \left(F_{q_k}\right)_{nc} + \left(F_{q_k}\right)_{\text{constraints}}} \qquad (k = 1, \ldots, n) \tag{4}$$

<u>Example: The Simple Pendulum</u>



- For the simple pendulum shown, $q_1 = x$ and $q_2 = y$ are used as the generalized coordinates, and the ***length constraint*** of the pendulum is ***relaxed*** in the formulation. Lagrange's equations can then be written in the form of Eq. (4).
- Here, $L = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) + mgy$, $\left(F_{q_k}\right)_{nc} = 0$, and the contributions of the constraint force to the right sides of the equations are

$$\boxed{\left(F_x\right)_{\text{constraint}} = \underset{\sim}{T} \cdot \left(\partial \underset{\sim}{v} / \partial \dot{x}\right) = T\left(-\left(x/\ell\right)\underset{\sim}{i} - \left(y/\ell\right)\underset{\sim}{j}\right) \cdot \partial\left(\dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}\right) \Big/ \partial \dot{x} = -T\left(x/\ell\right)}$$

$$\boxed{\left(F_y\right)_{\text{constraint}} = \underset{\sim}{T} \cdot \left(\partial \underset{\sim}{v} / \partial \dot{y}\right) = T\left(-\left(x/\ell\right)\underset{\sim}{i} - \left(y/\ell\right)\underset{\sim}{j}\right) \cdot \partial\left(\dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}\right) \Big/ \partial \dot{y} = -T\left(y/\ell\right)} \tag{6}$$

- Substituting into Lagrange's equations (4) and supplementing with the twice differentiated constraint equation gives the following equations of motion.

$$\boxed{\begin{aligned} m\ddot{x} + \left(\tfrac{x}{\ell}\right)T &= 0 \\ m\ddot{y} - mg + \left(\tfrac{y}{\ell}\right)T &= 0 \\ x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 &= 0 \end{aligned}} \tag{7}$$

- Using ***Lagrange multipliers***, it was shown in previous notes that the equations for the pendulum could be written as

$$\boxed{\begin{aligned} m\ddot{x} - \lambda x &= 0 \\ m\ddot{y} - mg - \lambda y &= 0 \\ x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 &= 0 \end{aligned}} \tag{8}$$

- Comparing Eqs. (7) and (8), it is clear that the ***Lagrange multiplier*** $\lambda = -T/\ell$, the tension force per unit pendulum length.