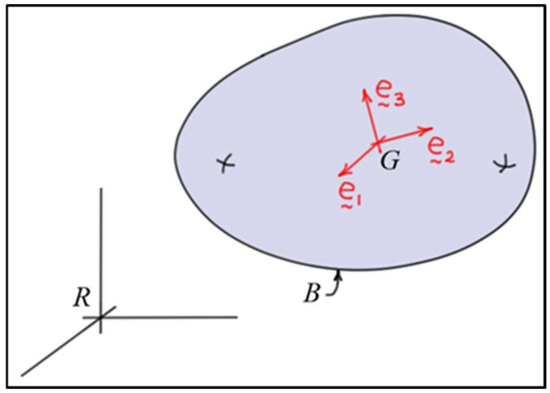
Multibody Dynamics
Examples using Kane's Equations

Examples

1. **Unconstrained Motion of a Rigid Body:** Find the equations of motion of a rigid body using Kane's equations. Use Cartesian coordinates to define the position of the mass center G and ***Euler parameters*** to define the orientation of the body. So, the vector of seven generalized coordinates is defined as follows.



$$\left[q_1,q_2,q_3,q_4,q_5,q_6,q_7\right] \triangleq \left[x_G,y_G,z_G,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\right]$$

Use the following independent set of generalized speeds.

$$\left[u_1,u_2,u_3,u_4,u_5,u_6\right] \triangleq \left[v_1',v_2',v_3',\omega_1',\omega_2',\omega_3'\right] \qquad (1)$$

Here, $v_i'\ (i=1,2,3)$ represent the ***body-fixed***, mass-center velocity components, and $\omega_i'\ (i=1,2,3)$ represent the ***body-fixed*** angular velocity components.

Solution:

Letting the $\underset{\sim}{e}_i \quad (i=1,2,3)$ represent the ***principal directions*** for the mass-center G, and using ***body-fixed*** components of ${}^R\underset{\sim}{v}_G$ and ${}^R\underset{\sim}{\omega}_B$, write

$$\boxed{{}^R\underset{\sim}{v}_G=\sum_{i=1}^{3}v_i'\,\underset{\sim}{e}_i} \qquad \boxed{{}^R\underset{\sim}{\omega}_B=\sum_{i=1}^{3}\omega_i'\,\underset{\sim}{e}_i}$$

Using these equations, the partial velocity and partial angular velocity vectors can be written as follows for $i=1,2,3$.

$$\partial\,{}^R\underset{\sim}{v}_G\big/\partial v_i'=\underset{\sim}{e}_i \qquad \partial\,{}^R\underset{\sim}{v}_G\big/\partial\omega_i'=0 \qquad \partial\,{}^R\underset{\sim}{\omega}_B\big/\partial\omega_i'=\underset{\sim}{e}_i \qquad \partial\,{}^R\underset{\sim}{\omega}_B\big/\partial v_i'=0$$

The acceleration of G can be found by differentiating ${}^R\underset{\sim}{v}_G$ using the derivative rule.

$$\begin{aligned}\underset{\sim}{a}_G &\triangleq \sum_{i=1}^{3}a_i'\,\underset{\sim}{e}_i=\frac{{}^Bd}{dt}\left(\underset{\sim}{v}_G\right)+\left({}^R\underset{\sim}{\omega}_B\times\underset{\sim}{v}_G\right)\\ &=\left(\dot v_1'+\omega_2'v_3'-\omega_3'v_2'\right)\underset{\sim}{e}_1+\left(\dot v_2'+\omega_3'v_1'-\omega_1'v_3'\right)\underset{\sim}{e}_2+\left(\dot v_3'+\omega_1'v_2'-\omega_2'v_1'\right)\underset{\sim}{e}_3\end{aligned}$$

Terms in the Equations of Motion:

$$m\,{}^R\underset{\sim}{a}_G\cdot\left(\partial\,{}^R\underset{\sim}{v}_G\big/\partial v_i'\right)=m\,a_i' \qquad \left(\underset{\approx}{I}_G\cdot{}^R\underset{\sim}{\alpha}_B\right)\cdot\partial\,{}^R\underset{\sim}{\omega}_B\big/\partial\omega_i'=\left(\sum_{j=1}^{3}I_j\dot\omega_j'\,\underset{\sim}{e}_j\right)\cdot\underset{\sim}{e}_i=I_i\dot\omega_i'$$

$$\left({}^{R}\underset{\sim}{\omega}_{B}\times\underset{\sim}{H}_{G}\right)\cdot\partial\,{}^{R}\underset{\sim}{\omega}_{B}\big/\partial\omega_i' = \left((I_3-I_2)\omega_2'\omega_3'\,\underset{\sim}{e}_1+(I_1-I_3)\omega_1'\omega_3'\,\underset{\sim}{e}_2+(I_2-I_1)\omega_1'\omega_2'\,\underset{\sim}{e}_3\right)\cdot\underset{\sim}{e}_i$$

$$=\begin{cases}(I_3-I_2)\omega_2'\omega_3' & (i=1)\\ (I_1-I_3)\omega_1'\omega_3' & (i=2)\\ (I_2-I_1)\omega_1'\omega_2' & (i=3)\end{cases}$$

<u>Equations of Motion</u>:

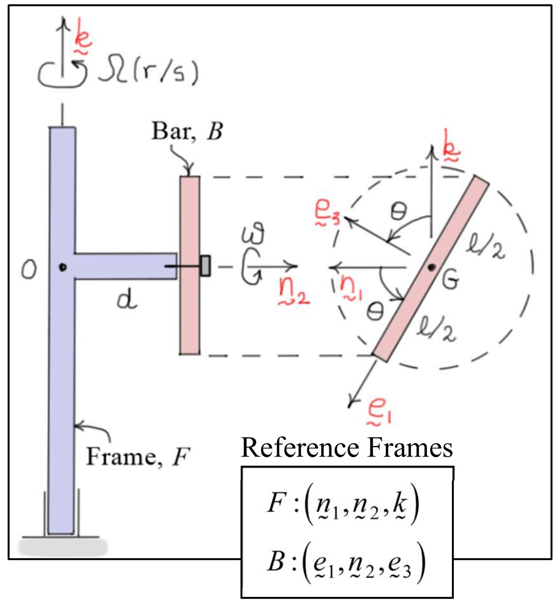
$$\boxed{\begin{aligned} m\,(\dot{v}_1'+\omega_2'v_3'-\omega_3'v_2') &= F_{v_1'}\\ m\,(\dot{v}_2'+\omega_3'v_1'-\omega_1'v_3') &= F_{v_2'}\\ m\,(\dot{v}_3'+\omega_1'v_2'-\omega_2'v_1') &= F_{v_3'}\end{aligned}} \quad \text{and} \quad \boxed{\begin{aligned} I_1\dot{\omega}_1'+(I_3-I_2)\omega_2'\omega_3' &= F_{\omega_1}\\ I_2\dot{\omega}_2'+(I_1-I_3)\omega_1'\omega_3' &= F_{\omega_2}\\ I_3\dot{\omega}_3'+(I_2-I_1)\omega_1'\omega_2' &= F_{\omega_3}\end{aligned}} \qquad (2)$$

Eqs. (2) are now ***supplemented*** with the ***kinematic differential equations***.

$$\begin{Bmatrix}\dot{\varepsilon}_1\\ \dot{\varepsilon}_2\\ \dot{\varepsilon}_3\\ \dot{\varepsilon}_4\end{Bmatrix}=\frac{1}{2}\begin{bmatrix}\varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & \varepsilon_1\\ \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & \varepsilon_2\\ -\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3\\ -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & \varepsilon_4\end{bmatrix}\begin{Bmatrix}\omega_1'\\ \omega_2'\\ \omega_3'\\ 0\end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix}\dot{x}_G\\ \dot{y}_G\\ \dot{z}_G\end{Bmatrix}=\left[R\right]^T\begin{Bmatrix}v_1'\\ v_2'\\ v_3'\end{Bmatrix} \qquad (3)$$

Together, Eqs. (2)-(3) represent a ***set*** of ***thirteen first-order***, ***ordinary differential equations*** for the ***four*** Euler parameters, ***three*** mass-center position coordinates, and the ***six*** generalized speeds defined by Eq. (1). Note the matrix $\left[R\right]^T$ converts vector components from the body frame into the base frame R.

2. **Example System II** (from Intermediate Dynamics): Find the equations of motion of the bar using Kane's equations, given that the frame F is light (massless), the bar B has mass m and length ℓ, the motor torque $M_\theta(t)$ is applied between the frame and the bar, and that the motor torque $M_\phi(t)$ is applied between the ground and the frame. Use $(u_1,u_2)=(v_G,\omega_2')$, where $v_G=-\underset{\sim}{n}_1\cdot\underset{\sim}{v}_G$ and $\omega_2'=\dot{\theta}=\underset{\sim}{\omega}_B\cdot\underset{\sim}{n}_2$ as the two ***independent generalized speeds***.



<u>Solution</u>: Using Kane's Equations

$$\boxed{\begin{aligned}
&\left(m_B\,\underset{\sim}{a}_{G_B}\cdot\frac{\partial \underset{\sim}{v}_{G_B}}{\partial v_G}\right)+\left[\left(\underset{\approx}{I}_{G_B}\cdot\underset{\sim}{\alpha}_B\right)+\left(\underset{\sim}{\omega}_B\times\underset{\sim}{H}_{G_B}\right)\right]\cdot\frac{\partial\underset{\sim}{\omega}_B}{\partial v_G}=F_{v_G}\\
&\left(m_B\,\underset{\sim}{a}_{G_B}\cdot\frac{\partial \underset{\sim}{v}_{G_B}}{\partial \omega_2'}\right)+\left[\left(\underset{\approx}{I}_{G_B}\cdot\underset{\sim}{\alpha}_B\right)+\left(\underset{\sim}{\omega}_B\times\underset{\sim}{H}_{G_B}\right)\right]\cdot\frac{\partial\underset{\sim}{\omega}_B}{\partial \omega_2'}=F_{\omega_2'}
\end{aligned}} \tag{4}$$

Here,

$\underset{\sim}{\omega}_F=(v_G/d)\underset{\sim}{k}$ $\qquad$ $\partial\underset{\sim}{\omega}_F/\partial v_G=\underset{\sim}{k}/d$ $\qquad$ $\partial\underset{\sim}{\omega}_F/\partial\omega_2'=\underset{\sim}{0}$

$\underset{\sim}{\omega}_B=\omega_2'\,\underset{\sim}{n}_2+(v_G/d)\underset{\sim}{k}$ $\qquad$ $\partial\underset{\sim}{\omega}_B/\partial v_G=\underset{\sim}{k}/d$ $\qquad$ $\partial\underset{\sim}{\omega}_B/\partial\omega_2'=\underset{\sim}{n}_2=\underset{\sim}{e}_2$

$\underset{\sim}{v}_G=-v_G\,\underset{\sim}{n}_1$ $\qquad$ $\partial\underset{\sim}{v}_G/\partial v_G=-\underset{\sim}{n}_1$ $\qquad$ $\partial\underset{\sim}{v}_G/\partial\omega_2'=\underset{\sim}{0}$

$\underset{\sim}{a}_G=-\dot{v}_G\,\underset{\sim}{n}_1-(v_G^2/d)\underset{\sim}{n}_2$ $\qquad$ (normal and tangential components)

$\underset{\sim}{\alpha}_F=(\dot{v}_G/d)\underset{\sim}{k}$

$\underset{\sim}{\alpha}_B=-\frac{1}{d}(\dot{v}_G S_\theta+v_G\omega_2' C_\theta)\underset{\sim}{e}_1+\dot{\omega}_2'\,\underset{\sim}{e}_2+\frac{1}{d}(\dot{v}_G C_\theta-v_G\omega_2' S_\theta)\underset{\sim}{e}_3$

$m\,\underset{\sim}{a}_G\cdot(\partial\underset{\sim}{v}_G/\partial v_G)=m\,\dot{v}_G$ $\qquad$ $m\,\underset{\sim}{a}_G\cdot(\partial\,\underset{\sim}{v}_G/\partial\omega_2')=0$

$\underset{\sim}{H}_G=\frac{1}{12}m\,\ell^2(\omega_2'\,\underset{\sim}{e}_2+(v_G C_\theta/d)\underset{\sim}{e}_3)$

$\underset{\approx}{I}_G\cdot\underset{\sim}{\alpha}_B=\frac{1}{12}m\,\ell^2\left(\dot{\omega}_2'\,\underset{\sim}{e}_2+\frac{1}{d}(\dot{v}_G C_\theta-v_G\omega_2' S_\theta)\underset{\sim}{e}_3\right)$

$\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G=\frac{1}{12}m\,\ell^2\left((v_G^2 S_\theta C_\theta/d^2)\underset{\sim}{e}_2-(v_G\omega_2'\,S_\theta/d)\underset{\sim}{e}_3\right)$

$\left(\underset{\approx}{I}_G\cdot\underset{\sim}{\alpha}_B\right)\cdot\left(\partial\underset{\sim}{\omega}_B/\partial v_G\right)=\frac{m\,\ell^2}{12d^2}(\dot{v}_G C_\theta-v_G\,\omega_2'\,S_\theta)C_\theta$

$\left(\underset{\approx}{I}_G\cdot\underset{\sim}{\alpha}_B\right)\cdot\left(\partial\underset{\sim}{\omega}_B/\partial\omega_2'\right)=\frac{1}{12}m\,\ell^2\dot{\omega}_2'$

$\left(\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G\right)\cdot\left(\partial\underset{\sim}{\omega}_B/\partial v_G\right)=-\frac{m\,\ell^2}{12d^2}v_G\,\omega_2'\,S_\theta C_\theta$

$\left(\underset{\sim}{\omega}_B\times\underset{\sim}{H}_G\right)\cdot\left(\partial\underset{\sim}{\omega}_B/\partial\omega_2'\right)=\frac{m\,\ell^2}{12d^2}v_G^2\,S_\theta\,C_\theta$

$$F_{v_G} = \left(M_\theta \underset{\sim}{n}_2\right)\cdot\left(\partial \underset{\sim}{\omega}_B / \partial v_G\right) + \left(-M_\theta \underset{\sim}{n}_2\right)\cdot\left(\partial \underset{\sim}{\omega}_F / \partial v_G\right) + \left(M_\phi \underset{\sim}{k}\right)\cdot\left(\partial \underset{\sim}{\omega}_F / \partial v_G\right) \;=\; M_\phi / d$$

$$F_{\omega_2'} = \left(M_\theta \underset{\sim}{n}_2\right)\cdot\left(\partial \underset{\sim}{\omega}_B / \partial \omega_2'\right) + \left(-M_\theta \underset{\sim}{n}_2\right)\cdot\left(\partial \underset{\sim}{\omega}_F / \partial \omega_2'\right) + \left(M_\phi \underset{\sim}{k}\right)\cdot\left(\partial \underset{\sim}{\omega}_F / \partial \omega_2'\right) \;=\; M_\theta$$

Substituting into Kane's equations gives the two differential equations of motion gives

$$\boxed{\begin{aligned} \left(md + \tfrac{mL^2}{12d} C_\theta^2\right)\dot{v}_G - \left(\tfrac{mL^2}{6d} S_\theta C_\theta\right) v_G \omega_2' &= M_\phi \\ \left(\tfrac{mL^2}{12}\right)\dot{\omega}_2' + \left(\tfrac{mL^2}{12d^2} S_\theta C_\theta\right) v_G^2 &= M_\theta \end{aligned}} \tag{5}$$

Eqs. (5) are now ***supplemented*** with the ***kinematic differential equations***.

$$\boxed{\dot{\theta} = \omega_2'} \qquad \text{and} \qquad \boxed{\dot{\phi} = v_G / d} \tag{6}$$

Eqs. (5) and (6) represent four, first-order ordinary differential equations in the variables θ, ϕ, v_G, and ω_2'.