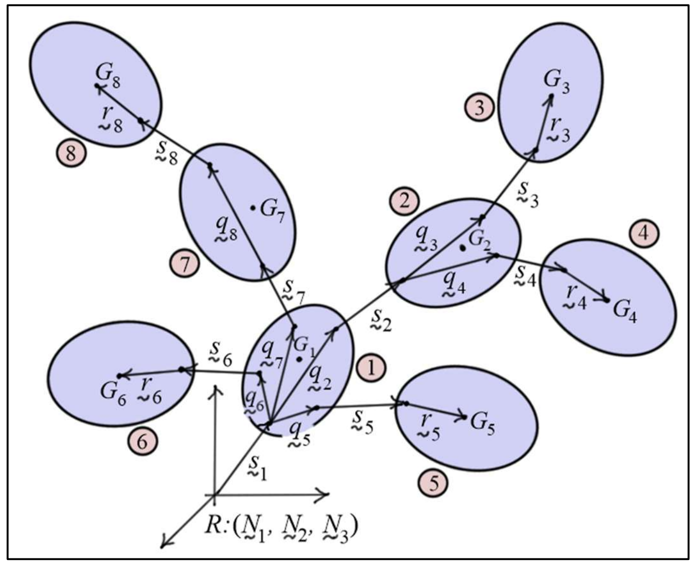
Multibody Dynamics
Equations of Motion for Unconstrained Multibody Systems

The ***explicit form*** of the equations of motion of a multibody system depends on:

- choice of generalized coordinates
- choice of generalized speeds
- method used to formulate equations
- constraints on system motion

In the notes that follow, ***Kane's equations*** are used to ***derive*** the ***equations of motion*** of ***unconstrained*** multibody systems using a ***combination*** of ***absolute*** and ***relative coordinates***. The choice of generalized coordinates and speeds are given below.



Multibody System with Eight Bodies

Body Numbering and the Body-Connection Array

As discussed in previous notes, the development of the kinematic and dynamic equations of motion of a multibody system can be ***structured*** using a ***body-connection array***. In the system shown above, the bodies are ***numbered*** by first choosing a ***reference body*** for the system and naming it body 1. Then, the remaining bodies are named in ***ascending progression*** away from body 1 through the branches of the system as shown in the diagram. The branches can be chosen in any order so long as the numbering progression is outward away from the system's reference body.

The body-connection array $\mathscr{L}(K)$ is formed by ***identifying***, for each body, its ***adjoining lower-numbered body*** which is one body ***closer*** to the ***system reference body***. For example, the lower-numbered body of body 8 is body 7, and the lower-numbered body of body 7 is the system reference body 1. The ***body-connection array*** for the example system above is $\boxed{\mathscr{L}(K) = \left(0,1,2,2,1,1,1,7\right)}$. Here, ***zero*** has been used to denote the ***inertial reference frame***.

The body connection array can be used to find the integers $u_K\ (K = 1, \ldots N)$ which represent the ***number of bodies below*** each body in the system. This is done by finding the integer u_K such that $\mathscr{L}^{u_K}(K) = 1$. For example, $\mathscr{L}^2\left(3\right) = \mathscr{L}\left(\mathscr{L}(3)\right) = \mathscr{L}\left(2\right) = 1$, so $u_3 = 2$. For the example eight body system, $u_K\ (K = 1, \ldots, 8) = \left(0,1,2,2,1,1,1,2\right)$.

Generalized Coordinates and Speeds (system with " N " bodies)

- ***Euler parameters*** ε_{Ki} $(K=1,\ldots,N;\ i=1,2,3,4)$ are used to measure the ***orientations*** of the bodies ***relative*** to the ***inertial frame*** $R(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3)$.
- ***Translation variables*** s'_{Ki} $(K=1,\ldots,N;\ i=1,2,3)$ are used to measure ***displacements*** of the bodies ***relative*** to their ***adjacent, lower-numbered bodies*** $(\mathfrak{L}(K))$. These variables represent the ***lower-body frame components*** of the translation vectors of the bodies $\left(\underset{\sim}{s}_K\ (K=1,\ldots N)\right)$.
- ***Absolute angular velocity components*** ω'_{Ki} $(K=1,\ldots,N;\ i=1,2,3)$ are used to measure the angular velocities of the bodies ***relative*** to the ***inertial frame*** $R(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3)$. These are the ***body-frame components*** of the angular velocity vectors of the bodies $\left({}^R\underset{\sim}{\omega}_K\right)$.

As described, there are " $7N$ " generalized coordinates, ε_{Ki} $(K=1,\ldots,N;\ i=1,2,3,4)$ and s'_{Ki} $(K=1,\ldots,N;\ i=1,2,3)$. To avoid the use of Lagrange multipliers, the " $6N$ " generalized speeds are defined to be ω'_{Ki} $(K=1,\ldots,N;\ i=1,2,3)$ and $\dot{s}'_{Ki}$ $(K=1,\ldots,N;\ i=1,2,3)$.

System State Vectors

Using the ***generalized coordinates*** and ***speeds*** defined above, the following ***system state vectors*** can be defined

$$\{\varepsilon\}_{4N\times 1}=[\varepsilon_{11},\varepsilon_{12},\varepsilon_{13},\varepsilon_{14},\ldots,\underbrace{\varepsilon_{K1},\varepsilon_{K2},\varepsilon_{K3},\varepsilon_{K4}}_{\{\varepsilon_K\}^T},\ldots,\varepsilon_{N1},\varepsilon_{N2},\varepsilon_{N3},\varepsilon_{N4}]^T$$

$$\{s'\}_{3N\times 1}=[s'_{11},s'_{12},s'_{13},\ldots,\underbrace{s'_{K1},s'_{K2},s'_{K3}}_{\{s'_K\}^T},\ldots,s'_{N1},s'_{N2},s'_{N3}]^T$$

$$\{\omega'\}_{3N\times 1}=[\omega'_{11},\omega'_{12},\omega'_{13},\ldots,\underbrace{\omega'_{K1},\omega'_{K2},\omega'_{K3}}_{\{{}^R\omega'_K\}^T},\ldots,\omega'_{N1},\omega'_{N2},\omega'_{N3}]^T$$

and

$$\{x\}_{7N\times 1}=\begin{Bmatrix}\{x_1\}\\ \{x_2\}\end{Bmatrix}=\begin{Bmatrix}\{\varepsilon\}_{4N\times 1}\\ \{s'\}_{3N\times 1}\end{Bmatrix}\qquad \{y\}_{6N\times 1}=\begin{Bmatrix}\{y_1\}\\ \{y_2\}\end{Bmatrix}=\begin{Bmatrix}\{\omega'\}_{3N\times 1}\\ \{\dot{s}'\}_{3N\times 1}\end{Bmatrix}\tag{1}$$

Transformation Matrices

The unit vectors $(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$ of body K can be written in terms of the inertial-frame unit vectors $(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3)$ using the coordinate transformation matrix $[R_K]$ of body K.

$$\boxed{\begin{Bmatrix}\underset{\sim}{e}_1\\ \underset{\sim}{e}_2\\ \underset{\sim}{e}_3\end{Bmatrix}=[R_K]\begin{Bmatrix}\underset{\sim}{N}_1\\ \underset{\sim}{N}_2\\ \underset{\sim}{N}_3\end{Bmatrix}}\tag{2}$$

Transformation matrix $[R_K]$ can be written in terms of the Euler parameters associated with body K as follows.

$$\left[R_K\right]=\begin{bmatrix}(\varepsilon_{K1}^2-\varepsilon_{K2}^2-\varepsilon_{K3}^2+\varepsilon_{K4}^2) & 2(\varepsilon_{K1}\varepsilon_{K2}+\varepsilon_{K3}\varepsilon_{K4}) & 2(\varepsilon_{K1}\varepsilon_{K3}-\varepsilon_{K2}\varepsilon_{K4}) \\ 2(\varepsilon_{K1}\varepsilon_{K2}-\varepsilon_{K3}\varepsilon_{K4}) & (-\varepsilon_{K1}^2+\varepsilon_{K2}^2-\varepsilon_{K3}^2+\varepsilon_{K4}^2) & 2(\varepsilon_{K2}\varepsilon_{K3}+\varepsilon_{K1}\varepsilon_{K4}) \\ 2(\varepsilon_{K1}\varepsilon_{K3}+\varepsilon_{K2}\varepsilon_{K4}) & 2(\varepsilon_{K2}\varepsilon_{K3}-\varepsilon_{K1}\varepsilon_{K4}) & (-\varepsilon_{K1}^2-\varepsilon_{K2}^2+\varepsilon_{K3}^2+\varepsilon_{K4}^2)\end{bmatrix} \tag{3}$$

Time Derivatives of the Transformation Matrices

In previous notes, the ***time derivatives*** of the ***transformation matrices*** from the bodies to the ***inertial frame*** were written in terms of ***skew-symmetric matrices*** formed from the components of the ***angular velocities*** of the bodies ***relative*** to the ***fixed frame***. Recall that the "prime" indicates the use of ***body-frame components***.

$$\left[\dot{R}_K\right]=\left[{}^R\tilde{\omega}'_K\right]^T\left[R_K\right] \quad \text{(components of } {}^R\underset{\sim}{\omega}_K \text{ are resolved in body } K\text{)} \tag{4}$$

Angular Velocity Components and Euler Parameters

The ***angular velocity components*** of body K can be written in terms of its Euler parameters as follows

$$\begin{Bmatrix}\omega'_{K1}\\ \omega'_{K2}\\ \omega'_{K3}\\ 0\end{Bmatrix}=2\begin{bmatrix}\varepsilon_{K4} & \varepsilon_{K3} & -\varepsilon_{K2} & -\varepsilon_{K1}\\ -\varepsilon_{K3} & \varepsilon_{K4} & \varepsilon_{K1} & -\varepsilon_{K2}\\ \varepsilon_{K2} & -\varepsilon_{K1} & \varepsilon_{K4} & -\varepsilon_{K3}\\ \varepsilon_{K1} & \varepsilon_{K2} & \varepsilon_{K3} & \varepsilon_{K4}\end{bmatrix}\begin{Bmatrix}\dot{\varepsilon}_{K1}\\ \dot{\varepsilon}_{K2}\\ \dot{\varepsilon}_{K3}\\ \dot{\varepsilon}_{K4}\end{Bmatrix}\triangleq 2\left[E'_K\right]\left\{\dot{\varepsilon}_K\right\} \tag{5}$$

Note the last equation is simply the ***derivative*** of the Euler parameter ***constraint equation***, $\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2=1$. The matrix $\left[E'_K\right]$ is an ***orthogonal matrix***, so the above equation can be easily ***inverted*** to give

$$\left\{\dot{\varepsilon}_K\right\}=\tfrac{1}{2}\left[E'_K\right]^T\left\{\omega'_K\right\}_{4\times1} \tag{6}$$

The subscript "4×1" has been added as a ***reminder*** that ***zero*** has been added as the ***fourth element*** of the angular velocity column vector. In practice, the fourth column of $\left[E'_K\right]^T$ and the fourth row of $\left\{\omega'_K\right\}_{4\times1}$ can be eliminated (because the fourth element of $\left\{\omega'_K\right\}_{4\times1}$ is zero).

Angular Velocity and Partial Angular Velocity

The ***body-fixed components*** of the ***angular velocities*** of the bodies can be written in terms of the ***partial angular velocity matrices*** in the partitioned form

$$\begin{aligned}\left\{{}^R\omega'_K\right\}&=\left[{}^R\omega'_{K,y}\right]_{3\times6N}\left\{y\right\}_{6N\times1}=\left[\left[{}^R\omega'_{K,y_1}\right]_{3\times3N}\quad\left[{}^R\omega'_{K,y_2}\right]_{3\times3N}\right]\begin{Bmatrix}\left\{y_1\right\}_{3N\times1}\\ \left\{y_2\right\}_{3N\times1}\end{Bmatrix}\\ &=\left[{}^R\omega'_{K,y_1}\right]_{3\times3N}\left\{y_1\right\}_{3N\times1}\end{aligned} \tag{7}$$

Here,

$$\left[{}^R\omega'_{K,y_1}\right]_{3\times3N}=\Big[\underbrace{[0]}_{1},\ldots,\underbrace{[0]}_{K-1},\underbrace{[I]}_{K},\underbrace{[0]}_{K+1},\ldots,\underbrace{[0]}_{N}\Big]_{3\times3N} \quad \text{and} \quad \left[{}^R\omega'_{K,y_2}\right]_{3\times3N}=\left[0\right]_{3\times3N} \tag{8}$$

Here $[I]$ represents the 3×3 identity matrix, $[0]$ represents the 3×3 zero matrix, and $\left[0\right]_{3\times 3N}$ represents a $3\times 3N$ zero matrix.

Angular Acceleration

The ***angular accelerations*** of the bodies are found by ***differentiating*** the angular velocities either in the ***inertial frame*** or in the ***body frame***.

$$\boxed{{}^R\underset{\sim}{\alpha}_K = \frac{{}^R d}{dt}\left({}^R\underset{\sim}{\omega}_K\right) = \frac{{}^K d}{dt}\left({}^R\underset{\sim}{\omega}_K\right)} \tag{9}$$

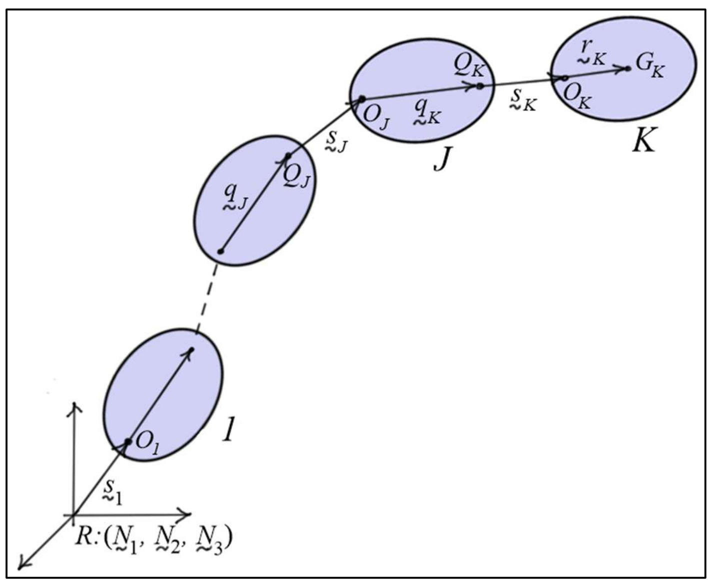
The ***body-frame components*** of ${}^R\underset{\sim}{\alpha}_K$ the ***angular acceleration*** of body K are found by ***differentiating*** the body-frame components of the angular velocity of body K in Eq. (7).

$$\boxed{\left\{{}^R\alpha'_K\right\} = \left\{{}^R\dot{\omega}'_K\right\} = \left[{}^R\omega'_{K,y}\right]\left\{\dot{y}\right\} + \underbrace{\left[{}^R\dot{\omega}'_{K,y}\right]}_{\text{zero}}\left\{y\right\} = \left[{}^R\omega'_{K,y}\right]\left\{\dot{y}\right\} = \left[{}^R\omega'_{K,y_1}\right]\left\{\dot{y}_1\right\}} \tag{10}$$

As noted in Eq. (10), the ***time derivatives*** of the ***partial angular velocity*** matrices are all ***zero***.

Mass-Center Position Vectors

Consider a ***typical branch*** of a multibody system as shown in the diagram. Each body K has a ***mass-center*** G_K, an ***origin*** O_K, and a ***reference point*** Q_K. The points G_K and O_K are fixed in body K, and the reference point Q_K is fixed in the adjacent, lower-numbered body J $\left(J = \mathscr{L}(K)\right)$. The point O_K is positioned relative to O_J the origin of body J using the position vectors $\underset{\sim}{q}_K$ and $\underset{\sim}{s}_K$.



Given that the vectors $\underset{\sim}{s}_K$ $(K = 1,\ldots,N)$ define the ***relative position*** between adjoining bodies, it is helpful to develop ***recursive relationships*** for calculating the positions, velocities, partial velocities, and accelerations of the mass-centers of the bodies. To this end, the position vector of O_K the origin of body K relative to the inertial system can be written as

$$\boxed{\underset{\sim}{p}_{O_K} = \underset{\sim}{p}_{O_J} + \underset{\sim}{q}_K + \underset{\sim}{s}_K} \quad \left(K = 1,\ldots,N\right) \tag{11}$$

Given that $\underset{\sim}{p}_{O_1} = \underset{\sim}{s}_1$ $\left(\underset{\sim}{q}_1 \triangleq \underset{\sim}{0}\right)$, Eq. (11) is a ***recursive relationship*** that can be used to build the position vectors of the origins of all the bodies of the system.

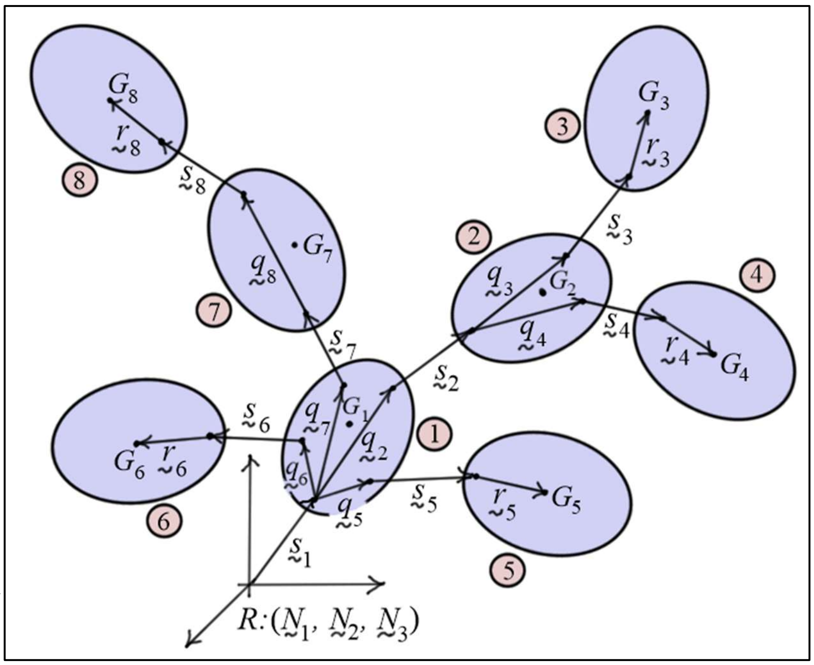
Resolving the components of $\underset{\sim}{p}_{O_K}$ and $\underset{\sim}{p}_{O_J}$ in the inertial system, and the components of $\underset{\sim}{q}_K$ and $\underset{\sim}{s}_K$ in body $J = \mathfrak{L}(K)$, Eq. (11) can be written in matrix form as follows.

$$\boxed{\left\{p_{O_K}\right\} = \left\{p_{O_J}\right\} + \left[R_J\right]^T \left(\left\{q'_K\right\} + \left\{s'_K\right\}\right)} \quad \text{(inertial components)} \qquad (12)$$

The matrix $\left[R_J\right]^T$ transforms the body J components of $\underset{\sim}{q}_K$ and $\underset{\sim}{s}_K$ into the inertial frame. Finally, resolving the components of $\underset{\sim}{r}_K$ in body K, the inertial components of the position vector of G_K can be written as

$$\boxed{\left\{p_{G_K}\right\} = \left\{p_{O_K}\right\} + \left[R_K\right]^T \left\{r'_K\right\} = \left\{p_{O_J}\right\} + \left[R_J\right]^T \left(\left\{q'_K\right\} + \left\{s'_K\right\}\right) + \left[R_K\right]^T \left\{r'_K\right\}} \qquad (13)$$

Consider now the eight-body example system. Using Eqs. (12) and (13), the ***position vectors*** of the body ***origins*** and ***mass-centers*** can be written as follows.



$$\boxed{\left\{p_{O_1}\right\} = \left\{s'_1\right\}}$$

$$\boxed{\begin{aligned}\left\{p_{G_1}\right\} &= \left\{p_{O_1}\right\} + \left[R_1\right]^T \left\{r'_1\right\} \\ &= \left\{s'_1\right\} + \left[R_1\right]^T \left\{r'_1\right\}\end{aligned}}$$

$$\boxed{\begin{aligned}\left\{p_{O_2}\right\} &= \left\{p_{O_1}\right\} + \left[R_1\right]^T \left(\left\{q'_2\right\} + \left\{s'_2\right\}\right) \\ &= \left\{s'_1\right\} + \left[R_1\right]^T \left(\left\{q'_2\right\} + \left\{s'_2\right\}\right)\end{aligned}}$$

$$\boxed{\begin{aligned}\left\{p_{G_2}\right\} &= \left\{p_{O_2}\right\} + \left[R_2\right]^T \left\{r'_2\right\} \\ &= \left\{s'_1\right\} + \left[R_1\right]^T \left(\left\{q'_2\right\} + \left\{s'_2\right\}\right) + \left[R_2\right]^T \left\{r'_2\right\}\end{aligned}}$$

$$\boxed{\begin{aligned}\left\{p_{O_3}\right\} &= \left\{p_{O_2}\right\} + \left[R_2\right]^T \left(\left\{q'_3\right\} + \left\{s'_3\right\}\right) \\ &= \left\{s'_1\right\} + \left[R_1\right]^T \left(\left\{q'_2\right\} + \left\{s'_2\right\}\right) + \left[R_2\right]^T \left(\left\{q'_3\right\} + \left\{s'_3\right\}\right)\end{aligned}}$$

$$\boxed{\begin{aligned}\left\{p_{G_3}\right\} &= \left\{p_{O_3}\right\} + \left[R_3\right]^T \left\{r'_3\right\} \\ &= \left\{s'_1\right\} + \left[R_1\right]^T \left(\left\{q'_2\right\} + \left\{s'_2\right\}\right) + \left[R_2\right]^T \left(\left\{q'_3\right\} + \left\{s'_3\right\}\right) + \left[R_3\right]^T \left\{r'_3\right\}\end{aligned}}$$

$$\boxed{\begin{aligned}\left\{p_{O_4}\right\} &= \left\{p_{O_2}\right\} + \left[R_2\right]^T \left(\left\{q'_4\right\} + \left\{s'_4\right\}\right) \\ &= \left\{s'_1\right\} + \left[R_1\right]^T \left(\left\{q'_2\right\} + \left\{s'_2\right\}\right) + \left[R_2\right]^T \left(\left\{q'_4\right\} + \left\{s'_4\right\}\right)\end{aligned}}$$

$$\boxed{\begin{aligned}\left\{p_{G_4}\right\} &= \left\{p_{O_4}\right\} + \left[R_4\right]^T \left\{r'_4\right\} \\ &= \left\{s'_1\right\} + \left[R_1\right]^T \left(\left\{q'_2\right\} + \left\{s'_2\right\}\right) + \left[R_2\right]^T \left(\left\{q'_4\right\} + \left\{s'_4\right\}\right) + \left[R_4\right]^T \left\{r'_4\right\}\end{aligned}}$$

$$\begin{aligned}\left\{p_{O_5}\right\} &= \left\{p_{O_1}\right\} + \left[R_1\right]^T \left(\left\{q_5'\right\} + \left\{s_5'\right\}\right) \\ &= \left\{s_1'\right\} + \left[R_1\right]^T \left(\left\{q_5'\right\} + \left\{s_5'\right\}\right)\end{aligned}$$

$$\begin{aligned}\left\{p_{G_5}\right\} &= \left\{p_{O_5}\right\} + \left[R_5\right]^T \left\{r_5'\right\} \\ &= \left\{s_1'\right\} + \left[R_1\right]^T \left(\left\{q_5'\right\} + \left\{s_5'\right\}\right) + \left[R_5\right]^T \left\{r_5'\right\}\end{aligned}$$

$$\begin{aligned}\left\{p_{O_6}\right\} &= \left\{p_{O_1}\right\} + \left[R_1\right]^T \left(\left\{q_6'\right\} + \left\{s_6'\right\}\right) \\ &= \left\{s_1'\right\} + \left[R_1\right]^T \left(\left\{q_6'\right\} + \left\{s_6'\right\}\right)\end{aligned}$$

$$\begin{aligned}\left\{p_{G_6}\right\} &= \left\{p_{O_6}\right\} + \left[R_6\right]^T \left\{r_6'\right\} \\ &= \left\{s_1'\right\} + \left[R_1\right]^T \left(\left\{q_6'\right\} + \left\{s_6'\right\}\right) + \left[R_6\right]^T \left\{r_6'\right\}\end{aligned}$$

$$\begin{aligned}\left\{p_{O_7}\right\} &= \left\{p_{O_1}\right\} + \left[R_1\right]^T \left(\left\{q_7'\right\} + \left\{s_7'\right\}\right) \\ &= \left\{s_1'\right\} + \left[R_1\right]^T \left(\left\{q_7'\right\} + \left\{s_7'\right\}\right)\end{aligned}$$

$$\begin{aligned}\left\{p_{G_7}\right\} &= \left\{p_{O_7}\right\} + \left[R_7\right]^T \left\{r_7'\right\} \\ &= \left\{s_1'\right\} + \left[R_1\right]^T \left(\left\{q_7'\right\} + \left\{s_7'\right\}\right) + \left[R_7\right]^T \left\{r_7'\right\}\end{aligned}$$

$$\begin{aligned}\left\{p_{O_8}\right\} &= \left\{p_{O_7}\right\} + \left[R_7\right]^T \left(\left\{q_8'\right\} + \left\{s_8'\right\}\right) \\ &= \left\{s_1'\right\} + \left[R_1\right]^T \left(\left\{q_7'\right\} + \left\{s_7'\right\}\right) + \left[R_7\right]^T \left(\left\{q_8'\right\} + \left\{s_8'\right\}\right)\end{aligned}$$

$$\begin{aligned}\left\{p_{G_8}\right\} &= \left\{p_{O_7}\right\} + \left[R_7\right]^T \left(\left\{q_8'\right\} + \left\{s_8'\right\}\right) + \left[R_8\right]^T \left\{r_8'\right\} \\ &= \left\{s_1'\right\} + \left[R_1\right]^T \left(\left\{q_7'\right\} + \left\{s_7'\right\}\right) + \left[R_7\right]^T \left(\left\{q_8'\right\} + \left\{s_8'\right\}\right) + \left[R_8\right]^T \left\{r_8'\right\}\end{aligned}$$

Mass-Center Velocities

Recursive relationships can also be developed for the ***mass-center velocities***. This is accomplished by first finding the ***velocities*** of the ***origins*** of the bodies as follows.

$$^{R}\underset{\sim}{v}_{O_K} = \frac{^{R}d\,\underset{\sim}{p}_{O_K}}{dt} = \frac{^{R}d}{dt}\left(\underset{\sim}{p}_{O_J} + \underset{\sim}{q}_K + \underset{\sim}{s}_K\right) = \frac{^{R}d\,\underset{\sim}{p}_{O_J}}{dt} + \frac{^{R}d}{dt}\left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right) = {^{R}\underset{\sim}{v}_{O_J}} + \frac{^{R}d}{dt}\left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right)$$

The last term can be expanded using the ***derivative rule*** (that relates the derivatives of a vector in different reference frames) as follows.

$$\frac{^{R}d}{dt}\left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right) = \frac{^{J}d}{dt}\left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right) + {^{R}\underset{\sim}{\omega}_J} \times \left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right) = \frac{^{J}d}{dt}\left(\underset{\sim}{s}_K\right) + {^{R}\underset{\sim}{\omega}_J} \times \left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right)$$

Combining these two results gives the following.

$$ {}^R\underset{\sim}{v}_{O_K} = {}^R\underset{\sim}{v}_{O_J} + \frac{{}^J d}{dt}\left(\underset{\sim}{s}_K\right) + {}^R\underset{\sim}{\omega}_J \times \left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right) \tag{14} $$

Resolving the velocity components in the inertial frame, Eq. (14) can be written in matrix form as follows.

$$ \left\{{}^R v_{O_K}\right\} = \left\{{}^R v_{O_J}\right\} + \left[R_J\right]^T \left(\left\{\dot{s}'_K\right\} + \left[{}^R\tilde{\omega}'_J\right]\left(\left\{q'_K\right\} + \left\{s'_K\right\}\right)\right) \quad \text{(inertial components)} \tag{15} $$

This result allows the ***velocities*** of the ***origins*** of the bodies to be calculated ***recursively***, starting with the velocity of O_1, the origin of reference body *1*.

Given the velocities of the origins of the bodies, the ***velocities*** of the ***mass-centers*** of the bodies can be calculated as follows.

$$ {}^R\underset{\sim}{v}_{G_K} = {}^R\underset{\sim}{v}_{O_K} + \left({}^R\underset{\sim}{\omega}_K \times \underset{\sim}{r}_K\right) \tag{16} $$

Resolving the components of ${}^R\underset{\sim}{v}_{G_K}$ in the inertial frame, the above equation can be written in matrix form as follows.

$$ \left\{{}^R v_{G_K}\right\} = \left\{{}^R v_{O_K}\right\} + \left[R_K\right]^T \left[{}^R\tilde{\omega}'_K\right]\left\{r'_K\right\} \quad \text{(inertial components)} \tag{17} $$

Applying these results to the ***example eight-body system*** gives the following results.

$$ \left\{{}^R v_{O_1}\right\} = \left\{\dot{s}'_1\right\} $$

$$ \begin{aligned} \left\{{}^R v_{G_1}\right\} &= \left\{{}^R v_{O_1}\right\} + \left[R_1\right]^T \left[{}^R\tilde{\omega}'_1\right]\left\{r'_1\right\} \\ &= \left\{\dot{s}'_1\right\} + \left[R_1\right]^T \left[{}^R\tilde{\omega}'_1\right]\left\{r'_1\right\} \end{aligned} $$

$$ \begin{aligned} \left\{{}^R v_{O_2}\right\} &= \left\{{}^R v_{O_1}\right\} + \left[R_1\right]^T \left(\left\{\dot{s}'_2\right\} + \left[{}^R\tilde{\omega}'_1\right]\left(\left\{q'_2\right\} + \left\{s'_2\right\}\right)\right) \\ &= \left\{\dot{s}'_1\right\} + \left[R_1\right]^T \left(\left\{\dot{s}'_2\right\} + \left[{}^R\tilde{\omega}'_1\right]\left(\left\{q'_2\right\} + \left\{s'_2\right\}\right)\right) \end{aligned} $$

$$ \begin{aligned} \left\{{}^R v_{G_2}\right\} &= \left\{{}^R v_{O_2}\right\} + \left[R_2\right]^T \left[{}^R\tilde{\omega}'_2\right]\left\{r'_2\right\} \\ &= \left\{\dot{s}'_1\right\} + \left[R_1\right]^T \left(\left\{\dot{s}'_2\right\} + \left[{}^R\tilde{\omega}'_1\right]\left(\left\{q'_2\right\} + \left\{s'_2\right\}\right)\right) + \left[R_2\right]^T \left[{}^R\tilde{\omega}'_2\right]\left\{r'_2\right\} \end{aligned} $$

$$ \begin{aligned} \left\{{}^R v_{O_3}\right\} &= \left\{{}^R v_{O_2}\right\} + \left[R_2\right]^T \left(\left\{\dot{s}'_3\right\} + \left[{}^R\tilde{\omega}'_2\right]\left(\left\{q'_3\right\} + \left\{s'_3\right\}\right)\right) \\ &= \left\{\dot{s}'_1\right\} + \left[R_1\right]^T \left(\left\{\dot{s}'_2\right\} + \left[{}^R\tilde{\omega}'_1\right]\left(\left\{q'_2\right\} + \left\{s'_2\right\}\right)\right) + \left[R_2\right]^T \left(\left\{\dot{s}'_3\right\} + \left[{}^R\tilde{\omega}'_2\right]\left(\left\{q'_3\right\} + \left\{s'_3\right\}\right)\right) \end{aligned} $$

$$ \begin{aligned} \left\{{}^R v_{G_3}\right\} &= \left\{{}^R v_{O_3}\right\} + \left[R_3\right]^T \left[{}^R\tilde{\omega}'_3\right]\left\{r'_3\right\} \\ &= \left\{\dot{s}'_1\right\} + \left[R_1\right]^T \left(\left\{\dot{s}'_2\right\} + \left[{}^R\tilde{\omega}'_1\right]\left(\left\{q'_2\right\} + \left\{s'_2\right\}\right)\right) + \left[R_2\right]^T \left(\left\{\dot{s}'_3\right\} + \left[{}^R\tilde{\omega}'_2\right]\left(\left\{q'_3\right\} + \left\{s'_3\right\}\right)\right) + \left[R_3\right]^T \left[{}^R\tilde{\omega}'_3\right]\left\{r'_3\right\} \end{aligned} $$

$$\left\{{}^{R}v_{O_4}\right\} = \left\{{}^{R}v_{O_2}\right\} + \left[R_2\right]^T \left(\left\{\dot{s}'_4\right\} + \left[{}^{R}\tilde{\omega}'_2\right]\left(\left\{q'_4\right\} + \left\{s'_4\right\}\right)\right)$$
$$= \left\{\dot{s}'_1\right\} + \left[R_1\right]^T \left(\left\{\dot{s}'_2\right\} + \left[{}^{R}\tilde{\omega}'_1\right]\left(\left\{q'_2\right\} + \left\{s'_2\right\}\right)\right) + \left[R_2\right]^T \left(\left\{\dot{s}'_4\right\} + \left[{}^{R}\tilde{\omega}'_2\right]\left(\left\{q'_4\right\} + \left\{s'_4\right\}\right)\right)$$

$$\left\{{}^{R}v_{G_4}\right\} = \left\{{}^{R}v_{O_4}\right\} + \left[R_4\right]^T \left[{}^{R}\tilde{\omega}'_4\right]\left\{r'_4\right\}$$
$$= \left\{\dot{s}'_1\right\} + \left[R_1\right]^T \left(\left\{\dot{s}'_2\right\} + \left[{}^{R}\tilde{\omega}'_1\right]\left(\left\{q'_2\right\} + \left\{s'_2\right\}\right)\right) + \left[R_2\right]^T \left(\left\{\dot{s}'_4\right\} + \left[{}^{R}\tilde{\omega}'_2\right]\left(\left\{q'_4\right\} + \left\{s'_4\right\}\right)\right) + \left[R_4\right]^T \left[{}^{R}\tilde{\omega}'_4\right]\left\{r'_4\right\}$$

$$\left\{{}^{R}v_{O_5}\right\} = \left\{{}^{R}v_{O_1}\right\} + \left[R_1\right]^T \left(\left\{\dot{s}'_5\right\} + \left[{}^{R}\tilde{\omega}'_1\right]\left(\left\{q'_5\right\} + \left\{s'_5\right\}\right)\right)$$
$$= \left\{\dot{s}'_1\right\} + \left[R_1\right]^T \left(\left\{\dot{s}'_5\right\} + \left[{}^{R}\tilde{\omega}'_1\right]\left(\left\{q'_5\right\} + \left\{s'_5\right\}\right)\right)$$

$$\left\{{}^{R}v_{G_5}\right\} = \left\{{}^{R}v_{O_5}\right\} + \left[R_5\right]^T \left[{}^{R}\tilde{\omega}'_5\right]\left\{r'_5\right\}$$
$$= \left\{\dot{s}'_1\right\} + \left[R_1\right]^T \left(\left\{\dot{s}'_5\right\} + \left[{}^{R}\tilde{\omega}'_1\right]\left(\left\{q'_5\right\} + \left\{s'_5\right\}\right)\right) + \left[R_5\right]^T \left[{}^{R}\tilde{\omega}'_5\right]\left\{r'_5\right\}$$

$$\left\{{}^{R}v_{O_6}\right\} = \left\{{}^{R}v_{O_1}\right\} + \left[R_1\right]^T \left(\left\{\dot{s}'_6\right\} + \left[{}^{R}\tilde{\omega}'_1\right]\left(\left\{q'_6\right\} + \left\{s'_6\right\}\right)\right)$$
$$= \left\{\dot{s}'_1\right\} + \left[R_1\right]^T \left(\left\{\dot{s}'_6\right\} + \left[{}^{R}\tilde{\omega}'_1\right]\left(\left\{q'_6\right\} + \left\{s'_6\right\}\right)\right)$$

$$\left\{{}^{R}v_{G_6}\right\} = \left\{{}^{R}v_{O_6}\right\} + \left[R_6\right]^T \left[{}^{R}\tilde{\omega}'_6\right]\left\{r'_6\right\}$$
$$= \left\{\dot{s}'_1\right\} + \left[R_1\right]^T \left(\left\{\dot{s}'_6\right\} + \left[{}^{R}\tilde{\omega}'_1\right]\left(\left\{q'_6\right\} + \left\{s'_6\right\}\right)\right) + \left[R_6\right]^T \left[{}^{R}\tilde{\omega}'_6\right]\left\{r'_6\right\}$$

$$\left\{{}^{R}v_{O_7}\right\} = \left\{{}^{R}v_{O_1}\right\} + \left[R_1\right]^T \left(\left\{\dot{s}'_7\right\} + \left[{}^{R}\tilde{\omega}'_1\right]\left(\left\{q'_7\right\} + \left\{s'_7\right\}\right)\right)$$
$$= \left\{\dot{s}'_1\right\} + \left[R_1\right]^T \left(\left\{\dot{s}'_7\right\} + \left[{}^{R}\tilde{\omega}'_1\right]\left(\left\{q'_7\right\} + \left\{s'_7\right\}\right)\right)$$

$$\left\{{}^{R}v_{G_7}\right\} = \left\{{}^{R}v_{O_7}\right\} + \left[R_7\right]^T \left[{}^{R}\tilde{\omega}'_7\right]\left\{r'_7\right\}$$
$$= \left\{\dot{s}'_1\right\} + \left[R_1\right]^T \left(\left\{\dot{s}'_7\right\} + \left[{}^{R}\tilde{\omega}'_1\right]\left(\left\{q'_7\right\} + \left\{s'_7\right\}\right)\right) + \left[R_7\right]^T \left[{}^{R}\tilde{\omega}'_7\right]\left\{r'_7\right\}$$

$$\left\{{}^{R}v_{O_8}\right\} = \left\{{}^{R}v_{O_7}\right\} + \left[R_7\right]^T \left(\left\{\dot{s}'_8\right\} + \left[{}^{R}\tilde{\omega}'_7\right]\left(\left\{q'_8\right\} + \left\{s'_8\right\}\right)\right)$$
$$= \left\{\dot{s}'_1\right\} + \left[R_1\right]^T \left(\left\{\dot{s}'_7\right\} + \left[{}^{R}\tilde{\omega}'_1\right]\left(\left\{q'_7\right\} + \left\{s'_7\right\}\right)\right) + \left[R_7\right]^T \left(\left\{\dot{s}'_8\right\} + \left[{}^{R}\tilde{\omega}'_7\right]\left(\left\{q'_8\right\} + \left\{s'_8\right\}\right)\right)$$

$$\left\{{}^{R}v_{G_8}\right\} = \left\{{}^{R}v_{O_8}\right\} + \left[R_8\right]^T \left[{}^{R}\tilde{\omega}'_8\right]\left\{r'_8\right\}$$
$$= \left\{\dot{s}'_1\right\} + \left[R_1\right]^T \left(\left\{\dot{s}'_7\right\} + \left[{}^{R}\tilde{\omega}'_1\right]\left(\left\{q'_7\right\} + \left\{s'_7\right\}\right)\right) + \left[R_7\right]^T \left(\left\{\dot{s}'_8\right\} + \left[{}^{R}\tilde{\omega}'_7\right]\left(\left\{q'_8\right\} + \left\{s'_8\right\}\right)\right) + \left[R_8\right]^T \left[{}^{R}\tilde{\omega}'_8\right]\left\{r'_8\right\}$$

Mass-Center Partial Velocities

The ***partial velocities*** of the ***mass centers*** of the bodies can be written in terms of the ***partial velocities*** of the ***origins*** of the bodies. To this end, rewrite Eq. (15) as follows.

$$\begin{aligned}
\left\{{}^{R}v_{O_K}\right\} &= \left\{{}^{R}v_{O_J}\right\} + \left[R_J\right]^T\left\{\dot{s}'_K\right\} + \left[R_J\right]^T\left[{}^{R}\tilde{\omega}'_J\right]\left(\left\{q'_K\right\}+\left\{s'_K\right\}\right) \\
&= \left[{}^{R}v_{O_J,y}\right]\{y\} - \left[R_J\right]^T\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\left[{}^{R}\omega'_{J,y}\right]\{y\} + \left[R_J\right]^T\left[{}^{J}v'_{O_K,y}\right]\{y\} \\
&= \left(\left[{}^{R}v_{O_J,y}\right] - \left[R_J\right]^T\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\left[{}^{R}\omega'_{J,y}\right] + \left[R_J\right]^T\left[{}^{J}v'_{O_K,y}\right]\right)\{y\} \\
&\triangleq \left[{}^{R}v_{O_K,y}\right]\{y\}
\end{aligned}$$

$$\Rightarrow \boxed{\left[{}^{R}v_{O_K,y}\right] = \left[{}^{R}v_{O_J,y}\right] - \left[R_J\right]^T\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\left[{}^{R}\omega'_{J,y}\right] + \left[R_J\right]^T\left[{}^{J}v'_{O_K,y}\right]} \qquad (18)$$

Here, $\left[{}^{R}v_{O_J,y}\right]_{3\times 6N}$ and $\left[{}^{R}v_{O_K,y}\right]_{3\times 6N}$ are the ***partial velocity matrices*** of the origin points O_J and O_K, and $\left[{}^{J}v'_{O_K,y}\right]$ can be partitioned and defined as follows.

$$\boxed{\left[{}^{J}v'_{O_K,y}\right]_{3\times 6N} = \left[\left[{}^{J}v'_{O_K,y_1}\right]_{3\times 3N}\left[{}^{J}v'_{O_K,y_2}\right]_{3\times 3N}\right] = \left[[0]_{3\times 3N}\left[{}^{J}v'_{O_K,y_2}\right]_{3\times 3N}\right]} \qquad (19)$$

with

$$\boxed{\left[{}^{J}v'_{O_K,y_2}\right]_{3\times 3N} = \left[\underbrace{[0]}_{1},\ldots,\underbrace{[0]}_{K-1},\underbrace{[I]}_{K},\underbrace{[0]}_{K+1},\ldots,\underbrace{[0]}_{N}\right]} \qquad (20)$$

In Eq. (20), $[0]$ represents the 3×3 ***zero*** matrix, and $[I]$ represents the 3×3 ***identity*** matrix.

Eqs. (18)-(20) provide a means to ***recursively*** calculate the ***partial velocity matrices*** of the ***origins*** of the bodies. Using this result, the ***partial velocity matrices*** of the ***mass-centers*** of the bodies can be calculated as follows. Returning to Eq. (17), write

$$\begin{aligned}
\left\{{}^{R}v_{G_K}\right\} &= \left\{{}^{R}v_{O_K}\right\} + \left[R_K\right]^T\left[{}^{R}\tilde{\omega}'_K\right]\left\{r'_K\right\} \\
&= \left\{{}^{R}v_{O_K}\right\} - \left[R_K\right]^T\left[\tilde{r}'_K\right]\left\{{}^{R}\omega'_K\right\} \\
&= \left(\left[{}^{R}v_{O_K,y}\right] - \left[R_K\right]^T\left[\tilde{r}'_K\right]\left[{}^{R}\omega'_{K,y}\right]\right)\{y\} \\
&\triangleq \left[{}^{R}v_{G_K,y}\right]\{y\}
\end{aligned}$$

$$\Rightarrow \boxed{\left[{}^{R}v_{G_K,y}\right] = \left[{}^{R}v_{O_K,y}\right] - \left[R_K\right]^T\left[\tilde{r}'_K\right]\left[{}^{R}\omega'_{K,y}\right]} \qquad (21)$$

Applying these results to the ***eight-body example*** system gives the following results.

$$\left[{}^R v_{O_1,y}\right]=\left[{}^R v'_{O_1,y}\right]=\left[[0]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right]$$

$$\begin{aligned}\left[{}^R v_{G_1,y}\right]&=\left[{}^R v_{O_1,y}\right]-[R_1]^T\left[\tilde{r}'_1\right]\left[{}^R\omega'_{1,y}\right]\\&=\left[[0]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right]\\&\quad-[R_1]^T\left[\tilde{r}'_1\right]\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R v_{O_2,y}\right]&=\left[{}^R v_{O_1,y}\right]-[R_1]^T\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)\left[{}^R\omega'_{1,y}\right]+[R_1]^T\left[{}^1 v'_{O_2,y}\right]\\&=\left[[0]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right]\\&\quad-[R_1]^T\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]\\&\quad+[R_1]^T\left[[0]_{3\times 24},[0],[I],[0],[0],[0],[0],[0],[0]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R v_{G_2,y}\right]&=\left[{}^R v_{O_2,y}\right]-[R_2]^T\left[\tilde{r}'_2\right]\left[{}^R\omega'_{2,y}\right]\\&=\left[[0]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right]\\&\quad-[R_1]^T\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]\\&\quad+[R_1]^T\left[[0]_{3\times 24},[0],[I],[0],[0],[0],[0],[0],[0]\right]\\&\quad-[R_2]^T\left[\tilde{r}'_2\right]\left[[0],[I],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R v_{O_3,y}\right]&=\left[{}^R v_{O_2,y}\right]-[R_2]^T\left(\left[\tilde{q}'_3\right]+\left[\tilde{s}'_3\right]\right)\left[{}^R\omega'_{2,y}\right]+[R_2]^T\left[{}^2 v'_{O_3,y}\right]\\&=\left[[0]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right]\\&\quad-[R_1]^T\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]\\&\quad+[R_1]^T\left[[0]_{3\times 24},[0],[I],[0],[0],[0],[0],[0],[0]\right]\\&\quad-[R_2]^T\left(\left[\tilde{q}'_3\right]+\left[\tilde{s}'_3\right]\right)\left[[0],[I],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]\\&\quad+[R_2]^T\left[[0]_{3\times 24},[0],[0],[I],[0],[0],[0],[0],[0]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^R v_{G_3,y}\right]&=\left[{}^R v_{O_3,y}\right]-[R_3]^T\left[\tilde{r}'_3\right]\left[{}^R\omega'_{3,y}\right]\\&=\left[[0]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right]\\&\quad-[R_1]^T\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]\\&\quad+[R_1]^T\left[[0]_{3\times 24},[0],[I],[0],[0],[0],[0],[0],[0]\right]\\&\quad-[R_2]^T\left(\left[\tilde{q}'_3\right]+\left[\tilde{s}'_3\right]\right)\left[[0],[I],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]\\&\quad+[R_2]^T\left[[0]_{3\times 24},[0],[0],[I],[0],[0],[0],[0],[0]\right]\\&\quad-[R_3]^T\left[\tilde{r}'_3\right]\left[[0],[0],[I],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]\end{aligned}$$

$$\left[{}^{R}v_{O_4,y}\right]=\left[{}^{R}v_{O_2,y}\right]-\left[R_2\right]^T\left(\left[\tilde{q}'_4\right]+\left[\tilde{s}'_4\right]\right)\left[{}^{R}\omega'_{2,y}\right]+\left[R_2\right]^T\left[{}^{2}v'_{O_4,y}\right]$$

$$=\left[[0]_{3\times24},[I],[0],[0],[0],[0],[0],[0],[0]\right]$$

$$-\left[R_1\right]^T\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right]$$

$$+\left[R_1\right]^T\left[[0]_{3\times24},[0],[I],[0],[0],[0],[0],[0],[0]\right]$$

$$-\left[R_2\right]^T\left(\left[\tilde{q}'_4\right]+\left[\tilde{s}'_4\right]\right)\left[[0],[I],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right]$$

$$+\left[R_2\right]^T\left[[0]_{3\times24},[0],[0],[0],[I],[0],[0],[0],[0]\right]$$

$$\left[{}^{R}v_{G_4,y}\right]=\left[{}^{R}v_{O_4,y}\right]-\left[R_4\right]^T\left[\tilde{r}'_4\right]\left[{}^{R}\omega'_{4,y}\right]$$

$$=\left[[0]_{3\times24},[I],[0],[0],[0],[0],[0],[0],[0]\right]$$

$$-\left[R_1\right]^T\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right]$$

$$+\left[R_1\right]^T\left[[0]_{3\times24},[0],[I],[0],[0],[0],[0],[0],[0]\right]$$

$$-\left[R_2\right]^T\left(\left[\tilde{q}'_4\right]+\left[\tilde{s}'_4\right]\right)\left[[0],[I],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right]$$

$$+\left[R_2\right]^T\left[[0]_{3\times24},[0],[0],[0],[I],[0],[0],[0],[0]\right]$$

$$-\left[R_4\right]^T\left[\tilde{r}'_4\right]\left[[0],[0],[0],[I],[0],[0],[0],[0],[0]_{3\times24}\right]$$

$$\left[{}^{R}v_{O_5,y}\right]=\left[{}^{R}v_{O_1,y}\right]-\left[R_1\right]^T\left(\left[\tilde{q}'_5\right]+\left[\tilde{s}'_5\right]\right)\left[{}^{R}\omega'_{1,y}\right]+\left[R_1\right]^T\left[{}^{1}v'_{O_5,y}\right]$$

$$=\left[[0]_{3\times24},[I],[0],[0],[0],[0],[0],[0],[0]\right]$$

$$-\left[R_1\right]^T\left(\left[\tilde{q}'_5\right]+\left[\tilde{s}'_5\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right]$$

$$+\left[R_1\right]^T\left[[0]_{3\times24},[0],[0],[0],[0],[I],[0],[0],[0]\right]$$

$$\left[{}^{R}v_{G_5,y}\right]=\left[{}^{R}v_{O_5,y}\right]-\left[R_5\right]^T\left[\tilde{r}'_5\right]\left[{}^{R}\omega'_{5,y}\right]$$

$$=\left[[0]_{3\times24},[I],[0],[0],[0],[0],[0],[0],[0]\right]$$

$$-\left[R_1\right]^T\left(\left[\tilde{q}'_5\right]+\left[\tilde{s}'_5\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right]$$

$$+\left[R_1\right]^T\left[[0]_{3\times24},[0],[0],[0],[0],[I],[0],[0],[0]\right]$$

$$-\left[R_5\right]^T\left[\tilde{r}'_5\right]\left[[0],[0],[0],[0],[I],[0],[0],[0],[0]_{3\times24}\right]$$

$$\left[{}^{R}v_{O_6,y}\right]=\left[{}^{R}v_{O_1,y}\right]-\left[R_1\right]^T\left(\left[\tilde{q}'_6\right]+\left[\tilde{s}'_6\right]\right)\left[{}^{R}\omega'_{1,y}\right]+\left[R_1\right]^T\left[{}^{1}v'_{O_6,y}\right]$$

$$=\left[[0]_{3\times24},[I],[0],[0],[0],[0],[0],[0],[0]\right]$$

$$-\left[R_1\right]^T\left(\left[\tilde{q}'_6\right]+\left[\tilde{s}'_6\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right]$$

$$+\left[R_1\right]^T\left[[0]_{3\times24},[0],[0],[0],[0],[0],[I],[0],[0]\right]$$

$$
\begin{aligned}
\left[{}^{R}v_{G_6,y}\right] &= \left[{}^{R}v_{O_6,y}\right] - \left[R_6\right]^T \left[\tilde{r}'_6\right]\left[{}^{R}\omega'_{6,y}\right] \\
&= \Big[[0]_{3\times 24}, [I], [0], [0], [0], [0], [0], [0], [0]\Big] \\
&\quad - \left[R_1\right]^T \left(\left[\tilde{q}'_6\right] + \left[\tilde{s}'_6\right]\right)\Big[[I], [0], [0], [0], [0], [0], [0], [0], [0]_{3\times 24}\Big] \\
&\quad + \left[R_1\right]^T \left[[0]_{3\times 24}, [0], [0], [0], [0], [0], [I], [0], [0]\right] \\
&\quad - \left[R_6\right]^T \left[\tilde{r}'_6\right]\Big[[0], [0], [0], [0], [0], [I], [0], [0], [0]_{3\times 24}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\left[{}^{R}v_{O_7,y}\right] &= \left[{}^{R}v_{O_1,y}\right] - \left[R_1\right]^T \left(\left[\tilde{q}'_7\right] + \left[\tilde{s}'_7\right]\right)\left[{}^{R}\omega'_{1,y}\right] + \left[R_1\right]^T \left[{}^{1}v'_{O_7,y}\right] \\
&= \Big[[0]_{3\times 24}, [I], [0], [0], [0], [0], [0], [0], [0]\Big] \\
&\quad - \left[R_1\right]^T \left(\left[\tilde{q}'_7\right] + \left[\tilde{s}'_7\right]\right)\Big[[I], [0], [0], [0], [0], [0], [0], [0], [0]_{3\times 24}\Big] \\
&\quad + \left[R_1\right]^T \left[[0]_{3\times 24}, [0], [0], [0], [0], [0], [0], [I], [0]\right]
\end{aligned}
$$

$$
\begin{aligned}
\left[{}^{R}v_{G_7,y}\right] &= \left[{}^{R}v_{O_7,y}\right] - \left[R_7\right]^T \left[\tilde{r}'_7\right]\left[{}^{R}\omega'_{7,y}\right] \\
&= \Big[[0]_{3\times 24}, [I], [0], [0], [0], [0], [0], [0], [0]\Big] \\
&\quad - \left[R_1\right]^T \left(\left[\tilde{q}'_7\right] + \left[\tilde{s}'_7\right]\right)\Big[[I], [0], [0], [0], [0], [0], [0], [0], [0]_{3\times 24}\Big] \\
&\quad + \left[R_1\right]^T \left[[0]_{3\times 24}, [0], [0], [0], [0], [0], [0], [I], [0]\right] \\
&\quad - \left[R_7\right]^T \left[\tilde{r}'_7\right]\Big[[0], [0], [0], [0], [0], [0], [I], [0], [0]_{3\times 24}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\left[{}^{R}v_{O_8,y}\right] &= \left[{}^{R}v_{O_7,y}\right] - \left[R_7\right]^T \left(\left[\tilde{q}'_8\right] + \left[\tilde{s}'_8\right]\right)\left[{}^{R}\omega'_{7,y}\right] + \left[R_7\right]^T \left[{}^{7}v'_{O_8,y}\right] \\
&= \Big[[0]_{3\times 24}, [I], [0], [0], [0], [0], [0], [0], [0]\Big] \\
&\quad - \left[R_1\right]^T \left(\left[\tilde{q}'_7\right] + \left[\tilde{s}'_7\right]\right)\Big[[I], [0], [0], [0], [0], [0], [0], [0], [0]_{3\times 24}\Big] \\
&\quad + \left[R_1\right]^T \left[[0]_{3\times 24}, [0], [0], [0], [0], [0], [0], [I], [0]\right] \\
&\quad - \left[R_7\right]^T \left(\left[\tilde{q}'_8\right] + \left[\tilde{s}'_8\right]\right)\Big[[0], [0], [0], [0], [0], [0], [I], [0], [0]_{3\times 24}\Big] \\
&\quad + \left[R_7\right]^T \left[[0]_{3\times 24}, [0], [0], [0], [0], [0], [0], [0], [I]\right]
\end{aligned}
$$

$$
\begin{aligned}
\left[{}^{R}v_{G_8,y}\right] &= \left[{}^{R}v_{O_8,y}\right] - \left[R_8\right]^T \left[\tilde{r}'_8\right]\left[{}^{R}\omega'_{8,y}\right] \\
&= \Big[[0]_{3\times 24}, [I], [0], [0], [0], [0], [0], [0], [0]\Big] \\
&\quad - \left[R_1\right]^T \left(\left[\tilde{q}'_7\right] + \left[\tilde{s}'_7\right]\right)\Big[[I], [0], [0], [0], [0], [0], [0], [0], [0]_{3\times 24}\Big] \\
&\quad + \left[R_1\right]^T \left[[0]_{3\times 24}, [0], [0], [0], [0], [0], [0], [I], [0]\right] \\
&\quad - \left[R_7\right]^T \left(\left[\tilde{q}'_8\right] + \left[\tilde{s}'_8\right]\right)\Big[[0], [0], [0], [0], [0], [0], [I], [0], [0]_{3\times 24}\Big] \\
&\quad + \left[R_7\right]^T \left[[0]_{3\times 24}, [0], [0], [0], [0], [0], [0], [0], [I]\right] \\
&\quad - \left[R_8\right]^T \left[\tilde{r}'_8\right]\Big[[0], [0], [0], [0], [0], [0], [0], [I], [0]_{3\times 24}\Big]
\end{aligned}
$$

Another Look at Velocities and Partial Velocities of the Mass-Centers of the Bodies

As a way of checking the above results, the velocity vectors and partial velocity matrices can also be found ***directly without using recursive relationships***. Direct relationships can be developed by first writing the ***inertial components*** of the position vectors of the mass-centers of bodies as follows.

$$\left\{p_{G_K}\right\} = \{s_1'\} + \left(\sum_{r=0}^{u_K-1}\left[R_{\mathfrak{L}^{r+1}(K)}\right]^T\left(\left\{q'_{\mathfrak{L}^r(K)}\right\}+\left\{s'_{\mathfrak{L}^r(K)}\right\}\right)\right) + \left[R_K\right]^T\{r_K'\} \tag{22}$$

Recall from above that u_K represents the number of bodies in the branch below body K, and as such, $\mathfrak{L}^{u_K}(K) = 1$.

The ***inertial components*** of the ***velocity*** of the ***mass-center*** of body K can be found by ***differentiating*** Eq. (22).

$$\begin{aligned}\left\{{}^R v_{G_K}\right\} &= \{\dot{s}_1'\} + \left(\sum_{r=0}^{u_K-1}\left[R_{\mathfrak{L}^{r+1}(K)}\right]^T\left\{\dot{s}'_{\mathfrak{L}^r(K)}\right\}\right)\\ &\quad + \left(\sum_{r=0}^{u_K-1}\left[\dot{R}_{\mathfrak{L}^{r+1}(K)}\right]^T\left(\left\{q'_{\mathfrak{L}^r(K)}\right\}+\left\{s'_{\mathfrak{L}^r(K)}\right\}\right)\right) + \left[\dot{R}_K\right]^T\{r_K'\}\end{aligned} \tag{23}$$

Here,

$$\left[\dot{R}_K\right]^T\{r_K'\} = \left[R_K\right]^T\left[{}^R\tilde{\omega}_K'\right]\{r_K'\} = -\left[R_K\right]^T\left[\tilde{r}_K'\right]\left\{{}^R\omega_K'\right\} = -\left[R_K\right]^T\left[\tilde{r}_K'\right]\left[{}^R\omega'_{K,y}\right]\{y\}$$

and

$$\begin{aligned}\sum_{r=0}^{u_K-1}\left[\dot{R}_{\mathfrak{L}^{r+1}(K)}\right]^T\left(\left\{q'_{\mathfrak{L}^r(K)}\right\}+\left\{s'_{\mathfrak{L}^r(K)}\right\}\right) &= \sum_{r=0}^{u_K-1}\left[R_{\mathfrak{L}^{r+1}(K)}\right]^T\left[{}^R\tilde{\omega}'_{\mathfrak{L}^{r+1}(K)}\right]\left(\left\{q'_{\mathfrak{L}^r(K)}\right\}+\left\{s'_{\mathfrak{L}^r(K)}\right\}\right)\\ &= -\sum_{r=0}^{u_K-1}\left[R_{\mathfrak{L}^{r+1}(K)}\right]^T\left(\left[\tilde{q}'_{\mathfrak{L}^r(K)}\right]+\left[\tilde{s}'_{\mathfrak{L}^r(K)}\right]\right)\left\{{}^R\omega'_{\mathfrak{L}^{r+1}(K)}\right\}\\ &= -\sum_{r=0}^{u_K-1}\left[R_{\mathfrak{L}^{r+1}(K)}\right]^T\left(\left[\tilde{q}'_{\mathfrak{L}^r(K)}\right]+\left[\tilde{s}'_{\mathfrak{L}^r(K)}\right]\right)\left[{}^R\omega'_{\mathfrak{L}^{r+1}(K),y}\right]\{y\}\end{aligned}$$

Substituting these two results into Eq. (23) gives the following.

$$\begin{aligned}\left\{{}^R v_{G_K}\right\} &= \{\dot{s}_1'\} + \sum_{r=0}^{u_K-1}\left[R_{\mathfrak{L}^{r+1}(K)}\right]^T\left\{\dot{s}'_{\mathfrak{L}^r(K)}\right\} - \sum_{r=0}^{u_K-1}\left[R_{\mathfrak{L}^{r+1}(K)}\right]^T\left(\left[\tilde{q}'_{\mathfrak{L}^r(K)}\right]+\left[\tilde{s}'_{\mathfrak{L}^r(K)}\right]\right)\left[{}^R\omega'_{\mathfrak{L}^{r+1}(K),y}\right]\{y\}\\ &\quad - \left[R_K\right]^T\left[\tilde{r}_K'\right]\left[{}^R\omega'_{K,y}\right]\{y\}\end{aligned} \tag{24}$$

By observation of Eq. (24), the ***inertial components*** of the ***velocity*** of the ***mass-center*** of body K can be written in the following partitioned form.

$$\left\{{}^R v_{G_K}\right\} = \left[\left[{}^R v_{G_K,y_1}\right]_{3\times 3N} \quad \left[{}^R v_{G_K,y_2}\right]_{3\times 3N}\right]\begin{Bmatrix}\{y_1\}_{3N\times 1}\\ \{y_2\}_{3N\times 1}\end{Bmatrix} \tag{25}$$

Here,

$$\begin{aligned}\left[{}^{R}v_{G_K,y_1}\right] = \Big[\, & \underbrace{-\left[R_1\right]^T\left(\left[\tilde{q}'_{\mathfrak{L}^{u_K-1}(K)}\right]+\left[\tilde{s}'_{\mathfrak{L}^{u_K-1}(K)}\right]\right)}_{\mathfrak{L}^{u_K}(K)=1},[0],\ldots,[0], \\ & \underbrace{-\left[R_{\mathfrak{L}^{u_K-1}(K)}\right]^T\left(\left[\tilde{q}'_{\mathfrak{L}^{u_K-2}(K)}\right]+\left[\tilde{s}'_{\mathfrak{L}^{u_K-2}(K)}\right]\right)}_{\mathfrak{L}^{u_K-1}(K)},[0],\ldots,[0], \\ & \underbrace{-\left[R_{\mathfrak{L}^{u_K-2}(K)}\right]^T\left(\left[\tilde{q}'_{\mathfrak{L}^{u_K-3}(K)}\right]+\left[\tilde{s}'_{\mathfrak{L}^{u_K-3}(K)}\right]\right)}_{\mathfrak{L}^{u_K-2}(K)},[0],\ldots,[0], \\ & \underbrace{-\left[R_{\mathfrak{L}^{2}(K)}\right]^T\left(\left[\tilde{q}'_{\mathfrak{L}(K)}\right]+\left[\tilde{s}'_{\mathfrak{L}(K)}\right]\right)}_{\mathfrak{L}^{2}(K)},[0],\ldots,[0],\underbrace{-\left[R_{\mathfrak{L}(K)}\right]^T\left(\left[\tilde{q}'_{K}\right]+\left[\tilde{s}'_{K}\right]\right)}_{\mathfrak{L}(K)}, \\ & [0],\ldots,[0],\underbrace{-\left[R_K\right]^T\left[\tilde{r}'_K\right]}_{K},[0],\ldots,[0]\,\Big]\end{aligned} \tag{26}$$

$$\begin{aligned}\left[{}^{R}v_{G_K,y_2}\right] = \Big[\, & \underbrace{[I]}_{\mathfrak{L}^{u_K}(K)=1},[0],\ldots,[0],\underbrace{\left[R_1\right]^T}_{\mathfrak{L}^{u_K-1}(K)},[0],\ldots,[0],\underbrace{\left[R_{\mathfrak{L}^{u_K-1}(K)}\right]^T}_{\mathfrak{L}^{u_K-2}(K)},[0],\ldots,[0],\underbrace{\left[R_{\mathfrak{L}^{3}(K)}\right]^T}_{\mathfrak{L}^{2}(K)}, \\ & [0],\ldots,[0],\underbrace{\left[R_{\mathfrak{L}^{2}(K)}\right]^T}_{\mathfrak{L}(K)},[0],\ldots,[0],\underbrace{\left[R_{\mathfrak{L}(K)}\right]^T}_{K},[0],\ldots,[0]\,\Big]\end{aligned} \tag{27}$$

In Eqs. (26) and (27) the ***locations*** of the ***nonzero elements*** are listed ***beneath*** the elements, $[I]$ represents the 3×3 ***identity*** matrix, and $[0]$ represents the 3×3 ***zero*** matrix.

To show recursive ***relationships*** give the ***same results*** as those obtained by using Eqs. (26) and (27), consider the results shown above for ***body 8*** of the example eight-body system. Using recursive ***relationships***, $\left[{}^{R}v_{G_8,y}\right]$ was found to be

$$\boxed{\begin{aligned}\left[{}^{R}v_{G_8,y}\right] = & \left[[0]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right] \\ & -\left[R_1\right]^T\left(\left[\tilde{q}'_7\right]+\left[\tilde{s}'_7\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\ & +\left[R_1\right]^T\left[[0]_{3\times 24},[0],[0],[0],[0],[0],[0],[I],[0]\right] \\ & -\left[R_7\right]^T\left(\left[\tilde{q}'_8\right]+\left[\tilde{s}'_8\right]\right)\left[[0],[0],[0],[0],[0],[0],[I],[0],[0]_{3\times 24}\right] \\ & +\left[R_7\right]^T\left[[0]_{3\times 24},[0],[0],[0],[0],[0],[0],[0],[I]\right] \\ & -\left[R_8\right]^T\left[\tilde{r}'_8\right]\left[[0],[0],[0],[0],[0],[0],[0],[I],[0]_{3\times 24}\right]\end{aligned}}$$

Rearranging terms gives

$$
\begin{aligned}
\left[{}^{R}v_{G_8,y}\right] = &-\left[R_1\right]^T\left(\left[\tilde{q}'_7\right]+\left[\tilde{s}'_7\right]\right)[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}] \\
&-\left[R_7\right]^T\left(\left[\tilde{q}'_8\right]+\left[\tilde{s}'_8\right]\right)[[0],[0],[0],[0],[0],[0],[I],[0],[0]_{3\times 24}] \\
&-\left[R_8\right]^T\left[\tilde{r}'_8\right][[0],[0],[0],[0],[0],[0],[0],[I],[0]_{3\times 24}] \\
&+\left[[0]_{3\times 24},[I],[0],[0],[0],[0],[0],[0],[0]\right] \\
&+\left[R_1\right]^T\left[[0]_{3\times 24},[0],[0],[0],[0],[0],[0],[I],[0]\right] \\
&+\left[R_7\right]^T\left[[0]_{3\times 24},[0],[0],[0],[0],[0],[0],[0],[I]\right] \\
\triangleq &\left[\left[{}^{R}v_{G_8,y_1}\right]_{3\times 24} \quad \left[{}^{R}v_{G_8,y_2}\right]_{3\times 24}\right]_{3\times 48}
\end{aligned}
$$

Note that the ***first three lines*** of this result contain matrices that have ***non-zero*** entries only in the ***first eight*** 3×3 submatrices, and the ***last three lines*** contain ***non-zero*** entries only in the ***last eight*** 3×3 submatrices. Given this observation, the submatrices of $\left[{}^{R}v_{G_8,y}\right]$ are identified to be

$$
\left[{}^{R}v_{G_8,y_1}\right] = -\left[\left[R_1\right]^T\left(\left[\tilde{q}'_7\right]+\left[\tilde{s}'_7\right]\right),[0],[0],[0],[0],[0],\left[R_7\right]^T\left(\left[\tilde{q}'_8\right]+\left[\tilde{s}'_8\right]\right),\left[R_8\right]^T\left[\tilde{r}'_8\right]\right]_{3\times 24}
$$

$$
\left[{}^{R}v_{G_8,y_2}\right] = \left[[I],[0],[0],[0],[0],[0],\left[R_1\right]^T,\left[R_7\right]^T\right]_{3\times 24}
$$

Alternately, using Eqs. (26) and (27) gives identical results.

$$
\left[{}^{R}v_{G_8,y_1}\right] = -\left[\left[R_1\right]^T\left(\left[\tilde{q}'_7\right]+\left[\tilde{s}'_7\right]\right),[0],[0],[0],[0],[0],\left[R_7\right]^T\left(\left[\tilde{q}'_8\right]+\left[\tilde{s}'_8\right]\right),\left[R_8\right]^T\left[\tilde{r}'_8\right]\right]_{3\times 24}
$$ ✓

$$
\left[{}^{R}v_{G_8,y_2}\right] = \left[[I],[0],[0],[0],[0],[0],\left[R_1\right]^T,\left[R_7\right]^T\right]_{3\times 24}
$$ ✓

Time Derivatives of the Partial Angular Velocity Matrices

The partial angular velocity matrices (Eq. (8)) were found to be constant matrices, so

$$
\left[\dot{\omega}'_{K,y_1}\right] = [0] \quad \text{and} \quad \left[\dot{\omega}'_{K,y_2}\right] = [0] \tag{28}
$$

Time Derivatives of the Partial Velocity Matrices of the Mass-Centers of the Bodies

The ***time derivatives*** of the ***partial velocity matrices*** of the ***mass-centers*** of the bodies are found by ***differentiating*** the partial velocity matrices given in Eq. (21).

$$
\left[{}^{R}\dot{v}_{G_K,y}\right] = \frac{d}{dt}\left(\left[{}^{R}v_{O_K,y}\right]-\left[R_K\right]^T\left[\tilde{r}'_K\right]\left[{}^{R}\omega'_{K,y}\right]\right) = \left[{}^{R}\dot{v}_{O_K,y}\right]-\left[\dot{R}_K\right]^T\left[\tilde{r}'_K\right]\left[{}^{R}\omega'_{K,y}\right]
$$

$$
\Rightarrow \boxed{\left[{}^{R}\dot{v}_{G_K,y}\right] = \left[{}^{R}\dot{v}_{O_K,y}\right]-\left[R_K\right]^T\left[{}^{R}\tilde{\omega}'_K\right]\left[\tilde{r}'_K\right]\left[{}^{R}\omega'_{K,y}\right]} \tag{29}
$$

The ***time derivatives*** of the ***partial velocities*** of the ***origins*** of the bodies are found by ***differentiating*** the partial velocities given in Eq. (18).

$$\left[{}^R\dot{v}_{O_K,y}\right]=\frac{d}{dt}\left(\left[{}^R v_{O_J,y}\right]-\left[R_J\right]^T\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\left[{}^R\omega'_{J,y}\right]+\left[R_J\right]^T\left[{}^J v'_{O_K,y}\right]\right)$$

$$=\left[{}^R\dot{v}_{O_J,y}\right]-\left[\dot{R}_J\right]^T\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\left[{}^R\omega'_{J,y}\right]-\left[R_J\right]^T\left[\tilde{\dot{s}}'_K\right]\left[{}^R\omega'_{J,y}\right]+\left[\dot{R}_J\right]^T\left[{}^J v'_{O_K,y}\right]$$

$$\Rightarrow \boxed{\begin{aligned}\left[{}^R\dot{v}_{O_K,y}\right]=\left[{}^R\dot{v}_{O_J,y}\right]-\left[R_J\right]^T\left(\left[\tilde{\omega}'_J\right]\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)+\left[\tilde{\dot{s}}'_K\right]\right)\left[{}^R\omega'_{J,y}\right]\\ +\left[R_J\right]^T\left[\tilde{\omega}'_J\right]\left[{}^J v'_{O_K,y}\right]\end{aligned}} \qquad (30)$$

Eq. (30) provides a means of ***recursively calculating*** the ***time derivatives*** of the ***partial velocity matrices*** of the ***origins*** of the bodies, and Eq. (29) provides a means of using those results to calculate the ***time derivatives*** of the ***partial velocity matrices*** of the ***mass-centers*** of the bodies.

To show how to use these results, consider again the ***example eight body system***. Using Eqs. (30) and (29), the ***time derivatives*** of the ***origins*** and ***mass-centers*** of the bodies can be calculated as follows.

$$\boxed{\left[{}^R\dot{v}_{O_1,y}\right]=[0]}$$

$$\boxed{\begin{aligned}\left[{}^R\dot{v}_{G_1,y}\right]&=\left[{}^R\dot{v}_{O_1,y}\right]-\left[R_1\right]^T\left[\tilde{\omega}'_1\right]\left[\tilde{r}'_1\right]\left[{}^R\omega'_{1,y}\right]=-\left[R_1\right]^T\left[\tilde{\omega}'_1\right]\left[\tilde{r}'_1\right]\left[{}^R\omega'_{1,y}\right]\\ &=-\left[R_1\right]^T\left[\tilde{\omega}'_1\right]\left[\tilde{r}'_1\right]\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right]\end{aligned}}$$

$$\boxed{\begin{aligned}\left[{}^R\dot{v}_{O_2,y}\right]&=\left[{}^R\dot{v}_{O_1,y}\right]-\left[R_1\right]^T\left(\left[\tilde{\omega}'_1\right]\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)+\left[\tilde{\dot{s}}'_2\right]\right)\left[{}^R\omega'_{1,y}\right]+\left[R_1\right]^T\left[\tilde{\omega}'_1\right]\left[{}^1 v'_{O_2,y}\right]\\ &=-\left[R_1\right]^T\left(\left[\tilde{\omega}'_1\right]\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)+\left[\tilde{\dot{s}}'_2\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right]\\ &\quad+\left[R_1\right]^T\left[\tilde{\omega}'_1\right]\left[[0]_{3\times24},[0],[I],[0],[0],[0],[0],[0],[0]\right]\end{aligned}}$$

$$\boxed{\begin{aligned}\left[{}^R\dot{v}_{G_2,y}\right]&=\left[{}^R\dot{v}_{O_2,y}\right]-\left[R_2\right]^T\left[\tilde{\omega}'_2\right]\left[\tilde{r}'_2\right]\left[{}^R\omega'_{2,y}\right]\\ &=-\left[R_1\right]^T\left(\left[\tilde{\omega}'_1\right]\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)+\left[\tilde{\dot{s}}'_2\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right]\\ &\quad+\left[R_1\right]^T\left[\tilde{\omega}'_1\right]\left[[0]_{3\times24},[0],[I],[0],[0],[0],[0],[0],[0]\right]\\ &\quad-\left[R_2\right]^T\left[\tilde{\omega}'_2\right]\left[\tilde{r}'_2\right]\left[[0],[I],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right]\end{aligned}}$$

$$\boxed{\begin{aligned}\left[{}^R\dot{v}_{O_3,y}\right]&=\left[{}^R\dot{v}_{O_2,y}\right]-\left[R_2\right]^T\left(\left[\tilde{\omega}'_2\right]\left(\left[\tilde{q}'_3\right]+\left[\tilde{s}'_3\right]\right)+\left[\tilde{\dot{s}}'_3\right]\right)\left[{}^R\omega'_{2,y}\right]+\left[R_2\right]^T\left[\tilde{\omega}'_2\right]\left[{}^2 v'_{O_3,y}\right]\\ &=-\left[R_1\right]^T\left(\left[\tilde{\omega}'_1\right]\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)+\left[\tilde{\dot{s}}'_2\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right]\\ &\quad+\left[R_1\right]^T\left[\tilde{\omega}'_1\right]\left[[0]_{3\times24},[0],[I],[0],[0],[0],[0],[0],[0]\right]\\ &\quad-\left[R_2\right]^T\left(\left[\tilde{\omega}'_2\right]\left(\left[\tilde{q}'_3\right]+\left[\tilde{s}'_3\right]\right)+\left[\tilde{\dot{s}}'_3\right]\right)\left[[0],[I],[0],[0],[0],[0],[0],[0],[0]_{3\times24}\right]\\ &\quad+\left[R_2\right]^T\left[\tilde{\omega}'_2\right]\left[[0]_{3\times24},[0],[0],[I],[0],[0],[0],[0],[0]\right]\end{aligned}}$$

$$\begin{aligned}
\left[{}^{R}\dot{v}_{G_3,y}\right] &= \left[{}^{R}\dot{v}_{O_3,y}\right] - \left[R_3\right]^T \left[\tilde{\omega}_3'\right]\left[\tilde{r}_3'\right]\left[{}^{R}\omega_{3,y}'\right] \\
&= -\left[R_1\right]^T \left(\left[\tilde{\omega}_1'\right]\left(\left[\tilde{q}_2'\right]+\left[\tilde{s}_2'\right]\right)+\left[\dot{\tilde{s}}_2'\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\
&\quad + \left[R_1\right]^T \left[\tilde{\omega}_1'\right]\left[[0]_{3\times 24},[0],[I],[0],[0],[0],[0],[0],[0]\right] \\
&\quad - \left[R_2\right]^T \left(\left[\tilde{\omega}_2'\right]\left(\left[\tilde{q}_3'\right]+\left[\tilde{s}_3'\right]\right)+\left[\dot{\tilde{s}}_3'\right]\right)\left[[0],[I],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\
&\quad + \left[R_2\right]^T \left[\tilde{\omega}_2'\right]\left[[0]_{3\times 24},[0],[0],[I],[0],[0],[0],[0],[0]\right] \\
&\quad - \left[R_3\right]^T \left[\tilde{\omega}_3'\right]\left[\tilde{r}_3'\right]\left[[0],[0],[I],[0],[0],[0],[0],[0],[0]_{3\times 24}\right]
\end{aligned}$$

$$\begin{aligned}
\left[{}^{R}\dot{v}_{O_4,y}\right] &= \left[{}^{R}\dot{v}_{O_2,y}\right] - \left[R_2\right]^T \left(\left[\tilde{\omega}_2'\right]\left(\left[\tilde{q}_4'\right]+\left[\tilde{s}_4'\right]\right)+\left[\dot{\tilde{s}}_4'\right]\right)\left[{}^{R}\omega_{2,y}'\right] + \left[R_2\right]^T\left[\tilde{\omega}_2'\right]\left[{}^{2}v_{O_4,y}'\right] \\
&= -\left[R_1\right]^T \left(\left[\tilde{\omega}_1'\right]\left(\left[\tilde{q}_2'\right]+\left[\tilde{s}_2'\right]\right)+\left[\dot{\tilde{s}}_2'\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\
&\quad + \left[R_1\right]^T \left[\tilde{\omega}_1'\right]\left[[0]_{3\times 24},[0],[I],[0],[0],[0],[0],[0],[0]\right] \\
&\quad - \left[R_2\right]^T \left(\left[\tilde{\omega}_2'\right]\left(\left[\tilde{q}_4'\right]+\left[\tilde{s}_4'\right]\right)+\left[\dot{\tilde{s}}_4'\right]\right)\left[[0],[I],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\
&\quad + \left[R_2\right]^T \left[\tilde{\omega}_2'\right]\left[[0]_{3\times 24},[0],[0],[0],[I],[0],[0],[0],[0]\right]
\end{aligned}$$

$$\begin{aligned}
\left[{}^{R}\dot{v}_{G_4,y}\right] &= \left[{}^{R}\dot{v}_{O_4,y}\right] - \left[R_4\right]^T \left[\tilde{\omega}_4'\right]\left[\tilde{r}_4'\right]\left[{}^{R}\omega_{4,y}'\right] \\
&= -\left[R_1\right]^T \left(\left[\tilde{\omega}_1'\right]\left(\left[\tilde{q}_2'\right]+\left[\tilde{s}_2'\right]\right)+\left[\dot{\tilde{s}}_2'\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\
&\quad + \left[R_1\right]^T \left[\tilde{\omega}_1'\right]\left[[0]_{3\times 24},[0],[I],[0],[0],[0],[0],[0],[0]\right] \\
&\quad - \left[R_2\right]^T \left(\left[\tilde{\omega}_2'\right]\left(\left[\tilde{q}_4'\right]+\left[\tilde{s}_4'\right]\right)+\left[\dot{\tilde{s}}_4'\right]\right)\left[[0],[I],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\
&\quad + \left[R_2\right]^T \left[\tilde{\omega}_2'\right]\left[[0]_{3\times 24},[0],[0],[0],[I],[0],[0],[0],[0]\right] \\
&\quad - \left[R_4\right]^T \left[\tilde{\omega}_4'\right]\left[\tilde{r}_4'\right]\left[[0],[0],[0],[I],[0],[0],[0],[0],[0]_{3\times 24}\right]
\end{aligned}$$

$$\begin{aligned}
\left[{}^{R}\dot{v}_{O_5,y}\right] &= \left[{}^{R}\dot{v}_{O_1,y}\right] - \left[R_1\right]^T \left(\left[\tilde{\omega}_1'\right]\left(\left[\tilde{q}_5'\right]+\left[\tilde{s}_5'\right]\right)+\left[\dot{\tilde{s}}_5'\right]\right)\left[{}^{R}\omega_{1,y}'\right] + \left[R_1\right]^T\left[\tilde{\omega}_1'\right]\left[{}^{1}v_{O_5,y}'\right] \\
&= -\left[R_1\right]^T \left(\left[\tilde{\omega}_1'\right]\left(\left[\tilde{q}_5'\right]+\left[\tilde{s}_5'\right]\right)+\left[\dot{\tilde{s}}_5'\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\
&\quad + \left[R_1\right]^T \left[\tilde{\omega}_1'\right]\left[[0]_{3\times 24},[0],[0],[0],[0],[I],[0],[0],[0]\right]
\end{aligned}$$

$$\begin{aligned}
\left[{}^{R}\dot{v}_{G_5,y}\right] &= \left[{}^{R}\dot{v}_{O_5,y}\right] - \left[R_5\right]^T \left[\tilde{\omega}_5'\right]\left[\tilde{r}_5'\right]\left[{}^{R}\omega_{5,y}'\right] \\
&= -\left[R_1\right]^T \left(\left[\tilde{\omega}_1'\right]\left(\left[\tilde{q}_5'\right]+\left[\tilde{s}_5'\right]\right)+\left[\dot{\tilde{s}}_5'\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\
&\quad + \left[R_1\right]^T \left[\tilde{\omega}_1'\right]\left[[0]_{3\times 24},[0],[0],[0],[0],[I],[0],[0],[0]\right] \\
&\quad - \left[R_5\right]^T \left[\tilde{\omega}_5'\right]\left[\tilde{r}_5'\right]\left[[0],[0],[0],[0],[I],[0],[0],[0],[0]_{3\times 24}\right]
\end{aligned}$$

$$\begin{aligned}\left[{}^{R}\dot{v}_{O_6,y}\right] &= \left[{}^{R}\dot{v}_{O_1,y}\right] - \left[R_1\right]^T\left(\left[\tilde{\omega}_1'\right]\left(\left[\tilde{q}_6'\right]+\left[\tilde{s}_6'\right]\right)+\left[\tilde{\dot{s}}_6'\right]\right)\left[{}^{R}\omega_{1,y}'\right] + \left[R_1\right]^T\left[\tilde{\omega}_1'\right]\left[{}^{1}v_{O_6,y}'\right] \\ &= -\left[R_1\right]^T\left(\left[\tilde{\omega}_1'\right]\left(\left[\tilde{q}_6'\right]+\left[\tilde{s}_6'\right]\right)+\left[\tilde{\dot{s}}_6'\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\ &\quad + \left[R_1\right]^T\left[\tilde{\omega}_1'\right]\left[[0]_{3\times 24},[0],[0],[0],[0],[0],[I],[0],[0]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^{R}\dot{v}_{G_6,y}\right] &= \left[{}^{R}\dot{v}_{O_6,y}\right] - \left[R_6\right]^T\left[\tilde{\omega}_6'\right]\left[\tilde{r}_6'\right]\left[{}^{R}\omega_{6,y}'\right] \\ &= -\left[R_1\right]^T\left(\left[\tilde{\omega}_1'\right]\left(\left[\tilde{q}_6'\right]+\left[\tilde{s}_6'\right]\right)+\left[\tilde{\dot{s}}_6'\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\ &\quad + \left[R_1\right]^T\left[\tilde{\omega}_1'\right]\left[[0]_{3\times 24},[0],[0],[0],[0],[0],[I],[0],[0]\right] \\ &\quad - \left[R_6\right]^T\left[\tilde{\omega}_6'\right]\left[\tilde{r}_6'\right]\left[[0],[0],[0],[0],[0],[I],[0],[0],[0]_{3\times 24}\right]\end{aligned}$$

$$\begin{aligned}\left[{}^{R}\dot{v}_{O_7,y}\right] &= \left[{}^{R}\dot{v}_{O_1,y}\right] - \left[R_1\right]^T\left(\left[\tilde{\omega}_1'\right]\left(\left[\tilde{q}_7'\right]+\left[\tilde{s}_7'\right]\right)+\left[\tilde{\dot{s}}_7'\right]\right)\left[{}^{R}\omega_{1,y}'\right] + \left[R_1\right]^T\left[\tilde{\omega}_1'\right]\left[{}^{1}v_{O_7,y}'\right] \\ &= -\left[R_1\right]^T\left(\left[\tilde{\omega}_1'\right]\left(\left[\tilde{q}_7'\right]+\left[\tilde{s}_7'\right]\right)+\left[\tilde{\dot{s}}_7'\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\ &\quad + \left[R_1\right]^T\left[\tilde{\omega}_1'\right]\left[[0]_{3\times 24},[0],[0],[0],[0],[0],[0],[I],[0]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^{R}\dot{v}_{G_7,y}\right] &= \left[{}^{R}\dot{v}_{O_7,y}\right] - \left[R_7\right]^T\left[\tilde{\omega}_7'\right]\left[\tilde{r}_7'\right]\left[{}^{R}\omega_{7,y}'\right] \\ &= -\left[R_1\right]^T\left(\left[\tilde{\omega}_1'\right]\left(\left[\tilde{q}_7'\right]+\left[\tilde{s}_7'\right]\right)+\left[\tilde{\dot{s}}_7'\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\ &\quad + \left[R_1\right]^T\left[\tilde{\omega}_1'\right]\left[[0]_{3\times 24},[0],[0],[0],[0],[0],[0],[I],[0]\right] \\ &\quad - \left[R_7\right]^T\left[\tilde{\omega}_7'\right]\left[\tilde{r}_7'\right]\left[[0],[0],[0],[0],[0],[0],[I],[0],[0]_{3\times 24}\right]\end{aligned}$$

$$\begin{aligned}\left[{}^{R}\dot{v}_{O_8,y}\right] &= \left[{}^{R}\dot{v}_{O_7,y}\right] - \left[R_7\right]^T\left(\left[\tilde{\omega}_7'\right]\left(\left[\tilde{q}_8'\right]+\left[\tilde{s}_8'\right]\right)+\left[\tilde{\dot{s}}_8'\right]\right)\left[{}^{R}\omega_{7,y}'\right] + \left[R_7\right]^T\left[\tilde{\omega}_7'\right]\left[{}^{7}v_{O_8,y}'\right] \\ &= -\left[R_1\right]^T\left(\left[\tilde{\omega}_1'\right]\left(\left[\tilde{q}_7'\right]+\left[\tilde{s}_7'\right]\right)+\left[\tilde{\dot{s}}_7'\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\ &\quad + \left[R_1\right]^T\left[\tilde{\omega}_1'\right]\left[[0]_{3\times 24},[0],[0],[0],[0],[0],[0],[I],[0]\right] \\ &\quad - \left[R_7\right]^T\left(\left[\tilde{\omega}_7'\right]\left(\left[\tilde{q}_8'\right]+\left[\tilde{s}_8'\right]\right)+\left[\tilde{\dot{s}}_8'\right]\right)\left[[0],[0],[0],[0],[0],[0],[I],[0],[0]_{3\times 24}\right] \\ &\quad + \left[R_7\right]^T\left[\tilde{\omega}_7'\right]\left[[0]_{3\times 24},[0],[0],[0],[0],[0],[0],[0],[I]\right]\end{aligned}$$

$$\begin{aligned}\left[{}^{R}\dot{v}_{G_8,y}\right] &= \left[{}^{R}\dot{v}_{O_8,y}\right] - \left[R_8\right]^T\left[\tilde{\omega}_8'\right]\left[\tilde{r}_8'\right]\left[{}^{R}\omega_{8,y}'\right] \\ &= -\left[R_1\right]^T\left(\left[\tilde{\omega}_1'\right]\left(\left[\tilde{q}_7'\right]+\left[\tilde{s}_7'\right]\right)+\left[\tilde{\dot{s}}_7'\right]\right)\left[[I],[0],[0],[0],[0],[0],[0],[0],[0]_{3\times 24}\right] \\ &\quad + \left[R_1\right]^T\left[\tilde{\omega}_1'\right]\left[[0]_{3\times 24},[0],[0],[0],[0],[0],[0],[I],[0]\right] \\ &\quad - \left[R_7\right]^T\left(\left[\tilde{\omega}_7'\right]\left(\left[\tilde{q}_8'\right]+\left[\tilde{s}_8'\right]\right)+\left[\tilde{\dot{s}}_8'\right]\right)\left[[0],[0],[0],[0],[0],[0],[I],[0],[0]_{3\times 24}\right] \\ &\quad + \left[R_7\right]^T\left[\tilde{\omega}_7'\right]\left[[0]_{3\times 24},[0],[0],[0],[0],[0],[0],[0],[I]\right] \\ &\quad - \left[R_8\right]^T\left[\tilde{\omega}_8'\right]\left[\tilde{r}_8'\right]\left[[0],[0],[0],[0],[0],[0],[0],[I],[0]_{3\times 24}\right]\end{aligned}$$

These results are ***identical*** to those found by simply ***differentiating*** the results given above for the partial velocities of the origins and mass-centers of the bodies.

Generalized Forces

Let the forces and torques acting on each body of the system be replaced by an ***equivalent force system*** consisting of a ***single force*** $\underset{\sim}{F}_K$ acting at the ***mass-center*** G_K and a ***single moment*** $\underset{\sim}{M}_K$. The ***generalized forces*** for the system can then be written as

$$F_{y_i} = \sum_{K=1}^{N}\left(\left(\underset{\sim}{F}_K \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_K}}{\partial y_i}\right) + \left(\underset{\sim}{M}_K \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i}\right)\right)$$

or, in matrix form, the column vector of generalized forces is

$$\boxed{\left\{F_y\right\}_{6N\times 1} = \sum_{K=1}^{N}\left(\left[{}^R v_{G_K,y}\right]^T \left\{F_K\right\} + \left[{}^R \omega'_{K,y}\right]^T \left\{M'_K\right\}\right)} \qquad (31)$$

Here, $\{F_K\}$ represents the ***inertial*** components of the force vector $\underset{\sim}{F}_K$ and $\{M'_K\}$ represents the ***body-fixed*** components of the moment vector $\underset{\sim}{M}_K$.

Kane's Equations of Motion

Assuming all "$6N$" of the ***generalized speeds*** are ***independent***, Kane's equations of motion for the multibody system can be written as

$$\boxed{\sum_{K=1}^{N}\left(m_K\, {}^R\underset{\sim}{a}_{G_K} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_K}}{\partial y_i}\right) + \sum_{K=1}^{N}\left[\left(\underset{\approx}{I}_{G_K} \cdot {}^R\underset{\sim}{\alpha}_K\right) + \left({}^R\underset{\sim}{\omega}_K \times \underset{\sim}{H}_{G_K}\right)\right] \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i} = F_{y_i}} \quad (i = 1,\ldots,6N) \qquad (32)$$

Here, the generalized forces on the ***right side*** of the equation are the entries of the generalized force column vector of Eq. (31). The terms on the ***left side*** of the equation can be written as follows.

$${}^R\underset{\sim}{a}_{G_K} \;\rightarrow\; \left\{{}^R a_{G_K}\right\} = \left\{{}^R \dot{v}_{G_K}\right\} = \frac{d}{dt}\left(\left[{}^R v_{G_K,y}\right]\{y\}\right) = \left[{}^R v_{G_K,y}\right]\{\dot{y}\} + \left[{}^R \dot{v}_{G_K,y}\right]\{y\}$$

$${}^R\underset{\sim}{\alpha}_K \;\rightarrow\; \left\{{}^R \alpha'_K\right\} = \left\{{}^R \dot{\omega}'_K\right\} = \frac{d}{dt}\left(\left[{}^R \omega'_{K,y}\right]\{y\}\right) = \left[{}^R \omega'_{K,y}\right]\{\dot{y}\}$$

$$\sum_{K=1}^{N}\left(m_K\, {}^R\underset{\sim}{a}_{G_K} \cdot \frac{\partial\, {}^R\underset{\sim}{v}_{G_K}}{\partial y_i}\right) \;\rightarrow$$

$$\boxed{\sum_{K=1}^{N}\left(m_K \left[{}^R v_{G_K,y}\right]^T \left\{{}^R a_{G_K}\right\}\right) = \sum_{K=1}^{N}\left(m_K \left[{}^R v_{G_K,y}\right]^T \left[{}^R v_{G_K,y}\right]\{\dot{y}\} + m_K \left[{}^R v_{G_K,y}\right]^T \left[{}^R \dot{v}_{G_K,y}\right]\{y\}\right)} \qquad (33)$$

$$\sum_{K=1}^{N}\left(\underset{\approx}{I}_{G_K} \cdot {}^R\underset{\sim}{\alpha}_K\right) \cdot \frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i} \;\rightarrow$$

$$\boxed{\sum_{K=1}^{N}\left(\left[{}^R \omega'_{K,y}\right]^T \left[I'_{G_K}\right]\left\{{}^R \alpha'_K\right\}\right) = \sum_{K=1}^{N}\left(\left[{}^R \omega'_{K,y}\right]^T \left[I'_{G_K}\right]\left[{}^R \omega'_{K,y}\right]\{\dot{y}\}\right)} \qquad (34)$$

$$ {}^R\underset{\sim}{\omega}_K \times \underset{\sim}{H}_{G_K} \quad \rightarrow \quad \left[{}^R\tilde{\omega}'_K\right]\left[I'_{G_K}\right]\left\{{}^R\omega'_K\right\} \qquad \text{(body-fixed components)} $$

$$ \sum_{K=1}^{N}\left({}^R\underset{\sim}{\omega}_K \times \underset{\sim}{H}_{G_K}\right)\cdot\frac{\partial\, {}^R\underset{\sim}{\omega}_K}{\partial y_i} \quad \rightarrow \quad \boxed{\sum_{K=1}^{N}\left(\left[{}^R\omega'_{K,y}\right]^T\left[{}^R\tilde{\omega}'_K\right]\left[I'_{G_K}\right]\left\{{}^R\omega'_K\right\}\right)} \tag{35} $$

Substituting from Eqs. (31), (33), (34), and (35) into Eq. (32) gives the matrix form of the ***equations of motion*** for the ***multibody system***

$$ \boxed{[A]\{\dot{y}\}=\{f\}} \tag{36} $$

Here, the coefficient matrix $[A]$ is called the "***generalized mass matrix***". The matrix $[A]$ and column vector $\{f\}$ are defined as follows.

$$ \boxed{[A]=\sum_{K=1}^{N}\left(m_K\left[{}^R v_{G_K,y}\right]^T\left[{}^R v_{G_K,y}\right]+\left[{}^R\omega'_{K,y}\right]^T\left[I'_{G_K}\right]\left[{}^R\omega'_{K,y}\right]\right)} \tag{37} $$

$$ \boxed{\begin{aligned}\{f\}=&\sum_{K=1}^{N}\left[{}^R v_{G_K,y}\right]^T\left(\{F_K\}-m_K\left[{}^R\dot{v}_{G_K,y}\right]\{y\}\right)+\\ &\sum_{K=1}^{N}\left[{}^R\omega'_{K,y}\right]^T\left(\{M'_K\}-\left[{}^R\tilde{\omega}'_K\right]\left[I'_{G_K}\right]\left\{{}^R\omega'_K\right\}\right)\end{aligned}} \tag{38} $$

Eq. (36) represents "$6N$" ***first-order, ordinary differential equations*** for the "$13N$" variables defined by the ***system state vectors*** $\{x\}$ and $\{y\}$ of Eq. (1). To form a ***complete set*** of ***differential equations***, Eq. (36) must be ***supplemented*** with the following set of "$7N$" ***first-order, kinematical differential equations***.

$$ \boxed{\{\dot{x}_1\}=\{\dot{\varepsilon}\}=\frac{1}{2}\begin{bmatrix}[\bar{E}'_1]^T & [0] & [0] & [0] & [0]\\ [0] & [\bar{E}'_2]^T & [0] & \ddots & [0]\\ [0] & [0] & [\bar{E}'_3]^T & \ddots & [0]\\ [0] & \ddots & \ddots & \ddots & [0]\\ [0] & [0] & [0] & [0] & [\bar{E}'_N]^T\end{bmatrix}\{\omega'\}} \tag{39} $$

and

$$ \boxed{\{\dot{x}_2\}=\{y_2\}} \tag{40} $$

The matrices $[\bar{E}'_K]^T$ that appear on the diagonal of Equation (39) can be written as

$$ \boxed{[\bar{E}'_K]^T=\begin{bmatrix}\varepsilon_{K4} & -\varepsilon_{K3} & \varepsilon_{K2}\\ \varepsilon_{K3} & \varepsilon_{K4} & -\varepsilon_{K1}\\ -\varepsilon_{K2} & \varepsilon_{K1} & \varepsilon_{K4}\\ -\varepsilon_{K1} & -\varepsilon_{K2} & -\varepsilon_{K3}\end{bmatrix}} \tag{41} $$

Note that $[\bar{E}'_K]^T$ is formed from $[E'_K]^T$ (as defined by Eq. (5)) by removing its ***fourth column***. This allows the angular velocity vectors $\left\{{}^R\omega'_K\right\}$ to be taken as 3×1 vectors and the vector $\{\omega'\}$ is a $3N\times1$ vector. Recall that, for convenience, the angular velocity vector of Eq. (5) was treated as a 4×1 vector whose last element was zero which made it ***easy*** to ***invert*** the equation and solve for the derivatives of the Euler parameters.