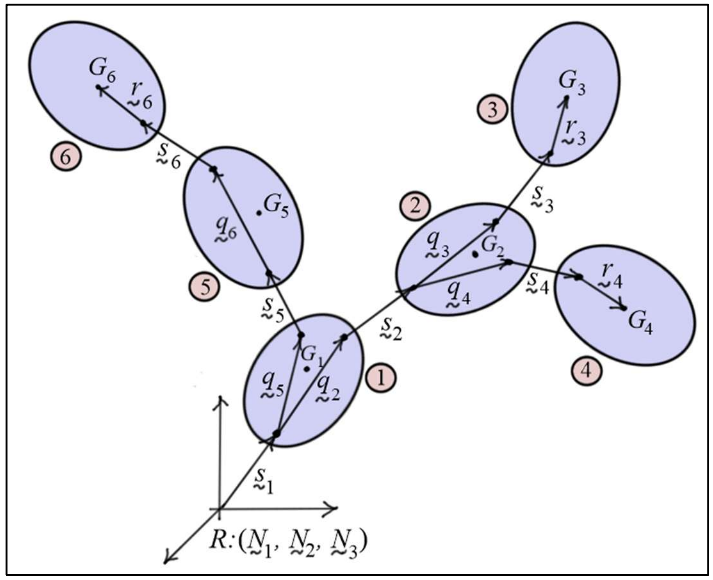
Multibody Dynamics
Equations of Motion for an Unconstrained Multibody – Example



The results below all apply to the ***six-body*** system shown. The ***orientations*** of the bodies relative to the ***inertial system*** are described using ***Euler parameters***, and the ***translation*** of the bodies ***relative*** to their ***lower-numbered body*** are defined by components of the vectors $\underset{\sim}{s}_K \; (K=1,\cdots,6)$ in the reference frame $\mathcal{L}(K)$.

The ***angular velocities*** of the bodies are all measured ***relative*** to the ***inertial frame*** and their components are resolved in the body frames.

In total, there are 42 first-order, kinematical differential equations and 36 first-order, dynamical differential equations.

Body Connection Array

Using the numbering of the bodies shown on the diagram, the body connection array (or lower-numbered body array) can be written as follows.

$$\mathcal{L}(K) = (0,\ 1,\ 2,\ 2,\ 1,\ 5)$$

Recall that ***zero*** represents the inertial frame. The array $u_K \; (K=1,\ldots,6)$ giving the number of lower bodies of each body in the system

$$(u_K) = (0,\ 1,\ 2,\ 2,\ 1,\ 2)$$

Recall that for each body, u_K is that integer such that $\mathcal{L}^{u_K}(K)=1$. In other words, it is the number of steps down from body *K* to reach the system's reference body (body 1).

System State Vectors

$$\{\varepsilon\}_{24\times1} = [\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{14}, \ldots, \varepsilon_{61}, \varepsilon_{62}, \varepsilon_{63}, \varepsilon_{64}]^T \qquad \{s'\}_{18\times1} = [s'_{11}, s'_{12}, s'_{13}, \ldots, s'_{61}, s'_{62}, s'_{63}]^T$$

$$\{\omega'\}_{18\times1} = [\omega'_{11}, \omega'_{12}, \omega'_{13}, \ldots, \omega'_{61}, \omega'_{62}, \omega'_{63}]^T$$

$$\{x\}_{42\times1} = \begin{Bmatrix} \{x_1\} \\ \{x_2\} \end{Bmatrix} = \begin{Bmatrix} \{\varepsilon\} \\ \{s'\} \end{Bmatrix} \qquad \{y\}_{36\times1} = \begin{Bmatrix} \{y_1\} \\ \{y_2\} \end{Bmatrix} = \begin{Bmatrix} \{\omega'\} \\ \{\dot{s}'\} \end{Bmatrix}$$

System Partial Angular Velocity Matrices

$$\left[\omega'_{1,y_1}\right]_{3\times18} = \left[\underbrace{[I]}_{1},\underbrace{[0]}_{2},\ldots,\underbrace{[0]}_{6}\right] \qquad \left[\omega'_{1,y_2}\right]_{3\times18} = [0]_{3\times18}$$

$$\left[\omega'_{2,y_1}\right]_{3\times18} = \left[\underbrace{[0]}_{1},\underbrace{[I]}_{2},\underbrace{[0]}_{3},\ldots,\underbrace{[0]}_{6}\right] \qquad \left[\omega'_{2,y_2}\right]_{3\times18} = [0]_{3\times18}$$

$$\left[\omega'_{3,y_1}\right]_{3\times18} = \left[\underbrace{[0]}_{1},\underbrace{[0]}_{2},\underbrace{[I]}_{3},\underbrace{[0]}_{4},\ldots,\underbrace{[0]}_{6}\right] \qquad \left[\omega'_{3,y_2}\right]_{3\times18} = [0]_{3\times18}$$

$$\vdots \qquad\qquad \vdots$$

$$\left[\omega'_{6,y_1}\right]_{3\times18} = \left[\underbrace{[0]}_{1},\underbrace{[0]}_{2},\ldots,\underbrace{[I]}_{6}\right] \qquad \left[\omega'_{6,y_2}\right]_{3\times18} = [0]_{3\times18}$$

Here, $[I]$ represents the 3×3 ***identity matrix***, and $\left[0\right]$ represents a 3×3 ***zero matrix***.

Position Vectors of the Mass Centers of the Bodies

$$\{p_1\} = \{s'_1\} + \left[R_1\right]^T \left\{r'_1\right\}$$

$$\{p_2\} = \{s'_1\} + \left[R_1\right]^T \left\{q'_2 + s'_2\right\} + \left[R_2\right]^T \left\{r'_2\right\}$$

$$\{p_3\} = \{s'_1\} + \left[R_1\right]^T \left\{q'_2 + s'_2\right\} + \left[R_2\right]^T \left\{q'_3 + s'_3\right\} + \left[R_3\right]^T \left\{r'_3\right\}$$

$$\{p_4\} = \{s'_1\} + \left[R_1\right]^T \left\{q'_2 + s'_2\right\} + \left[R_2\right]^T \left\{q'_4 + s'_4\right\} + \left[R_4\right]^T \left\{r'_4\right\}$$

$$\{p_5\} = \{s'_1\} + \left[R_1\right]^T \left\{q'_5 + s'_5\right\} + \left[R_5\right]^T \left\{r'_5\right\}$$

$$\{p_6\} = \{s'_1\} + \left[R_1\right]^T \left\{q'_5 + s'_5\right\} + \left[R_5\right]^T \left\{q'_6 + s'_6\right\} + \left[R_6\right]^T \left\{r'_6\right\}$$

Velocity Vectors of the Mass Centers of the Bodies

$$\{v_1\} = \{\dot{s}'_1\} - \left[R_1\right]^T \left[\tilde{r}'_1\right]\left[\omega'_{1,y}\right]\{y\}$$

$$\{v_2\} = \{\dot{s}'_1\} + \left[R_1\right]^T \{\dot{s}'_2\} - \left[R_1\right]^T \left[\tilde{q}'_2 + \tilde{s}'_2\right]\left[\omega'_{1,y}\right]\{y\} - \left[R_2\right]^T \left[\tilde{r}'_2\right]\left[\omega'_{2,y}\right]\{y\}$$

$$\begin{aligned}\{v_3\} = \{\dot{s}'_1\} + \left[R_1\right]^T \{\dot{s}'_2\} + \left[R_2\right]^T \{\dot{s}'_3\} - \\ \left[R_1\right]^T \left[\tilde{q}'_2 + \tilde{s}'_2\right]\left[\omega'_{1,y}\right]\{y\} - \left[R_2\right]^T \left[\tilde{q}'_3 + \tilde{s}'_3\right]\left[\omega'_{2,y}\right]\{y\} - \left[R_3\right]^T \left[\tilde{r}'_3\right]\left[\omega'_{3,y}\right]\{y\}\end{aligned}$$

$$\begin{aligned}\{v_4\} = \{\dot{s}'_1\} + \left[R_1\right]^T \{\dot{s}'_2\} + \left[R_2\right]^T \{\dot{s}'_4\} - \\ \left[R_1\right]^T \left[\tilde{q}'_2 + \tilde{s}'_2\right]\left[\omega'_{1,y}\right]\{y\} - \left[R_2\right]^T \left[\tilde{q}'_4 + \tilde{s}'_4\right]\left[\omega'_{2,y}\right]\{y\} - \left[R_4\right]^T \left[\tilde{r}'_4\right]\left[\omega'_{4,y}\right]\{y\}\end{aligned}$$

$$\{v_5\} = \{\dot{s}'_1\} + \left[R_1\right]^T \{\dot{s}'_5\} - \left[R_1\right]^T \left[\tilde{q}'_5 + \tilde{s}'_5\right]\left[\omega'_{1,y}\right]\{y\} - \left[R_5\right]^T \left[\tilde{r}'_5\right]\left[\omega'_{5,y}\right]\{y\}$$

$$\{v_6\} = \{\dot{s}_1'\} + \left[R_1\right]^T \{\dot{s}_5'\} + \left[R_5\right]^T \{\dot{s}_6'\} - \left[R_1\right]^T \left[\tilde{q}_5' + \tilde{s}_5'\right]\left[\omega_{1,y}'\right]\{y\} - \left[R_5\right]^T \left[\tilde{q}_6' + \tilde{s}_6'\right]\left[\omega_{5,y}'\right]\{y\} - \left[R_6\right]^T \left[\tilde{r}_6'\right]\left[\omega_{6,y}'\right]\{y\}$$

System Partial Velocity Matrices

$$\left[v_{1,y_1}\right]_{3\times 18} = \left[-\left[R_1\right]^T \left[\tilde{r}_1'\right],[0],[0],[0],[0],[0]\right]$$

$$\left[v_{2,y_1}\right]_{3\times 18} = \left[-\left[R_1\right]^T \left[\tilde{q}_2' + \tilde{s}_2'\right],-\left[R_2\right]^T \left[\tilde{r}_2'\right],[0],[0],[0],[0]\right]$$

$$\left[v_{3,y_1}\right]_{3\times 18} = \left[-\left[R_1\right]^T \left[\tilde{q}_2' + \tilde{s}_2'\right],-\left[R_2\right]^T \left[\tilde{q}_3' + \tilde{s}_3'\right],-\left[R_3\right]^T \left[\tilde{r}_3'\right],[0],[0],[0]\right]$$

$$\left[v_{4,y_1}\right]_{3\times 18} = \left[-\left[R_1\right]^T \left[\tilde{q}_2' + \tilde{s}_2'\right],-\left[R_2\right]^T \left[\tilde{q}_4' + \tilde{s}_4'\right],[0],-\left[R_4\right]^T \left[\tilde{r}_4'\right],[0],[0]\right]$$

$$\left[v_{5,y_1}\right]_{3\times 18} = \left[-\left[R_1\right]^T \left[\tilde{q}_5' + \tilde{s}_5'\right],[0],[0],[0],-\left[R_5\right]^T \left[\tilde{r}_5'\right],[0]\right]$$

$$\left[v_{6,y_1}\right]_{3\times 18} = \left[-\left[R_1\right]^T \left[\tilde{q}_5' + \tilde{s}_5'\right],[0],[0],[0],-\left[R_5\right]^T \left[\tilde{q}_6' + \tilde{s}_6'\right],-\left[R_6\right]^T \left[\tilde{r}_6'\right]\right]$$

$$\left[v_{1,y_2}\right]_{3\times 18} = \left[[I],[0],[0],[0],[0],[0]\right]$$

$$\left[v_{2,y_2}\right]_{3\times 18} = \left[[I],[R_1]^T,[0],[0],[0],[0]\right]$$

$$\left[v_{3,y_2}\right]_{3\times 18} = \left[[I],[R_1]^T,[R_2]^T,[0],[0],[0]\right]$$

$$\left[v_{4,y_2}\right]_{3\times 18} = \left[[I],[R_1]^T,[0],[R_2]^T,[0],[0]\right]$$

$$\left[v_{5,y_2}\right]_{3\times 18} = \left[[I],[0],[0],[0],[R_1]^T,[0]\right]$$

$$\left[v_{6,y_2}\right]_{3\times 18} = \left[[I],[0],[0],[0],[R_1]^T,[R_5]^T\right]$$

Time Derivatives of the Partial Angular Velocity Matrices

$$\left[\dot{\omega}_{K,y_1}'\right]_{3\times 18} = [0]_{3\times 18} \qquad \left[\dot{\omega}_{K,y_2}'\right]_{3\times 18} = [0]_{3\times 18} \qquad K = 1,\ldots,6$$

Time Derivatives of the Partial Velocity Matrices

$$\left[\dot{v}_{1,y_1}\right]_{3\times 18} = \left[-\left[R_1\right]^T \left[\tilde{\omega}_1'\right]\left[\tilde{r}_1'\right],[0],[0],[0],[0],[0]\right]$$

$$\left[\dot{v}_{2,y_1}\right]_{3\times 18} = \left[\left(-\left[R_1\right]^T \left[\dot{\tilde{s}}_2'\right] - \left[R_1\right]^T \left[\tilde{\omega}_1'\right]\left[\tilde{q}_2' + \tilde{s}_2'\right]\right),-\left[R_2\right]^T \left[\tilde{\omega}_2'\right]\left[\tilde{r}_2'\right],[0],[0],[0],[0]\right]$$

$$\left[\dot{v}_{3,y_1}\right]_{3\times 18}=\Big[\left(-\left[R_1\right]^T\left[\dot{\tilde{s}}_2'\right]-\left[R_1\right]^T\left[\tilde{\omega}_1'\right]\left[\tilde{q}_2'+\tilde{s}_2'\right]\right),$$
$$\left(-\left[R_2\right]^T\left[\dot{\tilde{s}}_3'\right]-\left[R_2\right]^T\left[\tilde{\omega}_2'\right]\left[\tilde{q}_3'+\tilde{s}_3'\right]\right),-\left[R_3\right]^T\left[\tilde{\omega}_3'\right]\left[\tilde{r}_3'\right],[0],[0],[0]\Big]$$

$$\left[\dot{v}_{4,y_1}\right]_{3\times 18}=\Big[\left(-\left[R_1\right]^T\left[\dot{\tilde{s}}_2'\right]-\left[R_1\right]^T\left[\tilde{\omega}_1'\right]\left[\tilde{q}_2'+\tilde{s}_2'\right]\right),$$
$$\left(-\left[R_2\right]^T\left[\dot{\tilde{s}}_4'\right]-\left[R_2\right]^T\left[\tilde{\omega}_2'\right]\left[\tilde{q}_4'+\tilde{s}_4'\right]\right),[0],-\left[R_4\right]^T\left[\tilde{\omega}_4'\right]\left[\tilde{r}_4'\right],[0],[0]\Big]$$

$$\left[\dot{v}_{5,y_1}\right]_{3\times 18}=\Big[\left(-\left[R_1\right]^T\left[\dot{\tilde{s}}_5'\right]-\left[R_1\right]^T\left[\tilde{\omega}_1'\right]\left[\tilde{q}_5'+\tilde{s}_5'\right]\right),[0],[0],[0],-\left[R_5\right]^T\left[\tilde{\omega}_5'\right]\left[\tilde{r}_5'\right],[0]\Big]$$

$$\left[\dot{v}_{6,y_1}\right]_{3\times 18}=\Big[\left(-\left[R_1\right]^T\left[\dot{\tilde{s}}_5'\right]-\left[R_1\right]^T\left[\tilde{\omega}_1'\right]\left[\tilde{q}_5'+\tilde{s}_5'\right]\right),[0],[0],[0],$$
$$\left(-\left[R_5\right]^T\left[\dot{\tilde{s}}_6'\right]-\left[R_5\right]^T\left[\tilde{\omega}_5'\right]\left[\tilde{q}_6'+\tilde{s}_6'\right]\right),-\left[R_6\right]^T\left[\tilde{\omega}_6'\right]\left[\tilde{r}_6'\right]\Big]$$

$$\left[\dot{v}_{1,y_2}\right]_{3\times 18}=[[0],[0],[0],[0],[0],[0]]$$

$$\left[\dot{v}_{2,y_2}\right]_{3\times 18}=\left[[0],[R_1]^T[\tilde{\omega}_1'],[0],[0],[0],[0]\right]$$

$$\left[\dot{v}_{3,y_2}\right]_{3\times 18}=\left[[0],[R_1]^T[\tilde{\omega}_1'],[R_2]^T[\tilde{\omega}_2'],[0],[0],[0]\right]$$

$$\left[\dot{v}_{4,y_2}\right]_{3\times 18}=\left[[0],[R_1]^T[\tilde{\omega}_1'],[0],[R_2]^T[\tilde{\omega}_2'],[0],[0]\right]$$

$$\left[\dot{v}_{5,y_2}\right]_{3\times 18}=\left[[0],[0],[0],[0],[R_1]^T[\tilde{\omega}_1'],[0]\right]$$

$$\left[\dot{v}_{6,y_2}\right]_{3\times 18}=\left[[0],[0],[0],[0],[R_1]^T[\tilde{\omega}_1'],[R_5]^T[\tilde{\omega}_5']\right]$$

Equations of Motion

Given the partial velocity and partial angular velocity matrices above, the dynamical equations of motion can be written as

$$\boxed{[A]\{\dot{y}\}=\{f\}} \qquad \text{(}[A]\text{ is the generalized mass matrix)}$$

Here,

$$\boxed{[A]=\sum_{K=1}^{6}\left(m_K\left[v_{K,y}\right]^T\left[v_{K,y}\right]+\left[\omega_{K,y}'\right]^T\left[I_K'\right]\left[\omega_{K,y}'\right]\right)}$$

$$\boxed{\begin{aligned}\{f\}=&\sum_{K=1}^{6}\left[v_{K,y}\right]^T\left(\{F_K\}-m_K\left[\dot{v}_{K,y}\right]\{y\}\right)+\\&\sum_{K=1}^{6}\left[\omega_{K,y}'\right]^T\left(\{M_K'\}-[\tilde{\omega}_K'][I_K']\{\omega_K'\}\right)\end{aligned}}$$

The above equations represent 36 dynamical differential equations that must be supplemented with 42 kinematical differential equations.