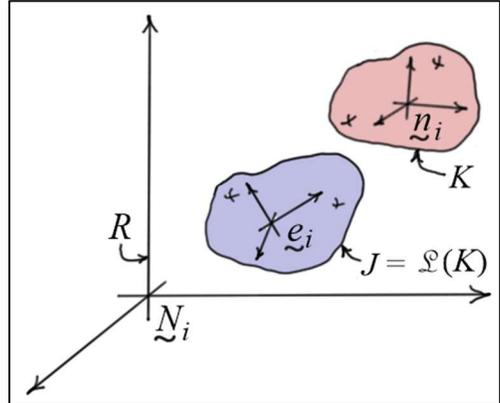


Multibody Dynamics

Time Derivative of Relative Transformation Matrices

Matrix Form of the Derivative of a Vector Fixed in a Rigid Body

Consider two bodies of a multibody system. The unit vector set $(\underline{n}_1, \underline{n}_2, \underline{n}_3)$ is fixed in body K , and the set $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ is fixed in body J . Body J is the lower-numbered body of K ($J = \mathcal{L}(K)$). Both bodies are moving in a fixed frame $R: (\underline{N}_1, \underline{N}_2, \underline{N}_3)$. If \underline{r} is a vector fixed in the body K , then, using the summation rule for angular velocities, the time derivative of \underline{r} can be written as



$$\frac{{}^R d\underline{r}}{dt} = \dot{\underline{r}} = {}^R \underline{\omega}_K \times \underline{r} = ({}^R \underline{\omega}_J + {}^J \underline{\omega}_K) \times \underline{r} = ({}^R \underline{\omega}_J \times \underline{r}) + ({}^J \underline{\omega}_K \times \underline{r}) \quad (1)$$

When performing the cross products, the individual vectors and the resulting cross products can be expressed in any reference frame. Two cases are considered below – components of ${}^J \underline{\omega}_K$ in body J (Case 1) and components of ${}^J \underline{\omega}_K$ in body K (Case 2). Components of vectors in body K have been annotated with a “prime”.

The transformation matrices associated with the two bodies ($[R_J]$, $[R_K]$, $[{}^J R_K]$) are defined by the following equations.

$$\{\underline{e}\} = [R_J] \{\underline{N}\} \quad \{\underline{n}\} = [R_K] \{\underline{N}\} \quad \{\underline{n}\} = [{}^J R_K] \{\underline{e}\}$$

Case 1:

Let $\dot{\underline{r}}$, ${}^R \underline{\omega}_J$, and ${}^R \underline{\omega}_K$ be expressed in $R: (\underline{N}_1, \underline{N}_2, \underline{N}_3)$, ${}^J \underline{\omega}_K$ be expressed in $J: (\underline{e}_1, \underline{e}_2, \underline{e}_3)$, and \underline{r} be expressed in $K: (\underline{n}_1, \underline{n}_2, \underline{n}_3)$. Then,

$$\begin{aligned} \{\underline{r}\} &= [R_K]^T \{r'\} & \{\dot{\underline{r}}\} &= [\dot{R}_K]^T \{r'\} \\ {}^R \underline{\omega}_K \times \underline{r} &\rightarrow [\tilde{\omega}_K] [R_K]^T \{r'\} & {}^R \underline{\omega}_J \times \underline{r} &\rightarrow [\tilde{\omega}_J] [R_K]^T \{r'\} \\ {}^J \underline{\omega}_K \times \underline{r} &\rightarrow [R_J]^T [{}^J \tilde{\omega}_K] [{}^J R_K]^T \{r'\} \end{aligned}$$

Substituting into Eq. (1) gives

$$\begin{aligned} [\dot{R}_K]^T &= [\tilde{\omega}_K] [R_K]^T = [\tilde{\omega}_J] [R_K]^T + [R_J]^T [{}^J \tilde{\omega}_K] [{}^J R_K]^T \\ \Rightarrow &([\tilde{\omega}_K] - [\tilde{\omega}_J]) [R_K]^T = [R_J]^T [{}^J \tilde{\omega}_K] [{}^J R_K]^T \end{aligned}$$

or

$$\boxed{[R_K]([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T) = [{}^J R_K][{}^J \tilde{\omega}_K]^T [R_J]} \quad (2)$$

Case 2:

Let \dot{r} , ${}^R \omega_J$, and ${}^R \omega_K$ be expressed in $R: (\underline{N}_1, \underline{N}_2, \underline{N}_3)$, and let ${}^J \omega_K$ and r be expressed in $K: (\underline{n}_1, \underline{n}_2, \underline{n}_3)$.

Then,

$$\begin{aligned} \boxed{\{r\} = [R_K]^T \{r'\}} & \quad \boxed{\{\dot{r}\} = [\dot{R}_K]^T \{r'\}} \\ \boxed{{}^R \omega_K \times r \rightarrow [\tilde{\omega}_K][R_K]^T \{r'\}} & \quad \boxed{{}^R \omega_J \times r \rightarrow [\tilde{\omega}_J][R_K]^T \{r'\}} \\ \boxed{{}^J \omega_K \times r \rightarrow [R_K]^T [{}^J \tilde{\omega}'_K] \{r'\}} & \end{aligned}$$

Substituting into Eq. (1) gives

$$\begin{aligned} \boxed{[\dot{R}_K]^T} &= [\tilde{\omega}_K][R_K]^T = [\tilde{\omega}_J][R_K]^T + [R_K]^T [{}^J \tilde{\omega}'_K] \\ \Rightarrow & \boxed{([\tilde{\omega}_K] - [\tilde{\omega}_J])[R_K]^T = [R_K]^T [{}^J \tilde{\omega}'_K]} \end{aligned}$$

or

$$\boxed{[R_K]([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T) = [{}^J \tilde{\omega}'_K]^T [R_K]} \quad (3)$$

Time Derivative of the Transformation Matrices Between Two Body Frames

The above results can be used to determine *two different forms* of the *time derivative* of the *relative transformation matrix* $[{}^J R_K]$ depending on what reference frame is used to express the relative angular velocity vector ${}^J \omega_K$. First, note the time derivative of $[{}^J R_K]$ can be written as follows.

$$\begin{aligned} [{}^J \dot{R}_K] &= \frac{{}^R d}{dt} [{}^J R_K] = \frac{{}^R d}{dt} ([R_K][R_J]^T) = [\dot{R}_K][R_J]^T + [R_K][\dot{R}_J]^T \\ &= [R_K][\tilde{\omega}_K]^T [R_J]^T + [R_K][\tilde{\omega}_J][R_J]^T \end{aligned}$$

As a skew-symmetric matrix, $[\tilde{\omega}_J] = -[\tilde{\omega}_J]^T$, so the above equation can be rewritten as

$$\Rightarrow \boxed{[{}^J \dot{R}_K] = [R_K]([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T)[R_J]^T} \quad (4)$$

Now, substituting from Eq. (2) into Eq. (4) gives

$$\begin{aligned} [{}^J \dot{R}_K] &= [R_K]([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T)[R_J]^T \\ &= [{}^J R_K][{}^J \tilde{\omega}_K]^T [R_J][R_J]^T \\ \Rightarrow & \boxed{[{}^J \dot{R}_K] = [{}^J R_K][{}^J \tilde{\omega}_K]^T} \quad (\text{components of } {}^J \omega_K \text{ are resolved in body } J) \end{aligned}$$

And, substituting from Eq. (3) into Eq. (4) gives

$$\begin{aligned}
\left[{}^J \dot{R}_K \right] &= [R_K] \left(\left[\tilde{\omega}_K \right]^T - \left[\tilde{\omega}_J \right]^T \right) [R_J]^T \\
&= \left[{}^J \tilde{\omega}'_K \right]^T [R_K] [R_J]^T \\
\Rightarrow \boxed{\left[{}^J \dot{R}_K \right]} &= \boxed{\left[{}^J \tilde{\omega}'_K \right]^T \left[{}^J R_K \right]} \quad (\text{components of } {}^J \omega_K \text{ are resolved in body } K)
\end{aligned}$$

Note that the results are the same when differentiating the transformation relative to a fixed frame. In summary,

Transformations into a fixed frame R :

$$\boxed{\left[\dot{R}_K \right] = [R_K] \left[{}^R \tilde{\omega}_K \right]^T = \left[{}^R \tilde{\omega}'_K \right]^T [R_K]}$$

$$\boxed{\left[\dot{R}_K \right]^T = \left[{}^R \tilde{\omega}_K \right] [R_K]^T = [R_K]^T \left[{}^R \tilde{\omega}'_K \right]}$$

Transformations into a moving frame J :

$$\boxed{\left[{}^J \dot{R}_K \right] = \left[{}^J R_K \right] \left[{}^J \tilde{\omega}_K \right]^T = \left[{}^J \tilde{\omega}'_K \right]^T \left[{}^J R_K \right]}$$

$$\boxed{\left[{}^J \dot{R}_K \right]^T = \left[{}^J \tilde{\omega}_K \right] \left[{}^J R_K \right]^T = \left[{}^J R_K \right]^T \left[{}^J \tilde{\omega}'_K \right]}$$