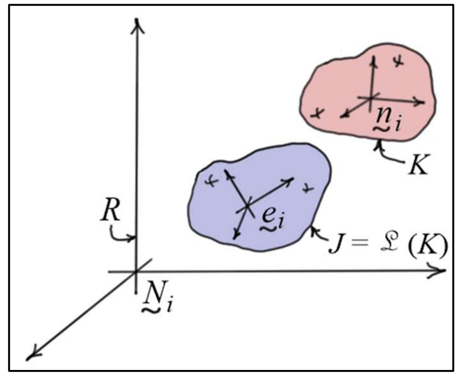
Multibody Dynamics
Time Derivative of Relative Transformation Matrices

Matrix Form of the Derivative of a Vector Fixed in a Rigid Body

Consider two bodies of a multibody system. The unit vector set $\left(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3\right)$ is fixed in body *K*, and the set $\left(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\right)$ is fixed in body *J*. Body *J* is the lower-numbered body of *K* ($J=\mathscr{L}(K)$). Both bodies are moving in a fixed frame $R:(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3)$. If $\underset{\sim}{r}$ is a vector fixed in the body *K*, then, using the summation rule for angular velocities, the time derivative of $\underset{\sim}{r}$ can be written as



$$\boxed{\frac{{}^R d\underset{\sim}{r}}{dt}=\dot{\underset{\sim}{r}}={}^R\underset{\sim}{\omega}_K\times\underset{\sim}{r}=\left({}^R\underset{\sim}{\omega}_J+{}^J\underset{\sim}{\omega}_K\right)\times\underset{\sim}{r}=\left({}^R\underset{\sim}{\omega}_J\times\underset{\sim}{r}\right)+\left({}^J\underset{\sim}{\omega}_K\times\underset{\sim}{r}\right)} \qquad (1)$$

When performing the cross products, the individual vectors and the resulting cross products can be expressed in any reference frame. Two cases are considered below – components of ${}^J\underset{\sim}{\omega}_K$ in body *J* (Case 1) and components of ${}^J\underset{\sim}{\omega}_K$ in body *K* (Case 2). Components of vectors in body *K* have been annotated with a "prime".

The transformation matrices associated with the two bodies ($[R_J]$, $[R_K]$, $\left[{}^J R_K\right]$) are defined by the following equations.

$$\boxed{\{\underset{\sim}{e}\}=[R_J]\{\underset{\sim}{N}\}} \quad \boxed{\{\underset{\sim}{n}\}=[R_K]\{\underset{\sim}{N}\}} \quad \boxed{\{\underset{\sim}{n}\}=\left[{}^J R_K\right]\{\underset{\sim}{e}\}}$$

Case 1:

Let $\dot{\underset{\sim}{r}}$, ${}^R\underset{\sim}{\omega}_J$, and ${}^R\underset{\sim}{\omega}_K$ be expressed in $R:(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3)$, ${}^J\underset{\sim}{\omega}_K$ be expressed in $J:\left(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\right)$, and $\underset{\sim}{r}$ be expressed in $K:\left(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3\right)$. Then,

$$\boxed{\{r\}=[R_K]^T\{r'\}} \quad \boxed{\{\dot{r}\}=\left[\dot{R}_K\right]^T\{r'\}}$$

$$\boxed{{}^R\underset{\sim}{\omega}_K\times\underset{\sim}{r} \;\;\to[\tilde{\omega}_K][R_K]^T\{r'\}} \quad \boxed{{}^R\underset{\sim}{\omega}_J\times\underset{\sim}{r} \;\;\to[\tilde{\omega}_J][R_K]^T\{r'\}}$$

$$\boxed{{}^J\underset{\sim}{\omega}_K\times\underset{\sim}{r} \;\;\to[R_J]^T\left[{}^J\tilde{\omega}_K\right]\left[{}^J R_K\right]^T\{r'\}}$$

Substituting into Eq. (1) gives

$$\boxed{\left[\dot{R}_K\right]^T=[\tilde{\omega}_K][R_K]^T=[\tilde{\omega}_J][R_K]^T+[R_J]^T\left[{}^J\tilde{\omega}_K\right]\left[{}^J R_K\right]^T}$$

$$\Rightarrow\boxed{\left([\tilde{\omega}_K]-[\tilde{\omega}_J]\right)[R_K]^T=[R_J]^T\left[{}^J\tilde{\omega}_K\right]\left[{}^J R_K\right]^T}$$

or

$$\boxed{[R_K]\left([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T\right) = \left[{}^J R_K\right]\left[{}^J\tilde{\omega}_K\right]^T [R_J]} \tag{2}$$

Case 2:

Let $\dot{\underset{\sim}{r}}$, ${}^R\underset{\sim}{\omega}_J$, and ${}^R\underset{\sim}{\omega}_K$ be expressed in $R:(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3)$, and let ${}^J\underset{\sim}{\omega}_K$ and $\underset{\sim}{r}$ be expressed in $K:\left(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3\right)$. Then,

$$\boxed{\{r\} = [R_K]^T\{r'\}} \quad \boxed{\{\dot{r}\} = \left[\dot{R}_K\right]^T\{r'\}}$$

$$\boxed{{}^R\underset{\sim}{\omega}_K \times \underset{\sim}{r} \quad \to [\tilde{\omega}_K][R_K]^T\{r'\}} \quad \boxed{{}^R\underset{\sim}{\omega}_J \times \underset{\sim}{r} \quad \to [\tilde{\omega}_J][R_K]^T\{r'\}}$$

$$\boxed{{}^J\underset{\sim}{\omega}_K \times \underset{\sim}{r} \quad \to [R_K]^T\left[{}^J\tilde{\omega}'_K\right]\{r'\}}$$

Substituting into Eq. (1) gives

$$\boxed{\left[\dot{R}_K\right]^T = [\tilde{\omega}_K][R_K]^T = [\tilde{\omega}_J][R_K]^T + [R_K]^T\left[{}^J\tilde{\omega}'_K\right]}$$

$$\Rightarrow \boxed{\left([\tilde{\omega}_K] - [\tilde{\omega}_J]\right)[R_K]^T = [R_K]^T\left[{}^J\tilde{\omega}'_K\right]}$$

or

$$\boxed{[R_K]\left([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T\right) = \left[{}^J\tilde{\omega}'_K\right]^T [R_K]} \tag{3}$$

Time Derivative of the Transformation Matrices Between Two Body Frames

The above results can be used to determine ***two different forms*** of the ***time derivative*** of the ***relative transformation matrix*** $\left[{}^J R_K\right]$ depending on what reference frame is used to express the relative angular velocity vector ${}^J\underset{\sim}{\omega}_K$. First, note the time derivative of $\left[{}^J R_K\right]$ can be written as follows.

$$\begin{aligned}\left[{}^J\dot{R}_K\right] &= \frac{{}^R d}{dt}\left[{}^J R_K\right] = \frac{{}^R d}{dt}\left([R_K][R_J]^T\right) = \left[\dot{R}_K\right][R_J]^T + [R_K]\left[\dot{R}_J\right]^T \\ &= [R_K][\tilde{\omega}_K]^T[R_J]^T + [R_K][\tilde{\omega}_J][R_J]^T\end{aligned}$$

As a skew-symmetric matrix, $[\tilde{\omega}_J] = -[\tilde{\omega}_J]^T$, so the above equation can be rewritten as

$$\Rightarrow \boxed{\left[{}^J\dot{R}_K\right] = [R_K]\left([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T\right)[R_J]^T} \tag{4}$$

Now, substituting from Eq. (2) into Eq. (4) gives

$$\begin{aligned}\left[{}^J\dot{R}_K\right] &= [R_K]\left([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T\right)[R_J]^T \\ &= \left[{}^J R_K\right]\left[{}^J\tilde{\omega}_K\right]^T[R_J][R_J]^T\end{aligned}$$

$\Rightarrow \boxed{\left[{}^J\dot{R}_K\right] = \left[{}^J R_K\right]\left[{}^J\tilde{\omega}_K\right]^T}$ (components of ${}^J\underset{\sim}{\omega}_K$ are resolved in body J)

And, substituting from Eq. (3) into Eq. (4) gives

$$\left[{}^J\dot{R}_K\right] = [R_K]\left([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T\right)[R_J]^T$$

$$= \left[{}^J\tilde{\omega}'_K\right]^T [R_K][R_J]^T$$

$$\Rightarrow \boxed{\left[{}^J\dot{R}_K\right] = \left[{}^J\tilde{\omega}'_K\right]^T \left[{}^J R_K\right]}$$ (components of ${}^J\underset{\sim}{\omega}_K$ are resolved in body K)