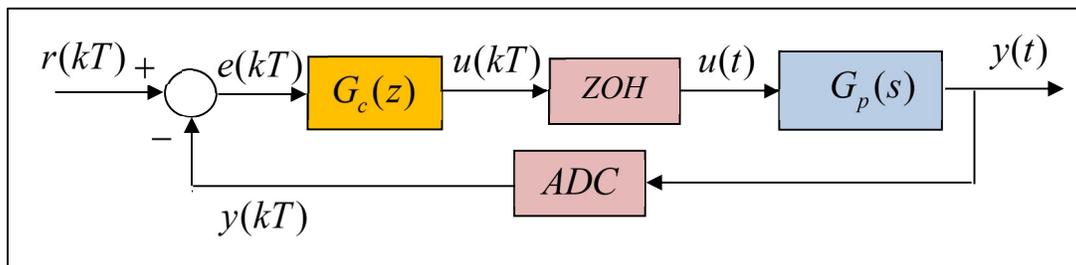


Introductory Motion and Control

Continuous and Equivalent Discrete Transfer Functions

The closed loop control system shown below has a **digital compensator** represented by the discrete transfer function $G_c(z)$ and a **continuous plant** represented by the continuous transfer function $G_p(s)$. The digital control signal ($u(kT)$) is shown to be converted into a continuous signal $u(t)$ using a **zero-order hold (ZOH)**.



Block Diagram of a Continuous/Discrete Closed-Loop System

One common method of finding $G_c(z)$ involves a two-step process.

1. First, design its continuous counterpart $G_c(s)$ using continuous compensator design techniques (root locus and /or Bode diagrams)
2. Transform the resulting transfer function into a discrete form

Some authors refer to this as **emulation**, because the **discrete compensator** emulates its **continuous counterpart**.

Tustin's approximation is one common method used to make this transformation. To **convert** a continuous transfer function $G(s)$ into a discrete transfer function $G(z)$ using Tustin's method, set $s = \frac{2}{T} \left(\frac{z-1}{z+1} \right)$ and

simplify. For example, the continuous phase lead compensator

$$G_c(s) = \frac{3(s+5)}{s+15}$$

becomes

$$G_c(z) = \frac{3 \left(\frac{2}{T} \left(\frac{z-1}{z+1} \right) + 5 \right)}{\frac{2}{T} \left(\frac{z-1}{z+1} \right) + 15} = \frac{3 \left(\frac{2}{T} (z-1) + 5(z+1) \right)}{\frac{2}{T} (z-1) + 15(z+1)} = \frac{\left(15 + \frac{6}{T} \right) z + \left(15 - \frac{6}{T} \right)}{\left(15 + \frac{2}{T} \right) z + \left(15 - \frac{2}{T} \right)}$$

For a sample time $T = 0.001$ (sec), the final form of the compensator is

$$G_c(z) = \frac{6015z - 5985}{2015z - 1985} = \frac{2.9851z - 2.97}{z - 0.9851}$$

It can be shown that when Tustin's approximation is applied to an **integral compensator**, the **trapezoidal rule** is used to approximate the integral.

MATLAB's "c2d" command can also be used to *convert* from *continuous* to *discrete* transfer functions. The methods of conversion include the *Tustin* approximation and a *zero-order hold* on the input to the transfer function.

```
>> num = 3*[1,5];
>> den = [1,15];
>> sys = tf(num,den)

Transfer function:
    3 s + 15
    -----
    s + 15

>> sysD = c2d(sys,0.001,'tustin')

Transfer function:
    2.985 z - 2.97
    -----
    z - 0.9851

Sampling time: 0.001
```

Given the *discrete transfer function*, the compensator can be written as a *difference equation* as follows.

$$G_c(z) = \frac{U(z)}{E(z)} = \frac{6015z - 5985}{2015z - 1985} = \frac{2.985z - 2.97}{z - 0.9851}$$

or

$$(z - 0.9851)U(z) = (2.985z - 2.97)E(z)$$

Multiplying by z^{-1} and solving for $U(z)$ gives

$$(1 - 0.9851z^{-1})U(z) = (2.985 - 2.97z^{-1})E(z)$$

or

$$U(z) = 0.9851z^{-1}U(z) + 2.985E(z) - 2.97z^{-1}E(z)$$

The terms $U(z)$ and $E(z)$ refer to the *compensator output* and the *error* at the *current time*, and the terms $z^{-1}U(z)$ and $z^{-1}E(z)$ refer to the *compensator output* and the *error* at the *previous time step*. Hence, the above equation is *equivalent* to the *difference equation*

$$u(k) = 0.9851 u(k-1) + 2.985 e(k) - 2.97 e(k-1)$$

Another common method is the **MPZ (matched pole-zero)** Method. The **MPZ** method is based on **mapping** the **poles** and **zeros** of the continuous transfer function using the relationship $z = e^{sT}$ and **preserving** the low frequency gain. For example, consider a **phase-lead** or **phase-lag** type **compensator** of the form

$$G_c(s) = K \left(\frac{s+a}{s+b} \right)$$

In this case, the *equivalent discrete* transfer function for a given sample time T is

$$G_c(z) = K' \left(\frac{z - e^{-aT}}{z - e^{-bT}} \right)$$

Here, the constant K' is found by applying the **final value theorems** to each transfer function and equating the results.

$$\lim_{s \rightarrow 0} \left(s \cdot \frac{1}{s} \cdot G_c(s) \right) = K \left(\frac{a}{b} \right) = \lim_{z \rightarrow 1} \left(\left(\frac{1 - z^{-1}}{1 - z^{-1}} \right) \cdot G_c(z) \right) = K' \left(\frac{1 - e^{-aT}}{1 - e^{-bT}} \right)$$

or

$$K' = K \left(\frac{a}{b} \right) \left(\frac{1 - e^{-bT}}{1 - e^{-aT}} \right)$$

The Tustin and *MPZ* methods provide **approximations** of the **original continuous transfer function**. The accuracy of the approximations is usually **application dependent**. **More details** on these methods can be found in Franklin, Powell, and Emami-Naeini, *Feedback Control of Dynamic Systems*, Prentice-Hall, 6th Ed. 2010.