An Introduction to
Three-Dimensional, Rigid Body Dynamics

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Volume III: Introduction to Multibody Kinematics

Unit 3

Angular Acceleration

Summary

This unit focuses on the ***matrix-based*** calculation of ***vector*** components of ***angular acceleration***. The calculations are performed using ***fixed frame*** and ***body frame*** components and are based on ***absolute*** and ***relative coordinates***. Both ***orientation angle derivatives*** and ***angular velocity components*** are used as ***generalized speeds***. As part of the calculations, a procedure for calculating the ***time derivatives*** of ***relative transformation matrices*** (between moving bodies) is also included.

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Introduction

This unit is a ***continuation*** of Unit 2 which covers matrix-based calculations of ***angular velocity vectors*** and ***partial angular velocity matrices***. As in Unit 2, calculations are completed using both ***absolute*** and ***relative coordinates***, vector components are resolved in both ***fixed*** and ***body*** (rotating) ***frames***, and both ***angle derivatives*** and ***angular velocity components*** are used as ***generalized*** speeds.

Angular Acceleration Using Absolute Coordinates

Angular Acceleration using Angle Derivatives as Generalized Speeds and a 1-2-3 Rotation Sequence

As calculated in Unit 2 using absolute coordinates, the ***fixed frame*** and ***body frame*** components of ${}^R\underset{\sim}{\omega}_B$ the ***angular velocity*** of body B in a multibody system can be written in matrix form as follows.

$$\{\omega_B\} \triangleq \begin{Bmatrix}\omega_{B1}\\ \omega_{B2}\\ \omega_{B3}\end{Bmatrix} = \begin{bmatrix}1 & 0 & S_{B2}\\ 0 & C_{B1} & -S_{B1}C_{B2}\\ 0 & S_{B1} & C_{B1}C_{B2}\end{bmatrix}\begin{Bmatrix}\dot{\theta}_{B1}\\ \dot{\theta}_{B2}\\ \dot{\theta}_{B3}\end{Bmatrix} \triangleq \left[{}^R\omega_{B,\dot{\theta}_B}\right]\{\dot{\theta}_B\} \quad \text{(fixed frame components)} \quad (1)$$

$$\{\omega'_B\} \triangleq \begin{Bmatrix}\omega'_{B1}\\ \omega'_{B2}\\ \omega'_{B3}\end{Bmatrix} = \begin{bmatrix}C_{B2}C_{B3} & S_{B3} & 0\\ -C_{B2}S_{B3} & C_{B3} & 0\\ S_{B2} & 0 & 1\end{bmatrix}\begin{Bmatrix}\dot{\theta}_{B1}\\ \dot{\theta}_{B2}\\ \dot{\theta}_{B3}\end{Bmatrix} \triangleq \left[{}^R\omega'_{B,\dot{\theta}_B}\right]\{\dot{\theta}_B\} \quad \text{(body frame components)} \quad (2)$$

Consequently, the ***fixed frame*** and ***body frame*** components of ${}^R\underset{\sim}{\alpha}_B$ the ***angular acceleration*** of the body B can be written in the following matrix forms. Recall that the angular velocity vector can be ***differentiated*** in ***either*** the ***fixed frame*** or the ***body frame*** to find the angular acceleration.

$$\begin{Bmatrix}\alpha_{B1}\\ \alpha_{B2}\\ \alpha_{B3}\end{Bmatrix} = \begin{Bmatrix}\dot{\omega}_{B1}\\ \dot{\omega}_{B2}\\ \dot{\omega}_{B3}\end{Bmatrix} = \left[{}^R\omega_{B,\dot{\theta}_B}\right]\{\ddot{\theta}_B\} + \left[{}^R\dot{\omega}_{B,\dot{\theta}_B}\right]\{\dot{\theta}_B\} \quad \text{(fixed frame components)} \quad (3)$$

$$\begin{Bmatrix}\alpha'_{B1}\\ \alpha'_{B2}\\ \alpha'_{B3}\end{Bmatrix} = \begin{Bmatrix}\dot{\omega}'_{B1}\\ \dot{\omega}'_{B2}\\ \dot{\omega}'_{B3}\end{Bmatrix} = \left[{}^R\omega'_{B,\dot{\theta}_B}\right]\{\ddot{\theta}_B\} + \left[{}^R\dot{\omega}'_{B,\dot{\theta}_B}\right]\{\dot{\theta}_B\} \quad \text{(body frame components)} \quad (4)$$

The ***time derivatives*** of the ***partial angular velocity matrices*** are found by differentiating the elements of the matrices to get the following.

$$\left[{}^R\dot{\omega}_{B,\dot{\theta}_B}\right] = \begin{bmatrix}0 & 0 & C_{B2}\dot{\theta}_{B2}\\ 0 & -S_{B1}\dot{\theta}_{B1} & \left(S_{B1}S_{B2}\dot{\theta}_{B2} - C_{B1}C_{B2}\dot{\theta}_{B1}\right)\\ 0 & C_{B1}\dot{\theta}_{B1} & -\left(S_{B1}C_{B2}\dot{\theta}_{B1} + C_{B1}S_{B2}\dot{\theta}_{B2}\right)\end{bmatrix} \quad \text{(fixed frame components)} \quad (5)$$

$$\left[{}^{R}\dot{\omega}'_{B,\dot{\theta}_B}\right]=\begin{bmatrix}-\left(S_{B2}C_{B3}\dot{\theta}_{B2}+C_{B2}S_{B3}\dot{\theta}_{B3}\right) & C_{B3}\dot{\theta}_{B3} & 0\\ \left(S_{B2}S_{B3}\dot{\theta}_{B2}-C_{B2}C_{B3}\dot{\theta}_{B3}\right) & -S_{B3}\dot{\theta}_{B3} & 0\\ C_{B2}\dot{\theta}_{B2} & 0 & 0\end{bmatrix} \quad \text{(body frame components)} \qquad (6)$$

Angular Acceleration using Angular Velocity Components as Generalized Speeds

If the ***angular velocity*** components are used as ***generalized speeds***, the angular accelerations are much simpler. Recalling that the partial velocity matrices associated with the angular velocity components are 3×3 ***identity matrices***, and that the angular velocity can be differentiated in ***either*** the fixed frame or the body frame, gives the following.

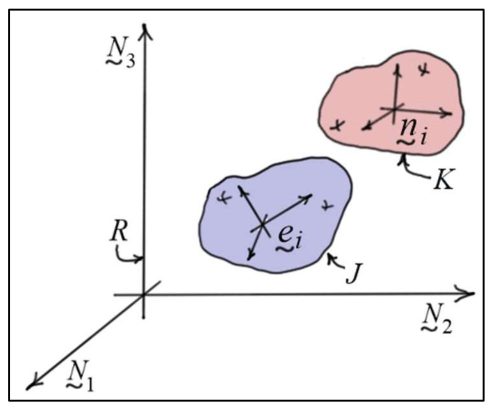
$$\begin{Bmatrix}\alpha_{B1}\\ \alpha_{B2}\\ \alpha_{B3}\end{Bmatrix}=\left[\omega_{B,\omega_B}\right]\left\{\dot{\omega}_B\right\}+\left[\dot{\omega}_{B,\omega_B}\right]\left\{\omega_B\right\}=\underbrace{\left[\omega_{B,\omega_B}\right]}_{\substack{3\times 3\text{ identity}\\ \text{matrix}}}\left\{\dot{\omega}_B\right\}=\left\{\dot{\omega}_B\right\} \quad \text{(fixed frame components)} \qquad (7)$$

$$\begin{Bmatrix}\alpha'_{B1}\\ \alpha'_{B2}\\ \alpha'_{B3}\end{Bmatrix}=\left[\omega'_{B,\omega'_B}\right]\left\{\dot{\omega}'_B\right\}+\left[\dot{\omega}'_{B,\omega'_B}\right]\left\{\omega'_B\right\}=\underbrace{\left[\omega'_{B,\omega'_B}\right]}_{\substack{3\times 3\text{ identity}\\ \text{matrix}}}\left\{\dot{\omega}'_B\right\}=\left\{\dot{\omega}'_B\right\} \quad \text{(body frame components)} \qquad (8)$$

Angular Acceleration Using Relative Coordinates

Time Derivative of Relative Transformation Matrices

Consider two bodies of a multibody system. The unit vector set $K:\left(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3\right)$ is fixed in body *K*, and the set $J:\left(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\right)$ is fixed in body *J*. Both bodies are moving in a fixed frame $R:\left(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3\right)$. If $\underset{\sim}{r}$ is a vector fixed in the body *K*, then, using the ***summation rule*** for angular velocities, the ***time derivative*** of $\underset{\sim}{r}$ can be written as



$$\frac{{}^{R}d\underset{\sim}{r}}{dt}=\dot{\underset{\sim}{r}}={}^{R}\underset{\sim}{\omega}_K\times\underset{\sim}{r}=\left({}^{R}\underset{\sim}{\omega}_J+{}^{J}\underset{\sim}{\omega}_K\right)\times\underset{\sim}{r}=\left({}^{R}\underset{\sim}{\omega}_J\times\underset{\sim}{r}\right)+\left({}^{J}\underset{\sim}{\omega}_K\times\underset{\sim}{r}\right) \qquad (9)$$

When performing the cross products, the individual vectors and the resulting cross products can be expressed in ***any reference frame***. ***Two cases*** are considered below – 1) components of ${}^{J}\underset{\sim}{\omega}_K$ in body *J* (Case 1), and 2) components of ${}^{J}\underset{\sim}{\omega}_K$ in body *K* (Case 2). Components of vectors in body *K* have been annotated with a prime (i.e., " ′ "). The transformation matrices associated with the two bodies ($\left[R_J\right]$, $\left[R_K\right]$, and $\left[{}^{J}R_K\right]$) are defined by the following equations.

$$\begin{Bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{Bmatrix} = [R_J] \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix} \qquad \begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix} = [R_K] \begin{Bmatrix} \underset{\sim}{N}_1 \\ \underset{\sim}{N}_2 \\ \underset{\sim}{N}_3 \end{Bmatrix} \qquad \begin{Bmatrix} \underset{\sim}{n}_1 \\ \underset{\sim}{n}_2 \\ \underset{\sim}{n}_3 \end{Bmatrix} = \left[{}^J R_K \right] \begin{Bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{Bmatrix}$$

Case 1:

Let $\dot{\underset{\sim}{r}}$, ${}^R\underset{\sim}{\omega}_J$, and ${}^R\underset{\sim}{\omega}_K$ be expressed in $R:\left(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3\right)$, ${}^J\underset{\sim}{\omega}_K$ be expressed in $J:\left(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\right)$, and $\underset{\sim}{r}$ be expressed in $K:\left(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3\right)$. The ***fixed frame*** components of $\underset{\sim}{r}$ and $\dot{\underset{\sim}{r}}$ can then be written as follows. Note that because $\underset{\sim}{r}$ is fixed in body *K*, its components $\{r'\}$ in that frame are ***constant***.

$$\{r\} = [R_K]^T \{r'\} \qquad \{\dot{r}\} = \left[\dot{R}_K\right]^T \{r'\} \tag{10}$$

Using the ***fixed frame components*** of ${}^R\underset{\sim}{\omega}_J$ and ${}^R\underset{\sim}{\omega}_K$ and the ***body J*** (base frame) ***components*** of ${}^J\underset{\sim}{\omega}_K$, the ***fixed frame components*** of the ***cross products*** in Equation (9) can be written as follows.

$${}^R\underset{\sim}{\omega}_K \times \underset{\sim}{r} \;\; \rightarrow \;\; [\tilde{\omega}_K][R_K]^T \{r'\} \qquad {}^R\underset{\sim}{\omega}_J \times \underset{\sim}{r} \;\; \rightarrow \;\; [\tilde{\omega}_J][R_K]^T \{r'\} \tag{11}$$

$${}^J\underset{\sim}{\omega}_K \times \underset{\sim}{r} \;\; \rightarrow \;\; [R_J]^T \left[{}^J\tilde{\omega}_K\right]\left[{}^J R_K\right]^T \{r'\} \tag{12}$$

Substituting from Equations (10) through (12) into Equation (9) gives

$$\left[\dot{R}_K\right]^T \{r'\} = [\tilde{\omega}_K][R_K]^T \{r'\} = [\tilde{\omega}_J][R_K]^T \{r'\} + [R_J]^T \left[{}^J\tilde{\omega}_K\right]\left[{}^J R_K\right]^T \{r'\}$$

The coefficient matrices of $\{r'\}$ on each side of the equation must be the same, so

$$[\tilde{\omega}_K][R_K]^T = [\tilde{\omega}_J][R_K]^T + [R_J]^T \left[{}^J\tilde{\omega}_K\right]\left[{}^J R_K\right]^T$$

$$\Rightarrow \left([\tilde{\omega}_K] - [\tilde{\omega}_J]\right)[R_K]^T = [R_J]^T \left[{}^J\tilde{\omega}_K\right]\left[{}^J R_K\right]^T$$

Taking the transpose of both sides of this result gives

$$[R_K]\left([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T\right) = \left[{}^J R_K\right]\left[{}^J\tilde{\omega}_K\right]^T [R_J] \tag{13}$$

Case 2:

Let $\dot{\underset{\sim}{r}}$, ${}^R\underset{\sim}{\omega}_J$, and ${}^R\underset{\sim}{\omega}_K$ be expressed in $R:\left(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3\right)$, and let ${}^J\underset{\sim}{\omega}_K$ and $\underset{\sim}{r}$ be expressed in $K:\left(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3\right)$. As in Case 1, the ***fixed frame*** components of $\underset{\sim}{r}$ and $\dot{\underset{\sim}{r}}$ can be written as follows.

$$\{r\} = [R_K]^T \{r'\} \qquad \{\dot{r}\} = \left[\dot{R}_K\right]^T \{r'\}$$

Using the ***fixed frame components*** of ${}^R\underset{\sim}{\omega}_J$ and ${}^R\underset{\sim}{\omega}_K$ and the ***body*** *K* ***components*** of ${}^J\underset{\sim}{\omega}_K$, the ***fixed frame components*** of the cross products in Equation (9) can be written as follows.

$$\boxed{{}^{R}\underset{\sim}{\omega}_K \times \underset{\sim}{r} \;\rightarrow\; [\tilde{\omega}_K][R_K]^T\{r'\}} \quad \boxed{{}^{R}\underset{\sim}{\omega}_J \times \underset{\sim}{r} \;\rightarrow\; [\tilde{\omega}_J][R_K]^T\{r'\}} \quad \boxed{{}^{J}\underset{\sim}{\omega}_K \times \underset{\sim}{r} \;\rightarrow\; [R_K]^T\left[{}^{J}\tilde{\omega}'_K\right]\{r'\}}$$

Substituting these results into Equation (9) and comparing the coefficient matrices of $\{r'\}$ on each side of the equation gives the following.

$$[\tilde{\omega}_K][R_K]^T = [\tilde{\omega}_J][R_K]^T + [R_K]^T\left[{}^{J}\tilde{\omega}'_K\right] \qquad \Rightarrow \left([\tilde{\omega}_K]-[\tilde{\omega}_J]\right)[R_K]^T = [R_K]^T\left[{}^{J}\tilde{\omega}'_K\right]$$

Taking the ***transpose*** of both sides of this result gives

$$\boxed{[R_K]\left([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T\right) = \left[{}^{J}\tilde{\omega}'_K\right]^T[R_K]} \qquad (14)$$

The results in Equations (13) and (14) can be used to determine ***two different forms*** of the ***time derivative*** of the ***relative transformation matrix*** $\left[{}^{J}R_K\right]$ depending on which reference frame is used to express the relative angular velocity vector ${}^{J}\underset{\sim}{\omega}_K$. First, recall from Unit 1 of this volume that $\left[{}^{J}R_K\right]$ can be written as follows.

$$\left[{}^{J}R_K\right] = [R_K][R_J]^T$$

So, the time derivative of $\left[{}^{J}R_K\right]$ can be calculated in terms of the derivatives of $[R_K]$ and $[R_J]^T$ which were calculated in Unit 1.

Using this approach, $\left[{}^{J}\dot{R}_K\right]$ can be calculated as follows.

$$\left[{}^{J}\dot{R}_K\right] = \frac{d}{dt}\left[{}^{J}R_K\right] = \frac{d}{dt}\left([R_K][R_J]^T\right) = \left[\dot{R}_K\right][R_J]^T + [R_K]\left[\dot{R}_J\right]^T$$
$$= [R_K][\tilde{\omega}_K]^T[R_J]^T + [R_K][\tilde{\omega}_J][R_J]^T$$

The skew-symmetric matrix, $[\tilde{\omega}_J] = -[\tilde{\omega}_J]^T$, so the above equation can be rewritten as follows.

$$\Rightarrow \boxed{\left[{}^{J}\dot{R}_K\right] = [R_K]\left([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T\right)[R_J]^T} \qquad (15)$$

Now, substituting from Equation (13) into Equation (15) gives

$$\left[{}^{J}\dot{R}_K\right] = [R_K]\left([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T\right)[R_J]^T = \left[{}^{J}R_K\right]\left[{}^{J}\tilde{\omega}_K\right]^T \underbrace{[R_J][R_J]^T}_{\text{identity matrix}}$$

$$\Rightarrow \boxed{\left[{}^{J}\dot{R}_K\right] = \left[{}^{J}R_K\right]\left[{}^{J}\tilde{\omega}_K\right]^T} \quad \text{(components of } {}^{J}\underset{\sim}{\omega}_K \text{ are resolved in } \textbf{\textit{body}}\ J\text{)} \qquad (16)$$

And, substituting from Equation (14) into Equation (15) gives

$$\left[{}^{J}\dot{R}_K\right] = [R_K]\left([\tilde{\omega}_K]^T - [\tilde{\omega}_J]^T\right)[R_J]^T = \left[{}^{J}\tilde{\omega}'_K\right]^T[R_K][R_J]^T$$

$$\Rightarrow \boxed{\left[{}^{J}\dot{R}_K\right] = \left[{}^{J}\tilde{\omega}'_K\right]^T\left[{}^{J}R_K\right]} \quad \text{(components of } {}^{J}\underset{\sim}{\omega}_K \text{ are resolved in } \textbf{\textit{body}}\ K\text{)} \qquad (17)$$

In Equation (16), the matrix $\left[{}^J\tilde{\omega}_K\right]$ is constructed using the ***body*** J components of the ***relative angular velocity*** vector ${}^J\underset{\sim}{\omega}_K$ $\left(\text{written here as } \hat{\omega}_{Ki}\ (i=1,2,3)\right)$, and in Equation (17) the matrix $\left[{}^J\tilde{\omega}'_K\right]$ is constructed using the ***body*** K components of ${}^J\underset{\sim}{\omega}_K$ $\left(\text{written here as } \hat{\omega}'_{Ki}\ (i=1,2,3)\right)$. That is,

$$\left[{}^J\tilde{\omega}_K\right]=\begin{bmatrix}0 & -\hat{\omega}_{K3} & \hat{\omega}_{K2}\\ \hat{\omega}_{K3} & 0 & -\hat{\omega}_{K1}\\ -\hat{\omega}_{K2} & \hat{\omega}_{K1} & 0\end{bmatrix} \quad \text{and} \quad \left[{}^J\tilde{\omega}'_K\right]=\begin{bmatrix}0 & -\hat{\omega}'_{K3} & \hat{\omega}'_{K2}\\ \hat{\omega}'_{K3} & 0 & -\hat{\omega}'_{K1}\\ -\hat{\omega}'_{K2} & \hat{\omega}'_{K1} & 0\end{bmatrix} \qquad (18)$$

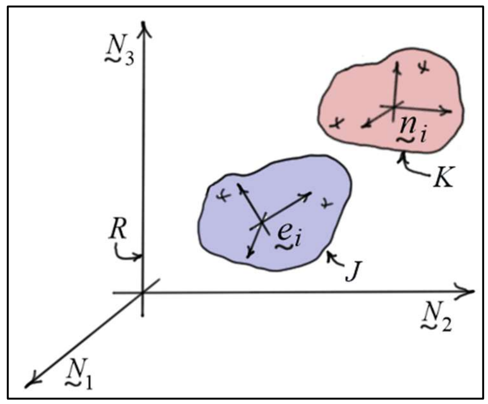
Body J Components   Body K Components

Note:

The results presented in Equations (16) and (17) for the time derivatives of ***relative transformation matrices*** are ***identical*** to those found for ***transformation matrices*** between ***moving bodies*** and a ***fixed frame*** as presented in Unit 1 of this volume.

Angular Acceleration using Angle Derivatives as Generalized Speeds and a 1-2-3 Rotation Sequence

Consider again the two-body system. As before, the ***orientation*** of ***body*** J is measured ***relative*** to the ***fixed frame*** R and the orientation of ***body*** K is measured ***relative*** to ***body*** J. The orientations are described by 1-2-3 ***orientation angle*** sequences. The next two sections cover calculation of the ***angular accelerations*** of the bodies using ***fixed frame*** and ***body frame*** components.



Fixed Frame Components:

As noted in Unit 2, the ***fixed frame*** components of the ***angular velocity*** of the body J can be written in terms of a set of orientation angles as follows.

$$\{\omega_J\}=\begin{bmatrix}1 & 0 & S_{J2}\\ 0 & C_{J1} & -S_{J1}C_{J2}\\ 0 & S_{J1} & C_{J1}C_{J2}\end{bmatrix}\begin{Bmatrix}\dot{\theta}_{J1}\\ \dot{\theta}_{J2}\\ \dot{\theta}_{J3}\end{Bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{Bmatrix}\dot{\theta}_{K1}\\ \dot{\theta}_{K2}\\ \dot{\theta}_{K3}\end{Bmatrix}$$
$$\triangleq\left[{}^R\omega_{J,\dot{\theta}_J}\right]\left\{\dot{\theta}_J\right\}+\left[{}^R\omega_{J,\dot{\theta}_K}\right]\left\{\dot{\theta}_K\right\}$$

Using these results, the ***fixed frame*** components of the ***angular accelerations*** of the body J can be written as follows.

$$\{\alpha_J\}=\{\dot{\omega}_J\}=\left[{}^R\omega_{J,\dot{\theta}_J}\right]\left\{\ddot{\theta}_J\right\}+\left[{}^R\dot{\omega}_{J,\dot{\theta}_J}\right]\left\{\dot{\theta}_J\right\} \qquad \text{(fixed frame components)} \qquad (19)$$

Here,

$$\left[{}^R\dot{\omega}_{J,\dot{\theta}_J}\right]=\frac{d}{dt}\begin{bmatrix}1 & 0 & S_{J2}\\ 0 & C_{J1} & -S_{J1}C_{J2}\\ 0 & S_{J1} & C_{J1}C_{J2}\end{bmatrix}=\begin{bmatrix}0 & 0 & C_{J2}\dot{\theta}_{J2}\\ 0 & -S_{J1}\dot{\theta}_{J1} & \left(S_{J1}S_{J2}\dot{\theta}_{J2}-C_{J1}C_{J2}\dot{\theta}_{J1}\right)\\ 0 & C_{J1}\dot{\theta}_{J1} & -\left(S_{J1}C_{J2}\dot{\theta}_{J1}+C_{J1}S_{J2}\dot{\theta}_{J2}\right)\end{bmatrix} \tag{20}$$

Using the ***summation rule*** for ***angular velocities***, ${}^R\underset{\sim}{\omega}_K$ the angular velocity of body K in R can be written in terms of ${}^R\underset{\sim}{\omega}_J$ the angular velocity of body J in R as follows.

$${}^R\underset{\sim}{\omega}_K={}^R\underset{\sim}{\omega}_J+{}^J\underset{\sim}{\omega}_K\triangleq{}^R\underset{\sim}{\omega}_J+\underset{\sim}{\hat{\omega}}_K \tag{21}$$

Using ***fixed frame components*** for ${}^R\underset{\sim}{\omega}_K$, the ***fixed frame components*** for ${}^R\underset{\sim}{\omega}_J$, and the ***body*** J (base frame) ***components*** for ${}^J\underset{\sim}{\omega}_K$, Equation (21) can be written in the following component form.

$$\{\omega_K\}=\{\omega_J\}+[R_J]^T\{\hat{\omega}_K\} \tag{22}$$

The ***fixed frame components*** of ${}^R\underset{\sim}{\alpha}_K$ the ***angular acceleration*** of body K can be found by ***differentiating*** Equation (22) as follows.

$$\{\alpha_K\}=\{\dot{\omega}_K\}=\{\dot{\omega}_J\}+[R_J]^T\{\dot{\hat{\omega}}_K\}+\left[\dot{R}_J\right]^T\{\hat{\omega}_K\}$$

$$\Rightarrow\{\alpha_K\}=\{\alpha_J\}+[R_J]^T\{\dot{\hat{\omega}}_K\}+[\tilde{\omega}_J][R_J]^T\{\hat{\omega}_K\} \tag{23}$$

This equation can be used to calculate $\{\alpha_K\}$ the ***fixed frame components*** of the ***angular acceleration*** of body K using the results for $\{\alpha_J\}$ the ***fixed frame components*** of the ***angular acceleration*** of body J.

The ***time derivatives*** of the body J frame components of ${}^J\underset{\sim}{\omega}_K\triangleq\underset{\sim}{\hat{\omega}}_K$ can be calculated as follows.

$$\{\hat{\omega}_K\}=\begin{bmatrix}1 & 0 & S_{K2}\\ 0 & C_{K1} & -S_{K1}C_{K2}\\ 0 & S_{K1} & C_{K1}C_{K2}\end{bmatrix}\left\{\dot{\hat{\theta}}_K\right\} \tag{24}$$

$$\left\{\dot{\hat{\omega}}_K\right\}=\frac{d}{dt}\left(\begin{bmatrix}1 & 0 & S_{K2}\\ 0 & C_{K1} & -S_{K1}C_{K2}\\ 0 & S_{K1} & C_{K1}C_{K2}\end{bmatrix}\left\{\dot{\hat{\theta}}_K\right\}\right)$$

$$\Rightarrow\left\{\dot{\hat{\omega}}_K\right\}=\begin{bmatrix}1 & 0 & S_{K2}\\ 0 & C_{K1} & -S_{K1}C_{K2}\\ 0 & S_{K1} & C_{K1}C_{K2}\end{bmatrix}\left\{\ddot{\hat{\theta}}_K\right\}+\begin{bmatrix}0 & 0 & C_{K2}\dot{\hat{\theta}}_{K2}\\ 0 & -S_{K1}\dot{\hat{\theta}}_{K1} & -C_{K1}C_{K2}\dot{\hat{\theta}}_{K1}+S_{K1}S_{K2}\dot{\hat{\theta}}_{K2}\\ 0 & C_{K1}\dot{\hat{\theta}}_{K1} & -S_{K1}C_{K2}\dot{\hat{\theta}}_{K1}-C_{K1}S_{K2}\dot{\hat{\theta}}_{K2}\end{bmatrix}\left\{\dot{\hat{\theta}}_K\right\} \tag{25}$$

Substituting from Equations (24) and (25) into Equation (23) gives an expression for the ***fixed frame components*** of ${}^R\underset{\sim}{\alpha}_K$ the ***angular acceleration*** of ***body*** K in terms of ${}^R\underset{\sim}{\alpha}_J$ the ***angular acceleration*** of ***body*** J.

Contrast this approach to ***differentiating*** the expression for ${}^R\underset{\sim}{\omega}_K$ in terms of the partial angular velocity matrices. In that case, write

$$\{\alpha_K\} = \{\dot{\omega}_K\} = \frac{d}{dt}\left(\left[{}^R\omega_{K,\dot{\theta}_J}\right]\{\dot{\theta}_J\} + \left[{}^R\omega_{K,\dot{\hat{\theta}}_K}\right]\{\dot{\hat{\theta}}_K\}\right)$$

$$\Rightarrow \boxed{\{\alpha_K\} = \left[{}^R\omega_{K,\dot{\theta}_J}\right]\{\ddot{\theta}_J\} + \left[{}^R\dot{\omega}_{K,\dot{\theta}_J}\right]\{\dot{\theta}_J\} + \left[{}^R\omega_{K,\dot{\hat{\theta}}_K}\right]\{\ddot{\hat{\theta}}_K\} + \left[{}^R\dot{\omega}_{K,\dot{\hat{\theta}}_K}\right]\{\dot{\hat{\theta}}_K\}} \qquad (26)$$

Here, using the results from Unit 2, the partial angular velocity matrices are as follows.

$$\boxed{\left[{}^R\omega_{K,\dot{\theta}_J}\right] = \begin{bmatrix} 1 & 0 & S_{J2} \\ 0 & C_{J1} & -S_{J1}C_{J2} \\ 0 & S_{J1} & C_{J1}C_{J2} \end{bmatrix}} \quad \boxed{\left[{}^R\omega_{K,\dot{\hat{\theta}}_K}\right] = [R_J]^T \begin{bmatrix} 1 & 0 & S_{K2} \\ 0 & C_{K1} & -S_{K1}C_{K2} \\ 0 & S_{K1} & C_{K1}C_{K2} \end{bmatrix}} \qquad (27)$$

Differentiating these expressions gives the following.

$$\boxed{\left[{}^R\dot{\omega}_{K,\dot{\theta}_J}\right] = \frac{d}{dt}\begin{bmatrix} 1 & 0 & S_{J2} \\ 0 & C_{J1} & -S_{J1}C_{J2} \\ 0 & S_{J1} & C_{J1}C_{J2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & C_{J2}\dot{\hat{\theta}}_{J2} \\ 0 & -S_{J1}\dot{\hat{\theta}}_{J1} & -C_{J1}C_{J2}\dot{\hat{\theta}}_{J1} + S_{J1}S_{J2}\dot{\hat{\theta}}_{J2} \\ 0 & C_{J1}\dot{\hat{\theta}}_{J1} & -S_{J1}C_{J2}\dot{\hat{\theta}}_{J1} - C_{J1}S_{J2}\dot{\hat{\theta}}_{J2} \end{bmatrix}} \qquad (28)$$

$$\boxed{\begin{aligned} \left[{}^R\dot{\omega}_{K,\dot{\hat{\theta}}_K}\right] &= \left[\dot{R}_J\right]^T \begin{bmatrix} 1 & 0 & S_{K2} \\ 0 & C_{K1} & -S_{K1}C_{K2} \\ 0 & S_{K1} & C_{K1}C_{K2} \end{bmatrix} + [R_J]^T \frac{d}{dt}\begin{bmatrix} 1 & 0 & S_{K2} \\ 0 & C_{K1} & -S_{K1}C_{K2} \\ 0 & S_{K1} & C_{K1}C_{K2} \end{bmatrix} \\ &= [\tilde{\omega}_J][R_J]^T \begin{bmatrix} 1 & 0 & S_{K2} \\ 0 & C_{K1} & -S_{K1}C_{K2} \\ 0 & S_{K1} & C_{K1}C_{K2} \end{bmatrix} + [R_J]^T \begin{bmatrix} 0 & 0 & C_{K2}\dot{\hat{\theta}}_{K2} \\ 0 & -S_{K1}\dot{\hat{\theta}}_{K1} & \left(S_{K1}S_{K2}\dot{\hat{\theta}}_{K2} - C_{K1}C_{K2}\dot{\hat{\theta}}_{K1}\right) \\ 0 & C_{K1}\dot{\hat{\theta}}_{K1} & -\left(S_{K1}C_{K2}\dot{\hat{\theta}}_{K1} + C_{K1}S_{K2}\dot{\hat{\theta}}_{K2}\right) \end{bmatrix} \end{aligned}} \qquad (29)$$

Substituting details into Equations (23) and (26), gives the ***same results*** for $\{\alpha_K\}$ as shown below in Equation (30).

$$\{\alpha_K\} = \begin{bmatrix} 1 & 0 & S_{J2} \\ 0 & C_{J1} & -S_{J1}C_{J2} \\ 0 & S_{J1} & C_{J1}C_{J2} \end{bmatrix} \{\ddot{\theta}_J\} + \begin{bmatrix} 0 & 0 & C_{J2}\dot{\theta}_{J2} \\ 0 & -S_{J1}\dot{\theta}_{J1} & \left(S_{J1}S_{J2}\dot{\theta}_{J2} - C_{J1}C_{J2}\dot{\theta}_{J1}\right) \\ 0 & C_{J1}\dot{\theta}_{J1} & -\left(S_{J1}C_{J2}\dot{\theta}_{J1} + C_{J1}S_{J2}\dot{\theta}_{J2}\right) \end{bmatrix} \{\dot{\theta}_J\}$$

$$+ [R_J]^T \left(\begin{bmatrix} 1 & 0 & S_{K2} \\ 0 & C_{K1} & -S_{K1}C_{K2} \\ 0 & S_{K1} & C_{K1}C_{K2} \end{bmatrix} \{\ddot{\hat{\theta}}_K\} + \begin{bmatrix} 0 & 0 & C_{K2}\dot{\hat{\theta}}_{K2} \\ 0 & -S_{K1}\dot{\hat{\theta}}_{K1} & -C_{K1}C_{K2}\dot{\hat{\theta}}_{K1} + S_{K1}S_{K2}\dot{\hat{\theta}}_{K2} \\ 0 & C_{K1}\dot{\hat{\theta}}_{K1} & -S_{K1}C_{K2}\dot{\hat{\theta}}_{K1} - C_{K1}S_{K2}\dot{\hat{\theta}}_{K2} \end{bmatrix} \{\dot{\hat{\theta}}_K\} \right) \quad (30)$$

$$+ [\tilde{\omega}_J][R_J]^T \begin{bmatrix} 1 & 0 & S_{K2} \\ 0 & C_{K1} & -S_{K1}C_{K2} \\ 0 & S_{K1} & C_{K1}C_{K2} \end{bmatrix} \{\dot{\hat{\theta}}_K\}$$

Body Frame Components:

As noted in Unit 2, the ***body frame*** components of the ***angular velocity*** of the body J can be written in terms of a set of 1-2-3 orientation angles as follows.

$$\{\omega'_J\} = \begin{bmatrix} C_{J2}C_{J3} & S_{J3} & 0 \\ -C_{J2}S_{J3} & C_{J3} & 0 \\ S_{J2} & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_{J1} \\ \dot{\theta}_{J2} \\ \dot{\theta}_{J3} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\hat{\theta}}_{K1} \\ \dot{\hat{\theta}}_{K2} \\ \dot{\hat{\theta}}_{K3} \end{Bmatrix} \quad \text{(body } J \text{ components)}$$

$$\triangleq \left[{}^R\omega'_{J,\dot{\theta}_J}\right]\{\dot{\theta}_J\} + \left[{}^R\omega'_{J,\dot{\hat{\theta}}_K}\right]\{\dot{\hat{\theta}}_K\}$$

The body J components of the angular acceleration of body J can be found by differentiating this result to give the following.

$$\{\alpha'_J\} = \{\dot{\omega}'_J\} = \left[{}^R\omega'_{J,\dot{\theta}_J}\right]\{\ddot{\theta}_J\} + \left[{}^R\dot{\omega}'_{J,\dot{\theta}_J}\right]\{\dot{\theta}_J\} \quad \text{(body frame components)} \quad (31)$$

Here,

$$\left[{}^R\dot{\omega}'_{J,\dot{\theta}_J}\right] = \frac{d}{dt}\begin{bmatrix} C_{J2}C_{J3} & S_{J3} & 0 \\ -C_{J2}S_{J3} & C_{J3} & 0 \\ S_{J2} & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\left(S_{J2}C_{J3}\dot{\theta}_{J2} + C_{J2}S_{J3}\dot{\theta}_{J3}\right) & C_{J3}\dot{\theta}_{J3} & 0 \\ S_{J2}S_{J3}\dot{\theta}_{J2} - C_{J2}C_{J3}\dot{\theta}_{J3} & -S_{J3}\dot{\theta}_{J3} & 0 \\ C_{J2}\dot{\theta}_{J2} & 0 & 0 \end{bmatrix} \quad (32)$$

Using ***body*** K ***frame*** components for ${}^R\underset{\sim}{\omega}_K$, ***body*** K ***frame*** components for ${}^J\underset{\sim}{\omega}_K$, and ***body*** J ***frame*** components for ${}^R\underset{\sim}{\omega}_J$, Equation (21) can be written in the following component form.

$$\{\omega'_K\} = \left[{}^J R_K\right]\{\omega'_J\} + \{\hat{\omega}'_K\} \quad (33)$$

The ***body frame components*** of ${}^R\underset{\sim}{\alpha}_K$ the ***angular acceleration*** of body K can be found by ***differentiating*** Equation (33) as follows.

$$\{\alpha'_K\} = \{\dot{\omega}'_K\} = \left[{}^J R_K\right]\{\dot{\omega}'_J\} + \left[{}^J \dot{R}_K\right]\{\omega'_J\} + \{\dot{\hat{\omega}}'_K\}$$

$$\Rightarrow \boxed{\{\alpha'_K\} = \left[{}^J R_K\right]\{\alpha'_J\} + \left[{}^J \tilde{\omega}'_K\right]^T \left[{}^J R_K\right]\{\omega'_J\} + \{\dot{\hat{\omega}}'_K\}} \tag{34}$$

This equation can be used to calculate $\{\alpha'_K\}$ the ***body frame components*** of the ***angular acceleration*** of body K using the results for $\{\alpha'_J\}$ the ***body frame components*** of the ***angular acceleration*** of body J.

The ***time derivatives*** of the ***body*** K ***frame components*** of ${}^J\underset{\sim}{\omega}_K \triangleq \underset{\sim}{\hat{\omega}}_K$ can be calculated as follows.

$$\{\dot{\hat{\omega}}'_K\} = \frac{d}{dt}\left(\begin{bmatrix} C_{K2}C_{K3} & S_{K3} & 0 \\ -C_{K2}S_{K3} & C_{K3} & 0 \\ S_{K2} & 0 & 1 \end{bmatrix}\{\dot{\hat{\theta}}_K\}\right)$$

$$\boxed{\{\dot{\hat{\omega}}'_K\} = \begin{bmatrix} C_{K2}C_{K3} & S_{K3} & 0 \\ -C_{K2}S_{K3} & C_{K3} & 0 \\ S_{K2} & 0 & 1 \end{bmatrix}\{\ddot{\hat{\theta}}_K\} + \begin{bmatrix} -\left(S_{K2}C_{K3}\dot{\hat{\theta}}_{K2} + C_{K2}S_{K3}\dot{\hat{\theta}}_{K3}\right) & C_{K3}\dot{\hat{\theta}}_{K3} & 0 \\ S_{K2}S_{K3}\dot{\hat{\theta}}_{K2} - C_{K2}C_{K3}\dot{\hat{\theta}}_{K3} & -S_{K3}\dot{\hat{\theta}}_{K3} & 0 \\ C_{K2}\dot{\hat{\theta}}_{K2} & 0 & 0 \end{bmatrix}\{\dot{\hat{\theta}}_K\}} \tag{35}$$

Contrast this approach to ***differentiating*** the expression for ${}^R\underset{\sim}{\omega}_K$ in terms of the partial angular velocity matrices. In that case, write

$$\begin{aligned} \{\omega'_K\} &= \left[{}^J R_K\right]\begin{bmatrix} C_{J2}C_{J3} & S_{J3} & 0 \\ -C_{J2}S_{J3} & C_{J3} & 0 \\ S_{J2} & 0 & 1 \end{bmatrix}\begin{Bmatrix} \dot{\theta}_{J1} \\ \dot{\theta}_{J2} \\ \dot{\theta}_{J3} \end{Bmatrix} + \begin{bmatrix} C_{K2}C_{K3} & S_{K3} & 0 \\ -C_{K2}S_{K3} & C_{K3} & 0 \\ S_{K2} & 0 & 1 \end{bmatrix}\begin{Bmatrix} \dot{\hat{\theta}}_{K1} \\ \dot{\hat{\theta}}_{K2} \\ \dot{\hat{\theta}}_{K3} \end{Bmatrix} \quad \text{(body } K \text{ components)} \\ &\triangleq \left[{}^R\omega'_{K,\dot{\theta}_J}\right]\{\dot{\theta}_J\} + \left[{}^R\omega'_{K,\dot{\hat{\theta}}_K}\right]\{\dot{\hat{\theta}}_K\} \end{aligned}$$

Using these results, the ***body frame*** components of the ***angular accelerations*** of the bodies can be calculated as follows.

$$\begin{aligned} \{\alpha'_K\} &= \{\dot{\omega}'_K\} \\ &= \left[{}^R\omega'_{K,\dot{\theta}_J}\right]\{\ddot{\theta}_J\} + \left[{}^R\omega'_{K,\dot{\hat{\theta}}_K}\right]\{\ddot{\hat{\theta}}_K\} + \left[{}^R\dot{\omega}'_{K,\dot{\theta}_J}\right]\{\dot{\theta}_J\} + \left[{}^R\dot{\omega}'_{K,\dot{\hat{\theta}}_K}\right]\{\dot{\hat{\theta}}_K\} \end{aligned} \tag{36}$$

Here,

$$\left[{}^R\dot{\omega}'_{K,\dot{\hat{\theta}}_K}\right]=\frac{d}{dt}\begin{bmatrix} C_{K2}C_{K3} & S_{K3} & 0 \\ -C_{K2}S_{K3} & C_{K3} & 0 \\ S_{K2} & 0 & 1 \end{bmatrix}=\begin{bmatrix} -\left(S_{K2}C_{K3}\dot{\hat{\theta}}_{K2}+C_{K2}S_{K3}\dot{\hat{\theta}}_{K3}\right) & C_{K3}\dot{\hat{\theta}}_{K3} & 0 \\ S_{K2}S_{K3}\dot{\hat{\theta}}_{K2}-C_{K2}C_{K3}\dot{\hat{\theta}}_{K3} & -S_{K3}\dot{\hat{\theta}}_{K3} & 0 \\ C_{K2}\dot{\hat{\theta}}_{K2} & 0 & 0 \end{bmatrix} \qquad (37)$$

$$\left[{}^R\dot{\omega}'_{K,\dot{\theta}_J}\right]=\frac{d}{dt}\left(\left[{}^JR_K\right]\left[{}^R\omega'_{J,\dot{\theta}_J}\right]\right)=\left[{}^J\dot{R}_K\right]\begin{bmatrix} C_{J2}C_{J3} & S_{J3} & 0 \\ -C_{J2}S_{J3} & C_{J3} & 0 \\ S_{J2} & 0 & 1 \end{bmatrix}+\left[{}^JR_K\right]\left[{}^R\dot{\omega}'_{J,\dot{\theta}_J}\right]$$

Or,

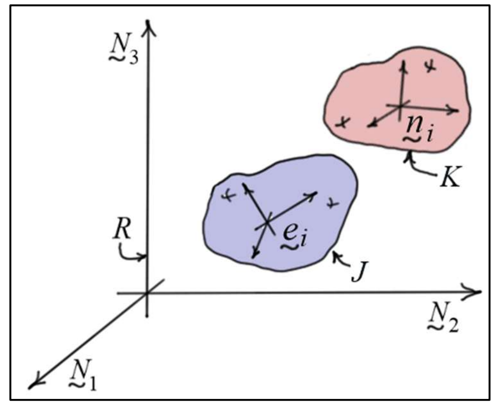
$$\left[{}^R\dot{\omega}'_{K,\dot{\theta}_J}\right]=\left[{}^J\tilde{\omega}'_K\right]^T\left[{}^JR_K\right]\begin{bmatrix} C_{J2}C_{J3} & S_{J3} & 0 \\ -C_{J2}S_{J3} & C_{J3} & 0 \\ S_{J2} & 0 & 1 \end{bmatrix}+\left[{}^JR_K\right]\begin{bmatrix} -\left(S_{J2}C_{J3}\dot{\theta}_{J2}+C_{J2}S_{J3}\dot{\theta}_{J3}\right) & C_{J3}\dot{\theta}_{J3} & 0 \\ S_{J2}S_{J3}\dot{\theta}_{J2}-C_{J2}C_{J3}\dot{\theta}_{J3} & -S_{J3}\dot{\theta}_{J3} & 0 \\ C_{J2}\dot{\theta}_{J2} & 0 & 0 \end{bmatrix} \qquad (38)$$

Substituting details into Equations (34) and (36), gives the ***same results*** for $\left\{\alpha'_K\right\}$ as shown below in Equation (39) below.

$$\begin{aligned}\left\{\alpha'_K\right\}=&\left[{}^JR_K\right]\begin{bmatrix} C_{J2}C_{J3} & S_{J3} & 0 \\ -C_{J2}S_{J3} & C_{J3} & 0 \\ S_{J2} & 0 & 1 \end{bmatrix}\left\{\ddot{\theta}_J\right\}+\left[{}^JR_K\right]\begin{bmatrix} -\left(S_{J2}C_{J3}\dot{\theta}_{J2}+C_{J2}S_{J3}\dot{\theta}_{J3}\right) & C_{J3}\dot{\theta}_{J3} & 0 \\ S_{J2}S_{J3}\dot{\theta}_{J2}-C_{J2}C_{J3}\dot{\theta}_{J3} & -S_{J3}\dot{\theta}_{J3} & 0 \\ C_{J2}\dot{\theta}_{J2} & 0 & 0 \end{bmatrix}\left\{\dot{\theta}_J\right\}\\ &+\left[{}^J\tilde{\omega}'_K\right]^T\left[{}^JR_K\right]\begin{bmatrix} C_{J2}C_{J3} & S_{J3} & 0 \\ -C_{J2}S_{J3} & C_{J3} & 0 \\ S_{J2} & 0 & 1 \end{bmatrix}\left\{\dot{\theta}_J\right\}\\ &+\begin{bmatrix} C_{K2}C_{K3} & S_{K3} & 0 \\ -C_{K2}S_{K3} & C_{K3} & 0 \\ S_{K2} & 0 & 1 \end{bmatrix}\left\{\ddot{\hat{\theta}}_K\right\}+\begin{bmatrix} -\left(S_{K2}C_{K3}\dot{\hat{\theta}}_{K2}+C_{K2}S_{K3}\dot{\hat{\theta}}_{K3}\right) & C_{K3}\dot{\hat{\theta}}_{K3} & 0 \\ S_{K2}S_{K3}\dot{\hat{\theta}}_{K2}-C_{K2}C_{K3}\dot{\hat{\theta}}_{K3} & -S_{K3}\dot{\hat{\theta}}_{K3} & 0 \\ C_{K2}\dot{\hat{\theta}}_{K2} & 0 & 0 \end{bmatrix}\left\{\dot{\hat{\theta}}_K\right\}\end{aligned} \qquad (39)$$

Angular Acceleration using Angular Velocity Components as Generalized Speeds

Consider again the two-body system. As before, the ***orientation*** of ***body*** J is measured ***relative*** to the ***fixed frame*** R and the orientation of ***body*** K is measured ***relative*** to ***body*** J. As noted in the previous section, the fixed frame components and body frame components of ${}^R\underset{\sim}{\alpha}_K$ the angular acceleration of body K can be written in terms of the fixed frame and body frame components of ${}^R\underset{\sim}{\alpha}_J$ the angular acceleration of body J as follows.



$$\{\alpha_K\} = \{\alpha_J\} + [R_J]^T\{\dot{\hat{\omega}}_K\} + [\tilde{\omega}_J][R_J]^T\{\hat{\omega}_K\} \qquad (40)$$

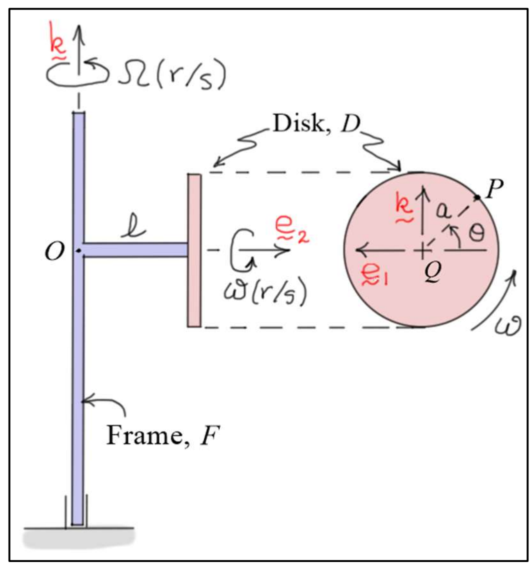
$$\{\alpha'_K\} = \left[{}^J R_K\right]\{\alpha'_J\} + \left[{}^J\tilde{\omega}'_K\right]^T\left[{}^J R_K\right]\{\omega'_J\} + \{\dot{\hat{\omega}}'_K\} \qquad (41)$$

Since the orientation of body J is measured relative to the fixed frame, the fixed frame and body frame components of ${}^R\underset{\sim}{\alpha}_J$ are simply

$$\{\alpha_J\} = \{\dot{\omega}_J\} \qquad \{\alpha'_J\} = \{\dot{\omega}'_J\}$$

Example 1

The system shown consists of two connected bodies – the vertical frame F and the disk D. Frame F rotates at a rate of $\dot{\phi} = \Omega$ (rad/s) about the fixed vertical direction (annotated by the unit vector $\underset{\sim}{k}$). Disk D is affixed to and rotates relative to F at a rate of $\dot{\theta} = \omega$ (rad/s) about the horizontal arm of F (direction annotated by the rotating unit vector $\underset{\sim}{e}_2$).



Reference frames: (all frames align when $\phi = \theta = 0$)

$R : (\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k})$ (fixed frame)

$F : (\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{k})$ (rotating with frame F)

$D : (\underset{\sim}{n}_1, \underset{\sim}{e}_2, \underset{\sim}{n}_3)$ (rotating with disk D)

In Example 1 of Unit 2, the following results were calculated. The ***fixed frame*** and ***body frame components*** of the ***angular velocity*** of D in the fixed frame R can be written as follows.

$$\{\omega_D\} = \begin{bmatrix} 0 & -S_\phi \\ 0 & C_\phi \\ 1 & 0 \end{bmatrix}\begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \end{Bmatrix} \triangleq \left[{}^R\omega_{D,\dot{\beta}}\right]\{\dot{\beta}\} \quad \text{(fixed frame components)} \qquad (42)$$

$$\{\omega'_D\} = \begin{Bmatrix} -S_\theta \dot{\phi} \\ \dot{\theta} \\ C_\theta \dot{\phi} \end{Bmatrix} = \begin{bmatrix} -S_\theta & 0 \\ 0 & 1 \\ C_\theta & 0 \end{bmatrix} \begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \end{Bmatrix} \triangleq \left[{}^R\omega'_{D,\dot{\beta}} \right] \{\dot{\beta}\} \quad \text{(body frame components)} \qquad (43)$$

Find $\{\alpha_D\}$ the ***fixed frame components*** and $\{\alpha'_D\}$ the ***disk frame components*** of the ***angular acceleration*** of D relative to R. Use $\{\beta\}$ as the column matrix of angles ϕ and θ.

Solution:

Differentiating Equations (42) and (43), the ***fixed frame*** and ***body frame components*** of the ***angular acceleration*** of D in the fixed frame R can be calculated as follows.

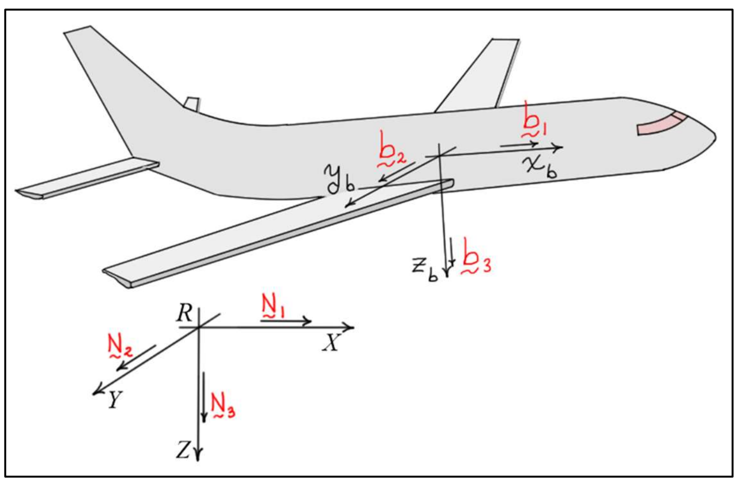
$$\{\alpha_D\} = \{\dot{\omega}_D\} = \left[{}^R\omega_{D,\dot{\beta}} \right] \{\ddot{\beta}\} + \left[{}^R\dot{\omega}_{D,\dot{\beta}} \right] \{\dot{\beta}\} \quad \text{with} \quad \left[{}^R\dot{\omega}_{D,\dot{\beta}} \right] = \frac{d}{dt} \begin{bmatrix} 0 & -S_\phi \\ 0 & C_\phi \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -C_\phi \dot{\phi} \\ 0 & -S_\phi \dot{\phi} \\ 0 & 0 \end{bmatrix}$$

$$\{\alpha'_D\} = \{\dot{\omega}'_D\} = \left[{}^R\omega'_{D,\dot{\beta}} \right] \{\ddot{\beta}\} + \left[{}^R\dot{\omega}'_{D,\dot{\beta}} \right] \{\dot{\beta}\} \quad \text{with} \quad \left[{}^R\dot{\omega}'_{D,\dot{\beta}} \right] = \frac{d}{dt} \begin{bmatrix} -S_\theta & 0 \\ 0 & 1 \\ C_\theta & 0 \end{bmatrix} = \begin{bmatrix} -C_\theta \dot{\theta} & 0 \\ 0 & 0 \\ -S_\theta \dot{\theta} & 0 \end{bmatrix}$$

Note that the angular velocity vector can be ***differentiated*** in either the ***fixed frame*** or the ***body frame*** to find the angular acceleration.

Example 2

The orientation of an aircraft A can be defined using a 3-2-1 body fixed rotation sequence. As before, the body axes $A:(x_b, y_b, z_b)$ are initially aligned with the fixed frame axes $R:(X, Y, Z)$. It is common to refer to these angles as ψ, θ, and ϕ. For small angles they are equivalent to the "yaw", "pitch", and "roll" angles of the aircraft. Complete the following expressing all results in matrix form. Use $\{\beta\}$ as the column matrix of angles ψ, θ, and ϕ. In Example 2 of Unit 2, the ***fixed frame*** and ***body frame components*** of the ***angular velocity*** of A relative to the fixed frame R were written as follows.



$$\{\omega_A\} = \begin{Bmatrix} -S_\psi \dot{\theta} + C_\psi C_\theta \dot{\phi} \\ C_\psi \dot{\theta} + S_\psi C_\theta \dot{\phi} \\ \dot{\psi} - S_\theta \dot{\phi} \end{Bmatrix} = \begin{bmatrix} 0 & -S_\psi & C_\psi C_\theta \\ 0 & C_\psi & S_\psi C_\theta \\ 1 & 0 & -S_\theta \end{bmatrix} \begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{Bmatrix} \triangleq \left[{}^R\omega_{A,\dot{\beta}} \right] \{\dot{\beta}\} \quad \text{(fixed frame components)} \qquad (44)$$

$$\left\{\omega'_A\right\} = \begin{Bmatrix} -S_\theta\dot{\psi}+\dot{\phi} \\ C_\theta S_\phi\dot{\psi}+C_\phi\dot{\theta} \\ C_\theta C_\phi\dot{\psi}-S_\phi\dot{\theta} \end{Bmatrix} = \begin{bmatrix} -S_\theta & 0 & 1 \\ C_\theta S_\phi & C_\phi & 0 \\ C_\theta C_\phi & -S_\phi & 0 \end{bmatrix}\begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{Bmatrix} \triangleq \left[{}^R\omega'_{A,\dot{\beta}}\right]\left\{\dot{\beta}\right\} \quad \text{(body frame components)} \qquad (45)$$

Using these results, find $\left\{\alpha_A\right\}$ the ***fixed frame components*** and $\left\{\alpha'_A\right\}$ the ***body frame components*** of the ***angular acceleration*** of *A* relative to the fixed frame *R*.

Solution:

Differentiating Equations (44) and (45), the ***fixed frame*** and ***body frame components*** of the ***angular acceleration*** of *A* in *R* can be calculated as follows.

$$\left\{\alpha_A\right\} = \left\{\dot{\omega}_A\right\} = \left[{}^R\omega_{A,\dot{\beta}}\right]\left\{\ddot{\beta}\right\} + \left[{}^R\dot{\omega}_{A,\dot{\beta}}\right]\left\{\dot{\beta}\right\} \quad \text{(fixed frame components)}$$

The time derivative of the partial angular velocity matrix can be calculated as follows.

$$\left[{}^R\dot{\omega}_{A,\dot{\beta}}\right] = \frac{d}{dt}\begin{bmatrix} 0 & -S_\psi & C_\psi C_\theta \\ 0 & C_\psi & S_\psi C_\theta \\ 1 & 0 & -S_\theta \end{bmatrix} = \begin{bmatrix} 0 & -C_\psi\dot{\psi} & -\left(S_\psi C_\theta\dot{\psi}+C_\psi S_\theta\dot{\theta}\right) \\ 0 & -S_\psi\dot{\psi} & C_\psi C_\theta\dot{\psi}-S_\psi S_\theta\dot{\theta} \\ 0 & 0 & -C_\theta\dot{\theta} \end{bmatrix}$$

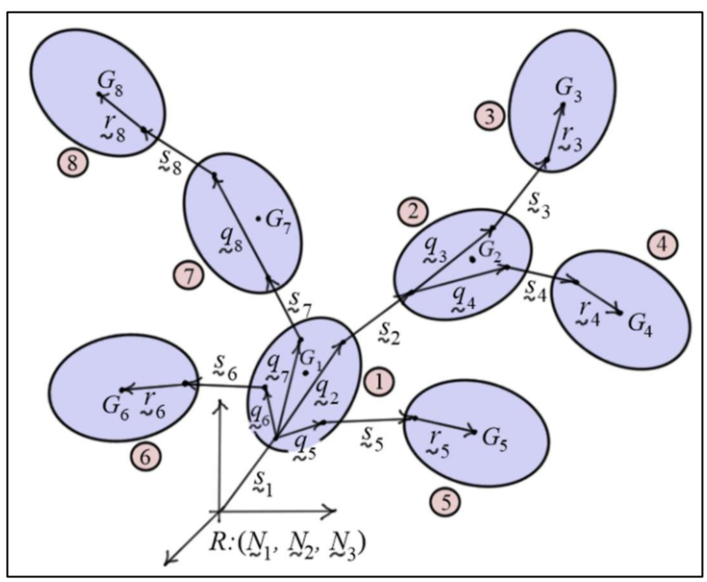
and

$$\left\{\alpha'_A\right\} = \left\{\dot{\omega}'_A\right\} = \left[{}^R\omega'_{A,\dot{\beta}}\right]\left\{\ddot{\beta}\right\} + \left[{}^R\dot{\omega}'_{A,\dot{\beta}}\right]\left\{\dot{\beta}\right\} \qquad (\textbf{\textit{body frame}} \text{ components})$$

The ***time derivative*** of the ***partial angular velocity matrix*** can be calculated as follows.

$$\left[{}^R\dot{\omega}'_{A,\dot{\beta}}\right] = \frac{d}{dt}\begin{bmatrix} -S_\theta & 0 & 1 \\ C_\theta S_\phi & C_\phi & 0 \\ C_\theta C_\phi & -S_\phi & 0 \end{bmatrix} = \begin{bmatrix} -C_\theta\dot{\theta} & 0 & 0 \\ -S_\theta S_\phi\dot{\theta}+C_\theta C_\phi\dot{\phi} & -S_\phi\dot{\phi} & 0 \\ -\left(S_\theta C_\phi\dot{\theta}+C_\theta S_\phi\dot{\phi}\right) & -C_\phi\dot{\phi} & 0 \end{bmatrix}$$

Example 3



The figure shows an eight-body system numbered using the numbering scheme presented in Unit 1. Body 1 is the system reference body, and the rest of the bodies are numbered in ascending progression outward along the branches. As structured, the lower-numbered body array for the system is as follows.

$$\mathscr{L}(1,\ldots,8) = (0,1,2,2,1,1,1,7)$$

The orientation of body 1 is defined relative to the fixed frame $R:\left(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3\right)$, and the orientations of all the other bodies are defined relative to their adjacent, lower-numbered bodies. Using ***relative angular velocity components*** as ***generalized speeds***, Example 3 of Unit 2 showed the ***fixed frame components*** of the ***angular velocities*** for all the bodies in the system could be written as follows.

$$\boxed{\{\omega_K\}_{3\times1} = \left[{}^R\omega_{K,\omega}\right]_{3\times24}\{\omega\}_{24\times1}} \quad (K=1,\ldots,8) \tag{46}$$

The ***system angular velocity matrix*** and the ***partial angular velocity matrices*** for all the bodies are as follows.

$$\boxed{\{\omega\}_{24\times1} = \left[(\hat{\omega}_1)_1 \quad (\hat{\omega}_1)_2 \quad (\hat{\omega}_1)_3 \quad (\hat{\omega}_2)_1 \quad (\hat{\omega}_2)_2 \quad (\hat{\omega}_2)_3 \quad \cdots \quad (\hat{\omega}_8)_1 \quad (\hat{\omega}_8)_2 \quad (\hat{\omega}_8)_3\right]^T}$$

$$\left[{}^R\omega_{K,\omega}\right]_{3\times24} \quad (K=1,\ldots,8)$$

$$\searrow$$

$$\begin{array}{l} K=1 \rightarrow \\ K=2 \rightarrow \\ K=3 \rightarrow \\ K=4 \rightarrow \\ K=5 \rightarrow \\ K=6 \rightarrow \\ K=7 \rightarrow \\ K=8 \rightarrow \end{array} \begin{bmatrix} [I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\ [I] & [R_1]^T & [0] & [0] & [0] & [0] & [0] & [0] \\ [I] & [R_1]^T & [R_2]^T & [0] & [0] & [0] & [0] & [0] \\ [I] & [R_1]^T & [0] & [R_2]^T & [0] & [0] & [0] & [0] \\ [I] & [0] & [0] & [0] & [R_1]^T & [0] & [0] & [0] \\ [I] & [0] & [0] & [0] & [0] & [R_1]^T & [0] & [0] \\ [I] & [0] & [0] & [0] & [0] & [0] & [R_1]^T & [0] \\ [I] & [0] & [0] & [0] & [0] & [0] & [R_1]^T & [R_7]^T \end{bmatrix} \tag{47}$$

Calculate $\{\alpha_K\}$ $(K=1,\ldots,8)$ the ***fixed frame components*** of the ***angular accelerations*** of all the bodies using ***two approaches***. In the ***first approach***, use Equation (40) relating the angular acceleration of a body to the angular acceleration of its lower-numbered body. In the ***second approach***, differentiate Equation (46).

Solution:

Approach #1:

Equation (40) above (repeated in Equation (48) below) provides a means of calculating the ***fixed frame components*** of the ***angular acceleration*** of ***each body*** K in the system using the ***fixed frame components*** of the ***angular acceleration*** of its ***lower numbered body*** J.

$$\boxed{\{\alpha_K\} = \{\alpha_J\} + [R_J]^T\{\dot{\hat{\omega}}_K\} + [\tilde{\omega}_J][R_J]^T\{\hat{\omega}_K\}} \quad (J=\mathfrak{L}(K)) \tag{48}$$

Body 1: $\boxed{\{\alpha_1\} = \{\dot{\omega}_1\} = \{\dot{\hat{\omega}}_1\}}$

Body 2: $\{\alpha_2\} = \{\alpha_1\} + \left[R_1\right]^T\{\dot{\hat{\omega}}_2\} + \left[\tilde{\omega}_1\right]\left[R_1\right]^T\{\hat{\omega}_2\}$

$$\boxed{\{\alpha_2\} = \{\dot{\hat{\omega}}_1\} + \left[R_1\right]^T\{\dot{\hat{\omega}}_2\} + \left[\tilde{\omega}_1\right]\left[R_1\right]^T\{\hat{\omega}_2\}}$$

Body 3: $\{\alpha_3\} = \{\alpha_2\} + \left[R_2\right]^T\{\dot{\hat{\omega}}_3\} + \left[\tilde{\omega}_2\right]\left[R_2\right]^T\{\hat{\omega}_3\}$

$$\boxed{\{\alpha_3\}=\{\dot{\hat{\omega}}_1\}+[R_1]^T\{\dot{\hat{\omega}}_2\}+[R_2]^T\{\dot{\hat{\omega}}_3\}+[\tilde{\omega}_1][R_1]^T\{\hat{\omega}_2\}+[\tilde{\omega}_2][R_2]^T\{\hat{\omega}_3\}}$$

Body 4: $\{\alpha_4\}=\{\alpha_2\}+[R_2]^T\{\dot{\hat{\omega}}_4\}+[\tilde{\omega}_2][R_2]^T\{\hat{\omega}_4\}$

$$\boxed{\{\alpha_4\}=\{\dot{\hat{\omega}}_1\}+[R_1]^T\{\dot{\hat{\omega}}_2\}+[R_2]^T\{\dot{\hat{\omega}}_4\}+[\tilde{\omega}_1][R_1]^T\{\hat{\omega}_2\}+[\tilde{\omega}_2][R_2]^T\{\hat{\omega}_4\}}$$

Body 5: $\{\alpha_5\}=\{\alpha_1\}+[R_1]^T\{\dot{\hat{\omega}}_5\}+[\tilde{\omega}_1][R_1]^T\{\hat{\omega}_5\}$

$$\boxed{\{\alpha_5\}=\{\dot{\hat{\omega}}_1\}+[R_1]^T\{\dot{\hat{\omega}}_5\}+[\tilde{\omega}_1][R_1]^T\{\hat{\omega}_5\}}$$

Body 6: $\{\alpha_6\}=\{\alpha_1\}+[R_1]^T\{\dot{\hat{\omega}}_6\}+[\tilde{\omega}_1][R_1]^T\{\hat{\omega}_6\}$

$$\boxed{\{\alpha_6\}=\{\dot{\hat{\omega}}_1\}+[R_1]^T\{\dot{\hat{\omega}}_6\}+[\tilde{\omega}_1][R_1]^T\{\hat{\omega}_6\}}$$

Body 7: $\{\alpha_7\}=\{\alpha_1\}+[R_1]^T\{\dot{\hat{\omega}}_7\}+[\tilde{\omega}_1][R_1]^T\{\hat{\omega}_7\}$

$$\boxed{\{\alpha_7\}=\{\dot{\hat{\omega}}_1\}+[R_1]^T\{\dot{\hat{\omega}}_7\}+[\tilde{\omega}_1][R_1]^T\{\hat{\omega}_7\}}$$

Body 8: $\{\alpha_8\}=\{\alpha_7\}+[R_7]^T\{\dot{\hat{\omega}}_8\}+[\tilde{\omega}_7][R_7]^T\{\hat{\omega}_8\}$

$$\boxed{\{\alpha_8\}=\{\dot{\hat{\omega}}_1\}+[R_1]^T\{\dot{\hat{\omega}}_7\}+[R_7]^T\{\dot{\hat{\omega}}_8\}+[\tilde{\omega}_1][R_1]^T\{\hat{\omega}_7\}+[\tilde{\omega}_7][R_7]^T\{\hat{\omega}_8\}}$$

Approach #2:

The ***fixed frame components*** of the ***angular accelerations*** of the bodies can also be found by ***differentiating*** Equation (46).

$$\boxed{\{\alpha_K\}=\{\dot{\omega}_K\}_{3\times1}=\left[{}^R\omega_{K,\omega}\right]_{3\times24}\{\dot{\omega}\}_{24\times1}+\left[{}^R\dot{\omega}_{K,\omega}\right]_{3\times24}\{\omega\}_{24\times1}}\quad (K=1,\ldots,8) \tag{49}$$

The ***time derivatives*** of the ***partial angular velocity matrices*** are as follows.

$\left[{}^R\dot{\omega}_{K,\omega}\right]_{3\times24}$ $(K=1,\ldots,8)$

↘

$$\boxed{\begin{matrix} K=1\to \\ K=2\to \\ K=3\to \\ K=4\to \\ K=5\to \\ K=6\to \\ K=7\to \\ K=8\to \end{matrix}\begin{bmatrix} [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\ [0] & [\dot{R}_1]^T & [0] & [0] & [0] & [0] & [0] & [0] \\ [0] & [\dot{R}_1]^T & [\dot{R}_2]^T & [0] & [0] & [0] & [0] & [0] \\ [0] & [\dot{R}_1]^T & [0] & [\dot{R}_2]^T & [0] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [\dot{R}_1]^T & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [0] & [\dot{R}_1]^T & [0] & [0] \\ [0] & [0] & [0] & [0] & [0] & [0] & [\dot{R}_1]^T & [0] \\ [0] & [0] & [0] & [0] & [0] & [0] & [\dot{R}_1]^T & [\dot{R}_7]^T \end{bmatrix}} \tag{50}$$

Using these results directly gives the following.

Body 1: $$\boxed{\left\{\alpha_1\right\} = \left[{}^{R}\omega_{1,\omega}\right]_{3\times24}\left\{\dot{\omega}\right\}_{24\times1} + \underbrace{\left[{}^{R}\dot{\omega}_{1,\omega}\right]_{3\times24}}_{\text{zero}}\left\{\omega\right\}_{24\times1} = \left\{\dot{\hat{\omega}}_1\right\}}$$

Body 2: $$\left\{\alpha_2\right\} = \left[{}^{R}\omega_{2,\omega}\right]_{3\times24}\left\{\dot{\omega}\right\}_{24\times1} + \left[{}^{R}\dot{\omega}_{2,\omega}\right]_{3\times24}\left\{\omega\right\}_{24\times1}$$

$$\boxed{\left\{\alpha_2\right\} = \left\{\dot{\hat{\omega}}_1\right\} + \left[R_1\right]^T\left\{\dot{\hat{\omega}}_2\right\} + \left[\tilde{\omega}_1\right]\left[R_1\right]^T\left\{\hat{\omega}_2\right\}}$$

Body 3: $$\left\{\alpha_3\right\} = \left[{}^{R}\omega_{3,\omega}\right]_{3\times24}\left\{\dot{\omega}\right\}_{24\times1} + \left[{}^{R}\dot{\omega}_{3,\omega}\right]_{3\times24}\left\{\omega\right\}_{24\times1}$$

$$\left\{\alpha_3\right\} = \left\{\dot{\hat{\omega}}_1\right\} + \left[R_1\right]^T\left\{\dot{\hat{\omega}}_2\right\} + \left[R_2\right]^T\left\{\dot{\hat{\omega}}_3\right\} + \left[\dot{R}_1\right]^T\left\{\hat{\omega}_2\right\} + \left[\dot{R}_2\right]^T\left\{\hat{\omega}_3\right\}$$

$$\boxed{\left\{\alpha_3\right\} = \left\{\dot{\hat{\omega}}_1\right\} + \left[R_1\right]^T\left\{\dot{\hat{\omega}}_2\right\} + \left[R_2\right]^T\left\{\dot{\hat{\omega}}_3\right\} + \left[\tilde{\omega}_1\right]\left[R_1\right]^T\left\{\hat{\omega}_2\right\} + \left[\tilde{\omega}_2\right]\left[R_2\right]^T\left\{\hat{\omega}_3\right\}}$$

Body 4: $$\left\{\alpha_4\right\} = \left[{}^{R}\omega_{4,\omega}\right]_{3\times24}\left\{\dot{\omega}\right\}_{24\times1} + \left[{}^{R}\dot{\omega}_{4,\omega}\right]_{3\times24}\left\{\omega\right\}_{24\times1}$$

$$\left\{\alpha_4\right\} = \left\{\dot{\hat{\omega}}_1\right\} + \left[R_1\right]^T\left\{\dot{\hat{\omega}}_2\right\} + \left[R_2\right]^T\left\{\dot{\hat{\omega}}_4\right\} + \left[\dot{R}_1\right]^T\left\{\hat{\omega}_2\right\} + \left[\dot{R}_2\right]^T\left\{\hat{\omega}_4\right\}$$

$$\boxed{\left\{\alpha_4\right\} = \left\{\dot{\hat{\omega}}_1\right\} + \left[R_1\right]^T\left\{\dot{\hat{\omega}}_2\right\} + \left[R_2\right]^T\left\{\dot{\hat{\omega}}_4\right\} + \left[\tilde{\omega}_1\right]\left[R_1\right]^T\left\{\hat{\omega}_2\right\} + \left[\tilde{\omega}_2\right]\left[R_2\right]^T\left\{\hat{\omega}_4\right\}}$$

Body 5: $$\left\{\alpha_5\right\} = \left[{}^{R}\omega_{5,\omega}\right]_{3\times24}\left\{\dot{\omega}\right\}_{24\times1} + \left[{}^{R}\dot{\omega}_{5,\omega}\right]_{3\times24}\left\{\omega\right\}_{24\times1}$$

$$\boxed{\left\{\alpha_5\right\} = \left\{\dot{\hat{\omega}}_1\right\} + \left[R_1\right]^T\left\{\dot{\hat{\omega}}_5\right\} + \left[\tilde{\omega}_1\right]\left[R_1\right]^T\left\{\hat{\omega}_5\right\}}$$

Body 6: $$\left\{\alpha_6\right\} = \left[{}^{R}\omega_{6,\omega}\right]_{3\times24}\left\{\dot{\omega}\right\}_{24\times1} + \left[{}^{R}\dot{\omega}_{6,\omega}\right]_{3\times24}\left\{\omega\right\}_{24\times1}$$

$$\boxed{\left\{\alpha_6\right\} = \left\{\dot{\hat{\omega}}_1\right\} + \left[R_1\right]^T\left\{\dot{\hat{\omega}}_6\right\} + \left[\tilde{\omega}_1\right]\left[R_1\right]^T\left\{\hat{\omega}_6\right\}}$$

Body 7: $$\left\{\alpha_7\right\} = \left[{}^{R}\omega_{7,\omega}\right]_{3\times24}\left\{\dot{\omega}\right\}_{24\times1} + \left[{}^{R}\dot{\omega}_{7,\omega}\right]_{3\times24}\left\{\omega\right\}_{24\times1}$$

$$\boxed{\left\{\alpha_7\right\} = \left\{\dot{\hat{\omega}}_1\right\} + \left[R_1\right]^T\left\{\dot{\hat{\omega}}_7\right\} + \left[\tilde{\omega}_1\right]\left[R_1\right]^T\left\{\hat{\omega}_7\right\}}$$

Body 8: $$\left\{\alpha_8\right\} = \left[{}^{R}\omega_{8,\omega}\right]_{3\times24}\left\{\dot{\omega}\right\}_{24\times1} + \left[{}^{R}\dot{\omega}_{8,\omega}\right]_{3\times24}\left\{\omega\right\}_{24\times1}$$

$$\left\{\alpha_8\right\} = \left\{\dot{\hat{\omega}}_1\right\} + \left[R_1\right]^T\left\{\dot{\hat{\omega}}_7\right\} + \left[R_7\right]^T\left\{\dot{\hat{\omega}}_8\right\} + \left[\dot{R}_1\right]^T\left\{\hat{\omega}_7\right\} + \left[\dot{R}_7\right]^T\left\{\hat{\omega}_8\right\}$$

$$\boxed{\left\{\alpha_8\right\} = \left\{\dot{\hat{\omega}}_1\right\} + \left[R_1\right]^T\left\{\dot{\hat{\omega}}_7\right\} + \left[R_7\right]^T\left\{\dot{\hat{\omega}}_8\right\} + \left[\tilde{\omega}_1\right]\left[R_1\right]^T\left\{\hat{\omega}_7\right\} + \left[\tilde{\omega}_7\right]\left[R_7\right]^T\left\{\hat{\omega}_8\right\}}$$

These results are ***identical*** to those found above.

Example 4

Consider again the eight-body system of Example 3. In Example 4 of Unit 2, the ***body frame components*** of the ***angular velocities*** for all the bodies in the system were written as follows.

$$\boxed{\left\{\omega'_K\right\}_{3\times1} = \left[{}^R\omega'_{K,\omega'}\right]_{3\times24}\left\{\omega'\right\}_{24\times1}} \quad \left(K=1,\ldots,8\right) \tag{51}$$

The system angular velocity matrix and the ***partial angular velocity matrices*** for all the bodies are as follows.

$$\boxed{\left\{\omega'\right\}_{24\times1} = \left[\left(\hat{\omega}'_1\right)_1 \quad \left(\hat{\omega}'_1\right)_2 \quad \left(\hat{\omega}'_1\right)_3 \quad \left(\hat{\omega}'_2\right)_1 \quad \left(\hat{\omega}'_2\right)_2 \quad \left(\hat{\omega}'_2\right)_3 \quad \cdots \quad \left(\hat{\omega}'_8\right)_1 \quad \left(\hat{\omega}'_8\right)_2 \quad \left(\hat{\omega}'_8\right)_3\right]^T}$$

$$\left[{}^R\omega'_{K,\omega'}\right]_{3\times24} \quad \left(K=1,\ldots,8\right)$$

$\searrow$

$$\boxed{\begin{matrix} K=1 \rightarrow \\ K=2 \rightarrow \\ K=3 \rightarrow \\ K=4 \rightarrow \\ K=5 \rightarrow \\ K=6 \rightarrow \\ K=7 \rightarrow \\ K=8 \rightarrow \end{matrix}\begin{bmatrix} [I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\ [{}^1R_2] & [I] & [0] & [0] & [0] & [0] & [0] & [0] \\ [{}^1R_3] & [{}^2R_3] & [I] & [0] & [0] & [0] & [0] & [0] \\ [{}^1R_4] & [{}^2R_4] & [0] & [I] & [0] & [0] & [0] & [0] \\ [{}^1R_5] & [0] & [0] & [0] & [I] & [0] & [0] & [0] \\ [{}^1R_6] & [0] & [0] & [0] & [0] & [I] & [0] & [0] \\ [{}^1R_7] & [0] & [0] & [0] & [0] & [0] & [I] & [0] \\ [{}^1R_8] & [0] & [0] & [0] & [0] & [0] & [{}^7R_8] & [I] \end{bmatrix}} \tag{52}$$

Calculate $\left\{\alpha'_K\right\}$ $\left(K=1,\ldots,8\right)$ the ***body frame components*** of the ***angular accelerations*** of all the bodies using ***two approaches***. In the ***first approach***, use Equation (41) relating the ***angular acceleration*** of ***each body*** K to the ***angular acceleration*** of its ***lower numbered body***. In the ***second approach***, ***differentiate*** Equation (51).

Solution:

Approach #1:

Equation (41) above (repeated in Equation (53) below) provides a means of calculating the ***body frame components*** of the ***angular acceleration*** of each body K in the system using the ***body frame components*** of the ***angular acceleration*** of its ***lower numbered body*** J.

$$\boxed{\left\{\alpha'_K\right\} = \left[{}^JR_K\right]\left\{\alpha'_J\right\} + \left[{}^J\tilde{\omega}'_K\right]^T\left[{}^JR_K\right]\left\{\omega'_J\right\} + \left\{\dot{\hat{\omega}}'_K\right\}} \quad \left(J=\mathfrak{L}(K)\right) \tag{53}$$

Applying this result to the eight-body system gives the following.

Body 1: $\boxed{\left\{\alpha'_1\right\} = \left\{\dot{\hat{\omega}}'_1\right\}}$

Body 2: $\left\{\alpha'_2\right\} = \left[{}^1R_2\right]\left\{\alpha'_1\right\} + \left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]\left\{\omega'_1\right\} + \left\{\dot{\hat{\omega}}'_2\right\}$

$$\boxed{\left\{\alpha'_2\right\} = \left[{}^1R_2\right]\left\{\dot{\hat{\omega}}'_1\right\} + \left\{\dot{\hat{\omega}}'_2\right\} + \left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]\left\{\hat{\omega}'_1\right\}}$$

Body 3: $\left\{\alpha_3'\right\}=\left[{}^2R_3\right]\left\{\alpha_2'\right\}+\left[{}^2\tilde{\omega}_3'\right]^T\left[{}^2R_3\right]\left\{\omega_2'\right\}+\left\{\dot{\hat{\omega}}_3'\right\}$

$$=\left[{}^2R_3\right]\left\{\alpha_2'\right\}+\left[{}^2\tilde{\omega}_3'\right]^T\left[{}^2R_3\right]\left(\left[{}^1R_2\right]\left\{\hat{\omega}_1'\right\}+\left\{\hat{\omega}_2'\right\}\right)+\left\{\dot{\hat{\omega}}_3'\right\}$$

$$=\left[{}^2R_3\right]\left(\left[{}^1R_2\right]\left\{\dot{\hat{\omega}}_1'\right\}+\left\{\dot{\hat{\omega}}_2'\right\}+\left[{}^1\tilde{\omega}_2'\right]^T\left[{}^1R_2\right]\left\{\hat{\omega}_1'\right\}\right)+\left\{\dot{\hat{\omega}}_3'\right\}$$

$$+\left[{}^2\tilde{\omega}_3'\right]^T\left[{}^2R_3\right]\left(\left[{}^1R_2\right]\left\{\hat{\omega}_1'\right\}+\left\{\hat{\omega}_2'\right\}\right)$$

$$\boxed{\begin{aligned}\left\{\alpha_3'\right\}=&\left[{}^2R_3\right]\left[{}^1R_2\right]\left\{\dot{\hat{\omega}}_1'\right\}+\left[{}^2R_3\right]\left\{\dot{\hat{\omega}}_2'\right\}+\left\{\dot{\hat{\omega}}_3'\right\}+\left[{}^2R_3\right]\left[{}^1\tilde{\omega}_2'\right]^T\left[{}^1R_2\right]\left\{\hat{\omega}_1'\right\}\\&+\left[{}^2\tilde{\omega}_3'\right]^T\left[{}^2R_3\right]\left(\left[{}^1R_2\right]\left\{\hat{\omega}_1'\right\}+\left\{\hat{\omega}_2'\right\}\right)\end{aligned}}$$

Body 4: $\left\{\alpha_4'\right\}=\left[{}^2R_4\right]\left\{\alpha_2'\right\}+\left[{}^2\tilde{\omega}_4'\right]^T\left[{}^2R_4\right]\left\{\omega_2'\right\}+\left\{\dot{\hat{\omega}}_4'\right\}$

$$=\left[{}^2R_4\right]\left(\left[{}^1R_2\right]\left\{\dot{\hat{\omega}}_1'\right\}+\left\{\dot{\hat{\omega}}_2'\right\}+\left[{}^1\tilde{\omega}_2'\right]^T\left[{}^1R_2\right]\left\{\hat{\omega}_1'\right\}\right)$$

$$+\left[{}^2\tilde{\omega}_4'\right]^T\left[{}^2R_4\right]\left(\left[{}^1R_2\right]\left\{\hat{\omega}_1'\right\}+\left\{\hat{\omega}_2'\right\}\right)+\left\{\dot{\hat{\omega}}_4'\right\}$$

$$\boxed{\begin{aligned}\left\{\alpha_4'\right\}=&\left[{}^2R_4\right]\left[{}^1R_2\right]\left\{\dot{\hat{\omega}}_1'\right\}+\left[{}^2R_4\right]\left\{\dot{\hat{\omega}}_2'\right\}+\left\{\dot{\hat{\omega}}_4'\right\}+\left[{}^2R_4\right]\left[{}^1\tilde{\omega}_2'\right]^T\left[{}^1R_2\right]\left\{\omega_1'\right\}\\&+\left[{}^2\tilde{\omega}_4'\right]^T\left[{}^2R_4\right]\left(\left[{}^1R_2\right]\left\{\hat{\omega}_1'\right\}+\left\{\hat{\omega}_2'\right\}\right)\end{aligned}}$$

Body 5: $\left\{\alpha_5'\right\}=\left[{}^1R_5\right]\left\{\alpha_1'\right\}+\left[{}^1\tilde{\omega}_5'\right]^T\left[{}^1R_5\right]\left\{\omega_1'\right\}+\left\{\dot{\hat{\omega}}_5'\right\}$

$$\boxed{\left\{\alpha_5'\right\}=\left[{}^1R_5\right]\left\{\dot{\hat{\omega}}_1'\right\}+\left\{\dot{\hat{\omega}}_5'\right\}+\left[{}^1\tilde{\omega}_5'\right]^T\left[{}^1R_5\right]\left\{\hat{\omega}_1'\right\}}$$

Body 6: $\left\{\alpha_6'\right\}=\left[{}^1R_6\right]\left\{\alpha_1'\right\}+\left[{}^1\tilde{\omega}_6'\right]^T\left[{}^1R_6\right]\left\{\omega_1'\right\}+\left\{\dot{\hat{\omega}}_6'\right\}$

$$\boxed{\left\{\alpha_6'\right\}=\left[{}^1R_6\right]\left\{\dot{\hat{\omega}}_1'\right\}+\left\{\dot{\hat{\omega}}_6'\right\}+\left[{}^1\tilde{\omega}_6'\right]^T\left[{}^1R_6\right]\left\{\hat{\omega}_1'\right\}}$$

Body 7: $\left\{\alpha_7'\right\}=\left[{}^1R_7\right]\left\{\alpha_1'\right\}+\left[{}^1\tilde{\omega}_7'\right]^T\left[{}^1R_7\right]\left\{\omega_1'\right\}+\left\{\dot{\hat{\omega}}_7'\right\}$

$$\boxed{\left\{\alpha_7'\right\}=\left[{}^1R_7\right]\left\{\dot{\hat{\omega}}_1'\right\}+\left\{\dot{\hat{\omega}}_7'\right\}+\left[{}^1\tilde{\omega}_7'\right]^T\left[{}^1R_7\right]\left\{\hat{\omega}_1'\right\}}$$

Body 8: $\left\{\alpha_8'\right\}=\left[{}^7R_8\right]\left\{\alpha_7'\right\}+\left[{}^7\tilde{\omega}_8'\right]^T\left[{}^7R_8\right]\left\{\omega_7'\right\}+\left\{\dot{\hat{\omega}}_8'\right\}$

$$=\left[{}^7R_8\right]\left(\left[{}^1R_7\right]\left\{\dot{\hat{\omega}}_1'\right\}+\left\{\dot{\hat{\omega}}_7'\right\}+\left[{}^1\tilde{\omega}_7'\right]^T\left[{}^1R_7\right]\left\{\hat{\omega}_1'\right\}\right)$$

$$+\left[{}^7\tilde{\omega}_8'\right]^T\left[{}^7R_8\right]\left(\left[{}^1R_7\right]\left\{\hat{\omega}_1'\right\}+\left\{\hat{\omega}_7'\right\}\right)+\left\{\dot{\hat{\omega}}_8'\right\}$$

$$\boxed{\begin{aligned}\left\{\alpha_8'\right\}=&\left[{}^7R_8\right]\left[{}^1R_7\right]\left\{\dot{\hat{\omega}}_1'\right\}+\left[{}^7R_8\right]\left\{\dot{\hat{\omega}}_7'\right\}+\left\{\dot{\hat{\omega}}_8'\right\}+\left[{}^7R_8\right]\left[{}^1\tilde{\omega}_7'\right]^T\left[{}^1R_7\right]\left\{\hat{\omega}_1'\right\}\\&+\left[{}^7\tilde{\omega}_8'\right]^T\left[{}^7R_8\right]\left(\left[{}^1R_7\right]\left\{\hat{\omega}_1'\right\}+\left\{\hat{\omega}_7'\right\}\right)\end{aligned}}$$

Approach #2:

The ***body frame components*** of the ***angular accelerations*** of the bodies can also be found by differentiating Equation (51).

$$\boxed{\left\{\alpha'_K\right\}_{3\times1}=\left[{}^R\omega'_{K,\omega'}\right]_{3\times24}\left\{\dot{\omega}'\right\}_{24\times1}+\left[{}^R\dot{\omega}'_{K,\omega'}\right]_{3\times24}\left\{\omega'\right\}_{24\times1}}\quad\left(K=1,\ldots,8\right)\qquad(54)$$

The ***time derivatives*** of the ***partial angular velocity matrices*** are as follows.

$$\left[{}^R\dot{\omega}'_{K,\omega'}\right]_{3\times24}\quad\left(K=1,\ldots,8\right)$$

$$\searrow$$

$$\boxed{\begin{array}{l}K=1\;\rightarrow\\K=2\;\rightarrow\\K=3\;\rightarrow\\K=4\;\rightarrow\\K=5\;\rightarrow\\K=6\;\rightarrow\\K=7\;\rightarrow\\K=8\;\rightarrow\end{array}\begin{bmatrix}[0]&[0]&[0]&[0]&[0]&[0]&[0]&[0]\\ [{}^1\dot{R}_2]&[0]&[0]&[0]&[0]&[0]&[0]&[0]\\ [{}^1\dot{R}_3]&[{}^2\dot{R}_3]&[0]&[0]&[0]&[0]&[0]&[0]\\ [{}^1\dot{R}_4]&[{}^2\dot{R}_4]&[0]&[0]&[0]&[0]&[0]&[0]\\ [{}^1\dot{R}_5]&[0]&[0]&[0]&[0]&[0]&[0]&[0]\\ [{}^1\dot{R}_6]&[0]&[0]&[0]&[0]&[0]&[0]&[0]\\ [{}^1\dot{R}_7]&[0]&[0]&[0]&[0]&[0]&[0]&[0]\\ [{}^1\dot{R}_8]&[0]&[0]&[0]&[0]&[0]&[{}^7\dot{R}_8]&[0]\end{bmatrix}}\qquad(55)$$

The ***time derivatives*** of the ***transformation matrices*** can be calculated as follows.

$$\boxed{\left[{}^1\dot{R}_2\right]=\left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]}\quad\boxed{\left[{}^1\dot{R}_5\right]=\left[{}^1\tilde{\omega}'_5\right]^T\left[{}^1R_5\right]}\quad\boxed{\left[{}^1\dot{R}_6\right]=\left[{}^1\tilde{\omega}'_6\right]^T\left[{}^1R_6\right]}\quad\boxed{\left[{}^1\dot{R}_7\right]=\left[{}^1\tilde{\omega}'_7\right]^T\left[{}^1R_7\right]}$$

$$\boxed{\left[{}^2\dot{R}_3\right]=\left[{}^2\tilde{\omega}'_3\right]^T\left[{}^2R_3\right]}\quad\boxed{\left[{}^2\dot{R}_4\right]=\left[{}^2\tilde{\omega}'_4\right]^T\left[{}^2R_4\right]}\quad\boxed{\left[{}^7\dot{R}_8\right]=\left[{}^7\tilde{\omega}'_8\right]^T\left[{}^7R_8\right]}$$

$$\boxed{\left[{}^1\dot{R}_3\right]=\frac{d}{dt}\left(\left[{}^2R_3\right]\left[{}^1R_2\right]\right)=\left[{}^2R_3\right]\left[{}^1\dot{R}_2\right]+\left[{}^2\dot{R}_3\right]\left[{}^1R_2\right]=\left[{}^2R_3\right]\left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]+\left[{}^2\tilde{\omega}'_3\right]^T\left[{}^2R_3\right]\left[{}^1R_2\right]}$$

$$\boxed{\left[{}^1\dot{R}_4\right]=\frac{d}{dt}\left(\left[{}^2R_4\right]\left[{}^1R_2\right]\right)=\left[{}^2R_4\right]\left[{}^1\dot{R}_2\right]+\left[{}^2\dot{R}_4\right]\left[{}^1R_2\right]=\left[{}^2R_4\right]\left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]+\left[{}^2\tilde{\omega}'_4\right]^T\left[{}^2R_4\right]\left[{}^1R_2\right]}$$

$$\boxed{\left[{}^1\dot{R}_8\right]=\frac{d}{dt}\left(\left[{}^7R_8\right]\left[{}^1R_7\right]\right)=\left[{}^7R_8\right]\left[{}^1\dot{R}_7\right]+\left[{}^7\dot{R}_8\right]\left[{}^1R_7\right]=\left[{}^7R_8\right]\left[{}^1\tilde{\omega}'_7\right]^T\left[{}^1R_7\right]+\left[{}^7\tilde{\omega}'_8\right]^T\left[{}^7R_8\right]\left[{}^1R_7\right]}$$

Substituting these results into Equation (54) gives the following results.

Body 1: $\boxed{\left\{\alpha'_1\right\}=\left\{\dot{\hat{\omega}}'_1\right\}}$

Body 2: $\boxed{\left\{\alpha'_2\right\}=\left[{}^1R_2\right]\left\{\dot{\hat{\omega}}'_1\right\}+\left\{\dot{\hat{\omega}}'_2\right\}+\left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]\left\{\hat{\omega}'_1\right\}}$

Body 3: $\left\{\alpha'_3\right\}=\left[{}^1R_3\right]\left\{\dot{\hat{\omega}}'_1\right\}+\left[{}^2R_3\right]\left\{\dot{\hat{\omega}}'_2\right\}+\left\{\dot{\hat{\omega}}'_3\right\}+\left[{}^1\dot{R}_3\right]\left\{\hat{\omega}'_1\right\}+\left[{}^2\dot{R}_3\right]\left\{\hat{\omega}'_2\right\}$

$$\boxed{\begin{aligned}\left\{\alpha'_3\right\}=&\left[{}^2R_3\right]\left[{}^1R_2\right]\left\{\dot{\hat{\omega}}'_1\right\}+\left[{}^2R_3\right]\left\{\dot{\hat{\omega}}'_2\right\}+\left\{\dot{\hat{\omega}}'_3\right\}+\left[{}^2R_3\right]\left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]\left\{\hat{\omega}'_1\right\}\\&+\left[{}^2\tilde{\omega}'_3\right]^T\left[{}^2R_3\right]\left(\left[{}^1R_2\right]\left\{\hat{\omega}'_1\right\}+\left\{\hat{\omega}'_2\right\}\right)\end{aligned}}$$

Body 4: $\{\alpha'_4\} = [{}^1R_4]\{\dot{\hat{\omega}}'_1\} + [{}^2R_4]\{\dot{\hat{\omega}}'_2\} + \{\dot{\hat{\omega}}'_4\} + [{}^1\dot{R}_4]\{\hat{\omega}'_1\} + [{}^2\dot{R}_4]\{\hat{\omega}'_2\}$

$$\{\alpha'_4\} = [{}^2R_4][{}^1R_2]\{\dot{\hat{\omega}}'_1\} + [{}^2R_4]\{\dot{\hat{\omega}}'_2\} + \{\dot{\hat{\omega}}'_4\} + [{}^2R_4][{}^1\tilde{\omega}'_2]^T[{}^1R_2]\{\hat{\omega}'_1\} + [{}^2\tilde{\omega}'_4]^T[{}^2R_4]\left([{}^1R_2]\{\hat{\omega}'_1\} + \{\hat{\omega}'_2\}\right)$$

Body 5: $\{\alpha'_5\} = [{}^1R_5]\{\dot{\hat{\omega}}'_1\} + \{\dot{\hat{\omega}}'_5\} + [{}^1\tilde{\omega}'_5]^T[{}^1R_5]\{\hat{\omega}'_1\}$

Body 6: $\{\alpha'_6\} = [{}^1R_6]\{\dot{\hat{\omega}}'_1\} + \{\dot{\hat{\omega}}'_6\} + [{}^1\tilde{\omega}'_6]^T[{}^1R_6]\{\hat{\omega}'_1\}$

Body 7: $\{\alpha'_7\} = [{}^1R_7]\{\dot{\hat{\omega}}'_1\} + \{\dot{\hat{\omega}}'_7\} + [{}^1\tilde{\omega}'_7]^T[{}^1R_7]\{\hat{\omega}'_1\}$

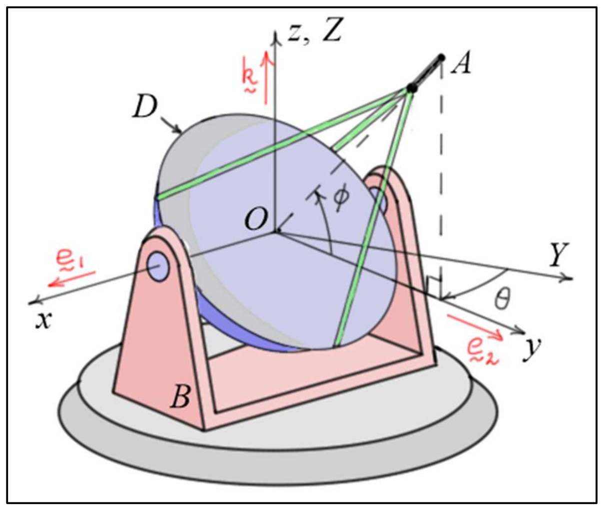
Body 8: $\{\alpha'_8\} = [{}^1R_8]\{\dot{\hat{\omega}}'_1\} + [{}^7R_8]\{\dot{\hat{\omega}}'_7\} + \{\dot{\hat{\omega}}'_8\} + [{}^1\dot{R}_8]\{\hat{\omega}'_1\} + [{}^7\dot{R}_8]\{\hat{\omega}'_7\}$

$$\{\alpha'_8\} = [{}^7R_8][{}^1R_7]\{\dot{\hat{\omega}}'_1\} + [{}^7R_8]\{\dot{\hat{\omega}}'_7\} + \{\dot{\hat{\omega}}'_8\} + [{}^7R_8][{}^1\tilde{\omega}'_7]^T[{}^1R_7]\{\hat{\omega}'_1\} + [{}^7\tilde{\omega}'_8]^T[{}^7R_8]\left([{}^1R_7]\{\hat{\omega}'_1\} + \{\hat{\omega}'_7\}\right)$$

These results are identical to those found above.

Exercises

3.1 The antenna system shown has two components, the base B and the antenna dish D. Base B with axes (x,y,z) rotates relative to the ground frame with axes (X,Y,Z) about the fixed Z (or z) axis. Dish D rotates relative to B about the rotating x axis annotated by the unit vector $\underset{\sim}{e}_1$. At any instant, the angle between the y axis annotated by the unit vector $\underset{\sim}{e}_2$ and the fixed Y axis is θ, and the angle between line segment OA and the rotating y axis is ϕ. The ground frame (X,Y,Z) is a fixed reference frame with origin at O. Given the diagram, the dish is oriented relative to the (X,Y,Z) axis system using a 3-1 rotation sequence. Note that, as shown in the diagram, the angle θ is ***negative***. When $\phi = \theta = 0$ all reference frames align. In Unit 2 Exercise 2.1, the fixed frame and body frame components of ${}^R\underset{\sim}{\omega}_D$ the angular velocity of D relative to the fixed frame were found to be as follows.



$$\{\omega_D\} = \begin{Bmatrix} C_\theta\dot{\phi} \\ -S_\theta\dot{\phi} \\ -\dot{\theta} \end{Bmatrix} = \begin{bmatrix} 0 & C_\theta \\ 0 & -S_\theta \\ -1 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ \dot{\phi} \end{Bmatrix} \triangleq [{}^R\omega_{D,\dot{\beta}}]\{\dot{\beta}\} \qquad \{\omega'_D\} = \begin{Bmatrix} \dot{\phi} \\ -S_\phi\dot{\theta} \\ -C_\phi\dot{\theta} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -S_\phi & 0 \\ -C_\phi & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ \dot{\phi} \end{Bmatrix} \triangleq [{}^R\omega'_{D,\dot{\beta}}]\{\dot{\beta}\}$$

Find $\{\alpha_D\}$ the ***fixed frame components*** and $\{\alpha'_D\}$ the ***dish-frame components*** of the ***angular acceleration*** of D in the fixed frame. Use the partial angular velocity matrices shown above.

Answers:

$$\left\{\alpha_D\right\}=\left\{\dot{\omega}_D\right\}=\left[{}^R\omega_{D,\dot{\beta}}\right]\left\{\ddot{\beta}\right\}+\left[{}^R\dot{\omega}_{D,\dot{\beta}}\right]\left\{\dot{\beta}\right\} \quad \text{with} \quad \left[{}^R\dot{\omega}_{D,\dot{\beta}}\right]=\begin{bmatrix} 0 & -S_\theta\,\dot{\theta} \\ 0 & -C_\theta\,\dot{\theta} \\ 0 & 0 \end{bmatrix}$$

$$\left\{\alpha'_D\right\}=\left\{\dot{\omega}'_D\right\}=\left[{}^R\omega'_{D,\dot{\beta}}\right]\left\{\ddot{\beta}\right\}+\left[{}^R\dot{\omega}'_{D,\dot{\beta}}\right]\left\{\dot{\beta}\right\} \quad \text{with} \quad \left[{}^R\dot{\omega}'_{D,\dot{\beta}}\right]=\begin{bmatrix} 0 & 0 \\ -C_\phi\,\dot{\phi} & 0 \\ S_\phi\,\dot{\phi} & 0 \end{bmatrix}$$

3.2 Extend the MATLAB script you developed in Exercise 2.2 to ***numerically*** evaluate the ***matrix equations*** for ***angular acceleration*** you derived in Exercise 3.1 using the data below.

$\theta=-30$ (deg) $\quad \phi=60$ (deg)

$\dot{\theta}=3\left(\text{rad/s}\right) \quad \dot{\phi}=7\left(\text{rad/s}\right) \quad \ddot{\theta}=-2\left(\text{rad/s}^2\right) \quad \ddot{\phi}=5\left(\text{rad/s}^2\right)$

In Exercise 2.2, the ***fixed frame*** and ***body frame components*** of ${}^R\underset{\sim}{\omega}_D$ the ***angular velocity*** of D in R and the ***fixed frame*** and ***body frame components*** of the ***partial angular velocity matrices*** were found to be as follows.

$$\left\{\omega_D\right\}=\begin{Bmatrix} 6.0622 \\ 3.5000 \\ -3.0000 \end{Bmatrix} \text{(rad/s)} \qquad \left[{}^R\omega_{D,\dot{\beta}}\right]=\begin{bmatrix} 0 & 0.86603 \\ 0 & 0.50000 \\ -1 & 0 \end{bmatrix}$$

$$\left\{\omega'_D\right\}=\begin{Bmatrix} 7.0000 \\ -2.5981 \\ -1.5000 \end{Bmatrix} \text{(rad/s)} \qquad \left[{}^R\omega'_{D,\dot{\beta}}\right]=\begin{bmatrix} 0 & 1 \\ -0.86603 & 0 \\ -0.50000 & 0 \end{bmatrix}$$

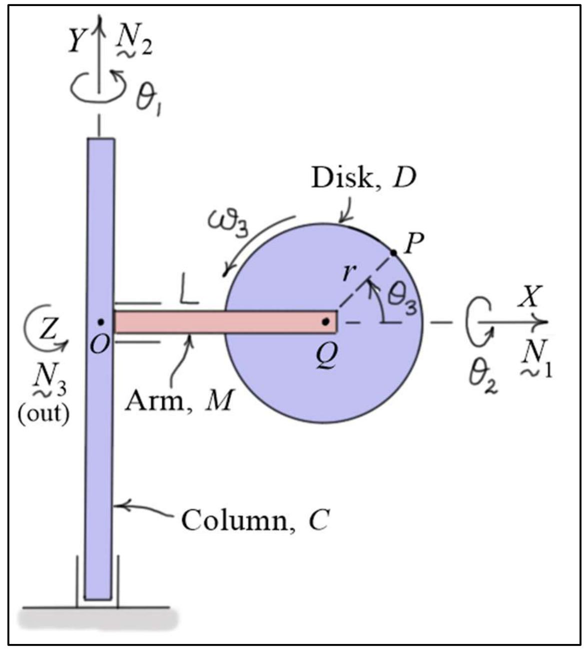
Calculate the ***angular acceleration vectors*** using the partial angular velocity matrices and their derivatives.

Answers:

$$\left\{\alpha_D\right\}=\begin{Bmatrix} 14.830 \\ -15.687 \\ 2.0000 \end{Bmatrix} \text{(rad/s}^2\text{)} \qquad \left[{}^R\dot{\omega}_{D,\dot{\beta}}\right]=\begin{bmatrix} 0 & 1.5000 \\ 0 & -2.5981 \\ 0 & 0 \end{bmatrix}$$

$$\left\{\alpha'_D\right\}=\begin{Bmatrix} 5.0000 \\ -8.7679 \\ 19.187 \end{Bmatrix} \text{(rad/s}^2\text{)} \qquad \left[{}^R\dot{\omega}'_{D,\dot{\beta}}\right]=\begin{bmatrix} 0 & 0 \\ -3.5000 & 0 \\ 6.0622 & 0 \end{bmatrix}$$

3.3 The system shown has ***three*** bodies, a vertical column C, a horizontal arm M, and a disk D. Disk D has radius r and is positioned relative to M using angle θ_3. Arm M has length L and is positioned relative to C using angle θ_2. Column C is positioned relative to the ***fixed frame*** (X,Y,Z) using angle θ_1. The unit vectors $\underset{\sim}{N}_i \; (i=1,2,3)$ are along the (X,Y,Z) directions. Given the diagram, disk D is positioned relative to (X,Y,Z) using a 2-1-3 ***body fixed*** rotation sequence. Using matrix notation, complete the following. In each case, find expressions for any general position where $\theta_1 \neq \theta_2 \neq \theta_3 \neq 0$. Note in the position shown in the diagram, that θ_1 and θ_2 are both zero. When $\theta_1 = \theta_2 = \theta_3 = 0$ all reference frames are aligned.



In Exercise 2.3, the ***fixed frame*** and ***body frame components*** of ${}^R\underset{\sim}{\omega}_D$ the ***angular velocity*** of D in R and the ***fixed frame*** and ***body frame components*** of the ***partial angular velocity matrices*** were found to be as follows.

$$\left\{\omega_D\right\} = \begin{Bmatrix} C_1\dot{\theta}_2 + S_1C_2\dot{\theta}_3 \\ \dot{\theta}_1 - S_2\dot{\theta}_3 \\ -S_1\dot{\theta}_2 + C_1C_2\dot{\theta}_3 \end{Bmatrix} = \begin{bmatrix} 0 & C_1 & S_1C_2 \\ 1 & 0 & -S_2 \\ 0 & -S_1 & C_1C_2 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} \triangleq \left[{}^R\omega_{D,\dot{\theta}}\right]\left\{\dot{\theta}\right\}$$

$$\left\{\omega'_D\right\} = \begin{Bmatrix} C_2S_3\dot{\theta}_1 + C_3\dot{\theta}_2 \\ C_2C_3\dot{\theta}_1 - S_3\dot{\theta}_2 \\ -S_2\dot{\theta}_1 + \dot{\theta}_3 \end{Bmatrix} = \begin{bmatrix} C_2S_3 & C_3 & 0 \\ C_2C_3 & -S_3 & 0 \\ -S_2 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} \triangleq \left[{}^R\omega'_{D,\dot{\theta}}\right]\left\{\dot{\theta}\right\}$$

Find $\left\{\alpha_D\right\}$ the ***fixed frame components*** and $\left\{\alpha'_D\right\}$ the ***body frame components*** of the ***angular acceleration*** of D in the fixed frame. Use the partial angular velocity matrices shown above.

Answers:

$$\left\{\alpha_D\right\} = \left\{\dot{\omega}_D\right\} = \left[{}^R\omega_{D,\dot{\theta}}\right]\left\{\ddot{\theta}\right\} + \left[{}^R\dot{\omega}_{D,\dot{\theta}}\right]\left\{\dot{\theta}\right\}$$

$$\left[{}^R\dot{\omega}_{D,\dot{\theta}}\right] = \frac{d}{dt}\begin{bmatrix} 0 & C_1 & S_1C_2 \\ 1 & 0 & -S_2 \\ 0 & -S_1 & C_1C_2 \end{bmatrix} = \begin{bmatrix} 0 & -S_1\dot{\theta}_1 & C_1C_2\dot{\theta}_1 - S_1S_2\dot{\theta}_2 \\ 0 & 0 & -C_2\dot{\theta}_2 \\ 0 & -C_1\dot{\theta}_1 & -S_1C_2\dot{\theta}_1 - C_1S_2\dot{\theta}_2 \end{bmatrix}$$

$$\left\{\alpha'_D\right\} = \left\{\dot{\omega}'_D\right\} = \left[{}^R\omega'_{D,\dot{\theta}}\right]\left\{\ddot{\theta}\right\} + \left[{}^R\dot{\omega}'_{D,\dot{\theta}}\right]\left\{\dot{\theta}\right\}$$

$$\left[{}^{R}\dot{\omega}'_{D,\dot{\theta}}\right] = \frac{d}{dt}\begin{bmatrix} C_2 S_3 & C_3 & 0 \\ C_2 C_3 & -S_3 & 0 \\ -S_2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -S_2 S_3 \dot{\theta}_2 + C_2 C_3 \dot{\theta}_3 & -S_3 \dot{\theta}_3 & 0 \\ -\left(S_2 C_3 \dot{\theta}_2 + C_2 S_3 \dot{\theta}_3\right) & -C_3 \dot{\theta}_3 & 0 \\ -C_2 \dot{\theta}_2 & 0 & 0 \end{bmatrix}$$

3.4 Extend the MATLAB script you developed in Exercise 2.4 to ***numerically evaluate*** the equations you derived in Exercise 3.3 using the data below.

$\theta_1 = 20$ (deg) $\quad \theta_2 = 40$ (deg) $\quad \theta_3 = 60$ (deg)

$\dot{\theta}_1 = 2$ (rad/s) $\quad \dot{\theta}_2 = -3$ (rad/s) $\quad \dot{\theta}_3 = 5$ (rad/s)

$\ddot{\theta}_1 = -5$ (rad/s^2) $\quad \ddot{\theta}_2 = 2$ (rad/s^2) $\quad \ddot{\theta}_3 = -3$ (rad/s^2)

In Exercise 2.4, the ***fixed frame*** and ***body frame components*** of ${}^{R}\underset{\sim}{\omega}_D$ the ***angular velocity*** of D in R and the ***fixed frame*** and ***body frame components*** of the ***partial angular velocity matrices*** were found to be as follows.

$$\left\{\omega_D\right\} = \begin{Bmatrix} -1.5091 \\ -1.2139 \\ 4.6253 \end{Bmatrix} \text{ (rad/s)} \qquad \left[{}^{R}\omega_{D,\dot{\theta}}\right] = \begin{bmatrix} 0 & 0.93969 & 0.26200 \\ 1 & 0 & -0.64279 \\ 0 & -0.34202 & 0.71985 \end{bmatrix}$$

$$\left\{\omega'_D\right\} = \begin{Bmatrix} -0.17317 \\ 3.3641 \\ 3.7144 \end{Bmatrix} \text{ (rad/s)} \qquad \left[{}^{R}\omega'_{D,\dot{\theta}}\right] = \begin{bmatrix} 0.66341 & 0.50000 & 0 \\ 0.38302 & -0.86603 & 0 \\ -0.64279 & 0 & 1 \end{bmatrix}$$

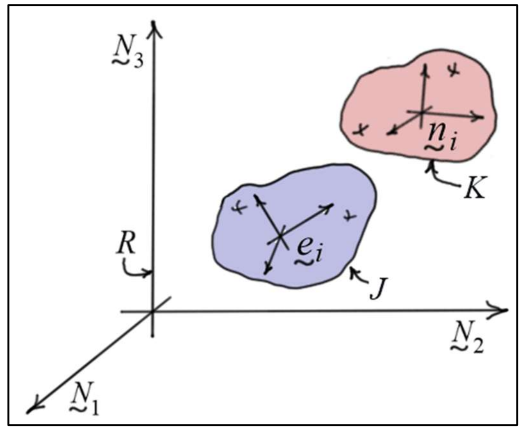
Calculate the ***fixed frame*** and ***body frame components*** of ${}^{R}\underset{\sim}{\alpha}_D$ the ***angular acceleration*** of disk D in R using the ***partial angular velocity matrices*** and their ***derivatives***.

Answers:

$$\left\{\alpha_D\right\} = \begin{Bmatrix} 13.642 \\ 8.4190 \\ 9.2349 \end{Bmatrix} \text{ (rad/s}^2\text{)} \qquad \left[{}^{R}\dot{\omega}_{D,\dot{\theta}}\right] = \begin{bmatrix} 0 & -0.68404 & 2.0992 \\ 0 & 0 & 2.2981 \\ 0 & -1.8794 & 1.2881 \end{bmatrix}$$

$$\left\{\alpha'_D\right\} = \begin{Bmatrix} 17.844 \\ -0.85294 \\ 4.8102 \end{Bmatrix} \text{ (rad/s}^2\text{)} \qquad \left[{}^{R}\dot{\omega}'_{D,\dot{\theta}}\right] = \begin{bmatrix} 3.5851 & -4.3301 & 0 \\ -2.3529 & -2.5000 & 0 \\ 2.2981 & 0 & 0 \end{bmatrix}$$

3.5 The two bodies shown are part of a multibody system. Body J is ***oriented*** with respect to the ***fixed frame*** R and body K is ***oriented*** with respect to ***body*** J both using 2-3-1 body fixed rotation sequences. The angles $\theta_{Ji}\ (i=1,2,3)$ give the orientation of ***body*** J ***relative*** to the ***fixed frame*** R, and the angles $\hat{\theta}_{Ki}\ (i=1,2,3)$ give the orientation of ***body*** K ***relative*** to ***body*** J.



Use the matrices $\{\theta_J\}_{3\times1}$ and $\{\theta_K\}_{3\times1}$ as the column matrices of angles $\theta_{Ji}\ (i=1,2,3)$ and $\hat{\theta}_{Ki}\ (i=1,2,3)$, respectively. In Exercise 2.5, the ***fixed frame*** and ***body frame components*** of the ***angular velocities*** of the two bodies in R and the ***fixed frame*** and ***body frame components*** of the ***partial angular velocity matrices*** were found to be as follows.

$$\{\omega_J\} = \begin{bmatrix} 0 & S_{J1} & C_{J1}C_{J2} \\ 1 & 0 & S_{J2} \\ 0 & C_{J1} & -S_{J1}C_{J2} \end{bmatrix} \begin{Bmatrix} \dot{\theta}_{J1} \\ \dot{\theta}_{J2} \\ \dot{\theta}_{J3} \end{Bmatrix} \triangleq \left[{}^R\omega_{J,\dot{\theta}_J}\right]\{\dot{\theta}_J\} + \underbrace{\left[{}^R\omega_{J,\dot{\theta}_K}\right]}_{\text{zero}}\{\dot{\theta}_K\}$$

$$\{\omega_K\} = \left[{}^R\omega_{K,\dot{\theta}_J}\right]\{\dot{\theta}_J\} + \left[{}^R\omega_{K,\dot{\theta}_K}\right]\{\dot{\theta}_K\}$$

$$\left[{}^R\omega_{K,\dot{\theta}_J}\right] = \left[{}^R\omega_{J,\dot{\theta}_J}\right] = \begin{bmatrix} 0 & S_{J1} & C_{J1}C_{J2} \\ 1 & 0 & S_{J2} \\ 0 & C_{J1} & -S_{J1}C_{J2} \end{bmatrix}$$

$$\left[{}^R\omega_{K,\dot{\theta}_K}\right] = [R_J]^T\left[{}^J\omega_{K,\dot{\theta}_K}\right] = [R_J]^T \begin{bmatrix} 0 & S_{K1} & C_{K1}C_{K2} \\ 1 & 0 & S_{K2} \\ 0 & C_{K1} & -S_{K1}C_{K2} \end{bmatrix}$$

$$\{\omega'_J\} = \begin{bmatrix} S_{J2} & 0 & 1 \\ C_{J2}C_{J3} & S_{J3} & 0 \\ -C_{J2}S_{J3} & C_{J3} & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_{J1} \\ \dot{\theta}_{J2} \\ \dot{\theta}_{J3} \end{Bmatrix} \triangleq \left[{}^R\omega'_{J,\dot{\theta}_J}\right]\{\dot{\theta}_J\} + \underbrace{\left[{}^R\omega'_{J,\dot{\theta}_K}\right]}_{\text{zero}}\{\dot{\theta}_K\}$$

$$\{\omega'_K\} = \left[{}^R\omega'_{K,\dot{\theta}_J}\right]\{\dot{\theta}_J\} + \left[{}^R\omega'_{K,\dot{\theta}_K}\right]\{\dot{\theta}_K\}$$

$$\left[{}^R\omega'_{K,\dot{\theta}_J}\right] = \left[{}^J R_K\right]\left[{}^R\omega'_{J,\dot{\theta}_J}\right] \qquad \left[{}^R\omega'_{K,\dot{\theta}_K}\right] = \left[{}^J\omega'_{K,\dot{\theta}_K}\right] = \begin{bmatrix} S_{K2} & 0 & 1 \\ C_{K2}C_{K3} & S_{K3} & 0 \\ -C_{K2}S_{K3} & C_{K3} & 0 \end{bmatrix}$$

a) Find $\{\alpha_J\}$ the ***fixed-frame components*** and $\{\alpha'_J\}$ the ***body-frame components*** of the ***angular acceleration*** of body J in R. Use the partial angular velocity matrices shown above.

b) Find $\{\alpha_K\}$ the ***fixed-frame components*** and $\{\alpha'_K\}$ the ***body-frame components*** of the ***angular acceleration*** of body K in R. Use ***two*** different approaches for the calculations. In the ***first approach***, calculate the angular acceleration components using the angular acceleration components of body J. In the ***second approach***, differentiate the angular velocity components in terms of the partial angular velocity matrices shown above.

Answers:

a)

$$\left\{\alpha_J\right\} = \begin{bmatrix} 0 & S_{J1} & C_{J1}C_{J2} \\ 1 & 0 & S_{J2} \\ 0 & C_{J1} & -S_{J1}C_{J2} \end{bmatrix} \left\{\ddot{\theta}_J\right\} + \begin{bmatrix} 0 & C_{J1}\dot{\theta}_{J1} & -\left(S_{J1}C_{J2}\dot{\theta}_{J1} + C_{J1}S_{J2}\dot{\theta}_{J2}\right) \\ 0 & 0 & C_{J2}\dot{\theta}_{J2} \\ 0 & -S_{J1}\dot{\theta}_{J1} & -C_{J1}C_{J2}\dot{\theta}_{J1} + S_{J1}S_{J2}\dot{\theta}_{J2} \end{bmatrix} \left\{\dot{\theta}_J\right\}$$

$$\left\{\alpha'_J\right\} = \begin{bmatrix} S_{J2} & 0 & 1 \\ C_{J2}C_{J3} & S_{J3} & 0 \\ -C_{J2}S_{J3} & C_{J3} & 0 \end{bmatrix} \left\{\ddot{\theta}_J\right\} + \begin{bmatrix} C_{J2}\dot{\theta}_{J2} & 0 & 0 \\ -\left(S_{J2}C_{J3}\dot{\theta}_{J2} + C_{J2}S_{J3}\dot{\theta}_{J3}\right) & C_{J3}\dot{\theta}_{J3} & 0 \\ S_{J2}S_{J3}\dot{\theta}_{J2} - C_{J2}C_{J3}\dot{\theta}_{J3} & -S_{J3}\dot{\theta}_{J3} & 0 \end{bmatrix} \left\{\dot{\theta}_J\right\}$$

b)

$$\begin{aligned} \left\{\alpha_K\right\} = {} & \begin{bmatrix} 0 & S_{J1} & C_{J1}C_{J2} \\ 1 & 0 & S_{J2} \\ 0 & C_{J1} & -S_{J1}C_{J2} \end{bmatrix} \left\{\ddot{\theta}_J\right\} + \begin{bmatrix} 0 & C_{J1}\dot{\theta}_{J1} & -\left(S_{J1}C_{J2}\dot{\theta}_{J1} + C_{J1}S_{J2}\dot{\theta}_{J2}\right) \\ 0 & 0 & C_{J2}\dot{\theta}_{J2} \\ 0 & -S_{J1}\dot{\theta}_{J1} & -C_{J1}C_{J2}\dot{\theta}_{J1} + S_{J1}S_{J2}\dot{\theta}_{J2} \end{bmatrix} \left\{\dot{\theta}_J\right\} \\ & + \left[R_J\right]^T \left(\begin{bmatrix} 0 & S_{K1} & C_{K1}C_{K2} \\ 1 & 0 & S_{K2} \\ 0 & C_{K1} & -S_{K1}C_{K2} \end{bmatrix} \left\{\ddot{\theta}_K\right\} + \begin{bmatrix} 0 & C_{K1}\dot{\theta}_{K1} & -\left(S_{K1}C_{K2}\dot{\theta}_{K1} + C_{K1}S_{K2}\dot{\theta}_{K2}\right) \\ 0 & 0 & C_{K2}\dot{\theta}_{K2} \\ 0 & -S_{K1}\dot{\theta}_{K1} & -C_{K1}C_{K2}\dot{\theta}_{K1} + S_{K1}S_{K2}\dot{\theta}_{K2} \end{bmatrix} \left\{\dot{\theta}_K\right\} \right) \\ & + \left[\tilde{\omega}_J\right]\left[R_J\right]^T \begin{bmatrix} 0 & S_{K1} & C_{K1}C_{K2} \\ 1 & 0 & S_{K2} \\ 0 & C_{K1} & -S_{K1}C_{K2} \end{bmatrix} \left\{\dot{\theta}_K\right\} \end{aligned}$$

$$\begin{aligned} \left\{\alpha'_K\right\} = {} & \left[{}^J R_K\right] \left(\begin{bmatrix} S_{J2} & 0 & 1 \\ C_{J2}C_{J3} & S_{J3} & 0 \\ -C_{J2}S_{J3} & C_{J3} & 0 \end{bmatrix} \left\{\ddot{\theta}_J\right\} + \begin{bmatrix} C_{J2}\dot{\theta}_{J2} & 0 & 0 \\ -\left(S_{J2}C_{J3}\dot{\theta}_{J2} + C_{J2}S_{J3}\dot{\theta}_{J3}\right) & C_{J3}\dot{\theta}_{J3} & 0 \\ S_{J2}S_{J3}\dot{\theta}_{J2} - C_{J2}C_{J3}\dot{\theta}_{J3} & -S_{J3}\dot{\theta}_{J3} & 0 \end{bmatrix} \left\{\dot{\theta}_J\right\} \right) \\ & + \left[{}^J\tilde{\omega}'_K\right]^T \left[{}^J R_K\right] \begin{bmatrix} S_{J2} & 0 & 1 \\ C_{J2}C_{J3} & S_{J3} & 0 \\ -C_{J2}S_{J3} & C_{J3} & 0 \end{bmatrix} \left\{\dot{\theta}_J\right\} \\ & + \begin{bmatrix} S_{K2} & 0 & 1 \\ C_{K2}C_{K3} & S_{K3} & 0 \\ -C_{K2}S_{K3} & C_{K3} & 0 \end{bmatrix} \left\{\ddot{\theta}_K\right\} + \begin{bmatrix} C_{K2}\dot{\theta}_{K2} & 0 & 0 \\ -\left(S_{K2}C_{K3}\dot{\theta}_{K2} + C_{K2}S_{K3}\dot{\theta}_{K3}\right) & C_{K3}\dot{\theta}_{K3} & 0 \\ S_{K2}S_{K3}\dot{\theta}_{K2} - C_{K2}C_{K3}\dot{\theta}_{K3} & -S_{K3}\dot{\theta}_{K3} & 0 \end{bmatrix} \left\{\dot{\theta}_K\right\} \end{aligned}$$

3.6 Extend the MATLAB script you developed in Exercise 2.6 to ***numerically evaluate*** the equations you derived in Exercise 3.5 using the data below.

$\theta_{J1} = 20$ (deg) $\quad \theta_{J2} = 40$ (deg) $\quad \theta_{J3} = 60$ (deg)

$\dot{\theta}_{J1} = 2$ (rad/s) $\quad \dot{\theta}_{J2} = -3$ (rad/s) $\quad \dot{\theta}_{J3} = 5$ (rad/s)

$\ddot{\theta}_{J1} = -5$ (rad/s^2) $\quad \ddot{\theta}_{J2} = 2$ (rad/s^2) $\quad \ddot{\theta}_{J3} = -3$ (rad/s^2)

$\hat{\theta}_{K1} = -30$ (deg) $\quad \hat{\theta}_{K2} = -20$ (deg) $\quad \hat{\theta}_{K3} = 40$ (deg)

$\dot{\hat{\theta}}_{K1} = -5$ (rad/s) $\quad \dot{\hat{\theta}}_{K2} = 4$ (rad/s) $\quad \dot{\hat{\theta}}_{K3} = 3$ (rad/s)

$\ddot{\hat{\theta}}_{K1} = -2$ (rad/s^2) $\quad \ddot{\hat{\theta}}_{K2} = 3$ (rad/s^2) $\quad \ddot{\hat{\theta}}_{K3} = -5$ (rad/s^2)

In Exercise 2.6, the ***fixed frame*** and ***body frame components*** of the ***angular velocities*** of bodies J and K in R and the ***fixed frame*** and ***body frame components*** of the ***partial angular velocity matrices*** were found to be as follows.

$$\{\omega_J\} = \begin{Bmatrix} 2.5732 \\ 5.2139 \\ -4.1291 \end{Bmatrix} \text{ (rad/s)} \qquad \left[{}^R\omega_{J,\dot{\theta}_J}\right] = \begin{bmatrix} 0 & 0.34202 & 0.71985 \\ 1 & 0 & 0.64279 \\ 0 & 0.93969 & -0.26200 \end{bmatrix} \qquad \left[{}^R\omega_{J,\dot{\theta}_K}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\{\omega_K\} = \begin{Bmatrix} 6.3088 \\ -0.043696 \\ -8.4492 \end{Bmatrix} \text{ (rad/s)} \qquad \left[{}^R\omega_{K,\dot{\theta}_J}\right] = \begin{bmatrix} 0 & 0.34202 & 0.71985 \\ 1 & 0 & 0.64279 \\ 0 & 0.93969 & -0.26200 \end{bmatrix}$$

$$\left[{}^R\omega_{K,\dot{\theta}_K}\right] = \begin{bmatrix} -0.0058133 & 0.24119 & 0.91392 \\ 0.38302 & -0.89593 & 0.080395 \\ 0.92372 & 0.37302 & -0.39785 \end{bmatrix}$$

$$\{\omega'_J\} = \begin{Bmatrix} 6.2856 \\ -1.8320 \\ -2.8268 \end{Bmatrix} \text{ (rad/s)} \qquad \left[{}^R\omega'_{J,\dot{\theta}_J}\right] = \begin{bmatrix} 0.64279 & 0 & 1 \\ 0.38302 & 0.86603 & 0 \\ -0.66341 & 0.50000 & 0 \end{bmatrix} \qquad \left[{}^R\omega'_{J,\dot{\theta}_K}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\{\omega'_K\} = \begin{Bmatrix} 9.1237 \\ -4.8847 \\ 2.0220 \end{Bmatrix} \text{ (rad/s)} \qquad \left[{}^R\omega'_{K,\dot{\theta}_J}\right] = \begin{bmatrix} 0.080395 & -0.061275 & 0.81380 \\ -0.24123 & 0.96724 & -0.094493 \\ -0.96713 & -0.24635 & -0.57341 \end{bmatrix}$$

$$\left[{}^R\omega'_{K,\dot{\theta}_K}\right] = \begin{bmatrix} -0.34202 & 0 & 1 \\ 0.71985 & 0.64279 & 0 \\ -0.60402 & 0.76604 & 0 \end{bmatrix}$$

Calculate the ***fixed frame*** and ***body frame components*** of the ***angular accelerations*** of the bodies in R using ***two*** different approaches. In the ***first approach***, calculate the angular acceleration components using the angular acceleration components of body *J*. In the ***second approach***, differentiate the angular velocity components in terms of the partial angular velocity matrices shown above.

Answers:

$$\left\{\alpha_J\right\}=\begin{Bmatrix}-0.67334\\-18.419\\-5.7786\end{Bmatrix}\ (\text{rad/s}^2)\qquad \left[{}^R\dot{\omega}_{J,\dot{\theta}_J}\right]=\begin{bmatrix}0 & 1.8794 & 1.2881\\0 & 0 & -2.2981\\0 & -0.68404 & -2.0992\end{bmatrix}\qquad \left[{}^R\dot{\omega}_{J,\dot{\theta}_K}\right]=[0]_{3\times 3}$$

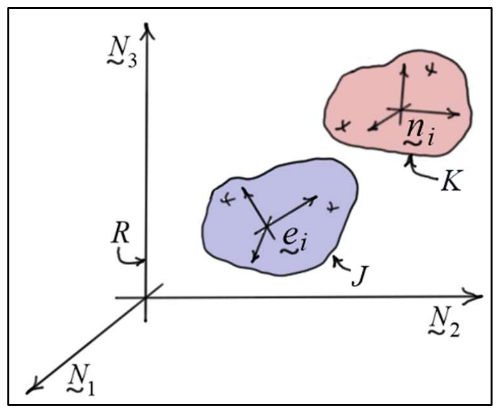
$$\left\{\alpha'_J\right\}=\begin{Bmatrix}-10.810\\-12.389\\10.137\end{Bmatrix}\ (\text{rad/s}^2)\qquad \left[{}^R\dot{\omega}'_{J,\dot{\theta}_J}\right]=\begin{bmatrix}-2.2981 & 0 & 0\\-2.3529 & 2.5 & 0\\-3.5851 & -4.3301 & 0\end{bmatrix}\qquad \left[{}^R\dot{\omega}'_{J,\dot{\theta}_K}\right]=[0]_{3\times 3}$$

$$\left\{\alpha_K\right\}=\begin{Bmatrix}-60.834\\-38.468\\-20.464\end{Bmatrix}\ (\text{rad/s}^2)\qquad \left[{}^R\dot{\omega}_{K,\dot{\theta}_J}\right]=\left[{}^R\dot{\omega}_{J,\dot{\theta}_J}\right]\qquad \left[{}^R\dot{\omega}_{K,\dot{\theta}_K}\right]=\begin{bmatrix}6.3978 & -6.6068 & 0.69666\\-2.3529 & -3.0806 & -5.2120\\1.0159 & -3.1271 & 0.54713\end{bmatrix}$$

$$\left\{\alpha'_K\right\}=\begin{Bmatrix}-50.548\\-33.411\\43.908\end{Bmatrix}\ (\text{rad/s}^2)\qquad \left[{}^R\dot{\omega}'_{K,\dot{\theta}_J}\right]=\begin{bmatrix}-5.2120 & 2.7422 & -1.1644\\-8.98640 & -1.9656 & -7.6522\\1.8082 & -8.3995 & -0.39158\end{bmatrix}$$

$$\left[{}^R\dot{\omega}'_{K,\dot{\theta}_K}\right]=\begin{bmatrix}3.7588 & 0 & 0\\-0.76406 & 2.2981 & 0\\-3.0389 & -1.9284 & 0\end{bmatrix}$$

3.7 Consider again the two body system of Exercise 3.5. As before, ***body*** J is ***oriented*** with respect to the ***fixed frame*** R and ***body*** K is ***oriented*** with respect to ***body*** J both using body fixed orientation angle sequences. The ***fixed frame*** and ***body frame*** components of ${}^R\underset{\sim}{\omega}_J$ the angular velocity of body J relative to the fixed frame are $\omega_{Ji}\,(i=1,2,3)$ and $\omega'_{Ji}\,(i=1,2,3)$, respectively. The ***fixed frame*** and ***body frame*** components of ${}^R\underset{\sim}{\omega}_K$ the angular velocity of body K relative to the fixed frame are $\omega_{Ki}\,(i=1,2,3)$ and $\omega'_{Ki}\,(i=1,2,3)$, respectively. The ***base frame*** (body J frame) and ***body frame*** components of ${}^J\underset{\sim}{\omega}_K$ the angular velocity of ***body*** K relative to ***body*** J are $\hat{\omega}_{Ki}\,(i=1,2,3)$ and $\hat{\omega}'_{Ki}\,(i=1,2,3)$, respectively. It was shown in Exercise 2.7 using the ***relative angular velocity components*** as ***generalized speeds*** that the ***fixed frame*** and ***body frame components*** of the ***angular velocities*** of the two bodies in R and the ***fixed frame*** and ***body frame components*** of the ***partial angular velocity matrices*** are as follows.



$$\left\{\omega_J\right\}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\omega_{J1}\\\omega_{J2}\\\omega_{J3}\end{Bmatrix}\triangleq\left[{}^R\omega_{J,\omega_J}\right]\begin{Bmatrix}\omega_{J1}\\\omega_{J2}\\\omega_{J3}\end{Bmatrix}+\underbrace{\left[{}^R\omega_{J,\hat{\omega}_K}\right]}_{\text{zero}}\begin{Bmatrix}\hat{\omega}_{K1}\\\hat{\omega}_{K2}\\\hat{\omega}_{K3}\end{Bmatrix}$$

$$\left\{\omega_K\right\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{\omega_J\right\} + \left[R_J\right]^T \left\{\hat{\omega}_K\right\} \triangleq \left[{}^R\omega_{K,\omega_J}\right]\left\{\omega_J\right\} + \left[{}^R\omega_{K,\hat{\omega}_K}\right]\left\{\hat{\omega}_K\right\}$$

$$\left\{\omega'_J\right\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \omega'_{J1} \\ \omega'_{J2} \\ \omega'_{J3} \end{Bmatrix} \triangleq \left[{}^R\omega'_{J,\omega'_J}\right] \begin{Bmatrix} \omega'_{J1} \\ \omega_{J2} \\ \omega_{J3} \end{Bmatrix} + \underbrace{\left[{}^R\omega'_{J,\hat{\omega}'_K}\right]}_{\text{zero}} \begin{Bmatrix} \hat{\omega}'_{K1} \\ \hat{\omega}'_{K2} \\ \hat{\omega}'_{K3} \end{Bmatrix}$$

$$\left\{\omega'_K\right\} = \left[{}^J R_K\right]\left\{\omega'_J\right\} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{\hat{\omega}'_K\right\} \triangleq \left[{}^R\omega'_{K,\omega'_J}\right]\left\{\omega'_J\right\} + \left[{}^R\omega'_{K,\hat{\omega}'_K}\right]\left\{\hat{\omega}'_K\right\}$$

a) Find $\left\{\alpha_J\right\}$ the ***fixed frame components*** and $\left\{\alpha'_J\right\}$ the ***body frame components*** of the ***angular acceleration*** of body J in R.

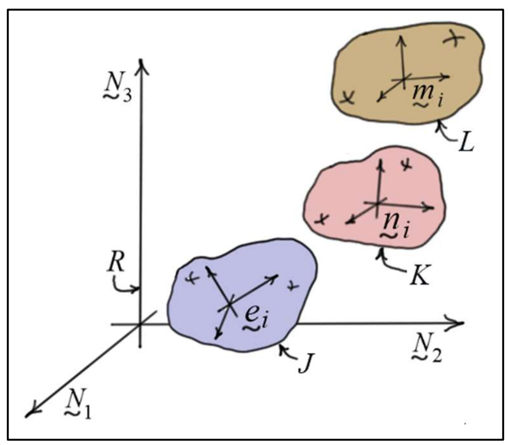
b) Find $\left\{\alpha_K\right\}$ the ***fixed frame components*** and $\left\{\alpha'_K\right\}$ the ***body frame components*** of the ***angular acceleration*** of body K in R.

Answers:

a) $\left\{\alpha_J\right\} = \left\{\dot{\omega}_J\right\} = \begin{Bmatrix} \dot{\omega}_{J1} \\ \dot{\omega}_{J2} \\ \dot{\omega}_{J3} \end{Bmatrix}$ $\quad$ $\left\{\alpha'_J\right\} = \left\{\dot{\omega}'_J\right\} = \begin{Bmatrix} \dot{\omega}'_{J1} \\ \dot{\omega}'_{J2} \\ \dot{\omega}'_{J3} \end{Bmatrix}$

b) $\left\{\alpha_K\right\} = \left\{\dot{\omega}_J\right\} + \left[R_J\right]^T \left\{\dot{\hat{\omega}}_K\right\} + \left[\tilde{\omega}_J\right]\left[R_J\right]^T \left\{\hat{\omega}_K\right\}$ $\quad$ $\left\{\alpha'_K\right\} = \left[{}^J R_K\right]\left\{\dot{\omega}'_J\right\} + \left\{\dot{\hat{\omega}}'_K\right\} + \left[{}^J\tilde{\omega}'_K\right]\left[{}^J R_K\right]\left\{\omega'_J\right\}$

3.8 Consider now a system with three bodies. As before, body J is ***oriented*** with respect to the ***fixed frame*** R. Body K is ***oriented*** with respect to ***body*** J, and body L is ***oriented*** with respect to ***body*** K. Body fixed orientation-angle sequences are used to describe the orientation of all three bodies relative to their base frames. The ***fixed frame*** and ***body frame*** components of ${}^R\underset{\sim}{\omega}_B$ $(B = J, K, L)$ the angular velocities of bodies relative to the fixed frame are ω_{Bi} $(B = J, K, L;\ i = 1,2,3)$ and ω'_{Bi} $(B = J, K, L;\ i = 1,2,3)$,



respectively. The ***base frame*** and ***body frame*** components of ${}^J\underset{\sim}{\omega}_K$ and ${}^K\underset{\sim}{\omega}_L$ the angular velocity of bodies K and L relative to their adjacent bodies are $\hat{\omega}_{Bi}$ $(B = K, L;\ i = 1,2,3)$ and $\hat{\omega}'_{Bi}$ $(B = K, L;\ i = 1,2,3)$, respectively. It was shown in Exercise 2.8 using the ***relative angular velocity components*** as ***generalized speeds*** that the ***fixed frame*** and ***body frame components*** of the ***angular velocities*** of the three bodies in R and the ***fixed frame*** and ***body frame components*** of the ***partial angular velocity matrices*** are as follows.

$$\left\{\omega_J\right\}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\omega_{J1}\\\omega_{J2}\\\omega_{J3}\end{Bmatrix}\triangleq\left[{}^R\omega_{J,\omega_J}\right]\begin{Bmatrix}\omega_{J1}\\\omega_{J2}\\\omega_{J3}\end{Bmatrix}+\underbrace{\left[{}^R\omega_{J,\hat{\omega}_K}\right]}_{\text{zero}}\begin{Bmatrix}\hat{\omega}_{K1}\\\hat{\omega}_{K2}\\\hat{\omega}_{K3}\end{Bmatrix}+\underbrace{\left[{}^R\omega_{J,\hat{\omega}_L}\right]}_{\text{zero}}\begin{Bmatrix}\hat{\omega}_{L1}\\\hat{\omega}_{L2}\\\hat{\omega}_{L3}\end{Bmatrix}$$

$$\left\{\omega_K\right\}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\left\{\omega_J\right\}+\left[R_J\right]^T\left\{\hat{\omega}_K\right\}\triangleq\left[{}^R\omega_{K,\omega_J}\right]\left\{\omega_J\right\}+\left[{}^R\omega_{K,\hat{\omega}_K}\right]\left\{\hat{\omega}_K\right\}+\underbrace{\left[{}^R\omega_{K,\hat{\omega}_L}\right]}_{\text{zero}}\left\{\hat{\omega}_L\right\}$$

$$\left\{\omega_L\right\}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\left\{\omega_J\right\}+\left[R_J\right]^T\left\{\hat{\omega}_K\right\}+\left[R_K\right]^T\left\{\hat{\omega}_L\right\}$$
$$\triangleq\left[{}^R\omega_{K,\omega_J}\right]\left\{\omega_J\right\}+\left[{}^R\omega_{K,\hat{\omega}_K}\right]\left\{\hat{\omega}_K\right\}+\left[{}^R\omega_{K,\hat{\omega}_L}\right]\left\{\hat{\omega}_L\right\}$$

$$\left\{\omega'_J\right\}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\omega'_{J1}\\\omega'_{J2}\\\omega'_{J3}\end{Bmatrix}\triangleq\left[{}^R\omega'_{J,\omega'_J}\right]\begin{Bmatrix}\omega'_{J1}\\\omega'_{J2}\\\omega'_{J3}\end{Bmatrix}+\underbrace{\left[{}^R\omega'_{J,\hat{\omega}'_K}\right]}_{\text{zero}}\begin{Bmatrix}\hat{\omega}'_{K1}\\\hat{\omega}'_{K2}\\\hat{\omega}'_{K3}\end{Bmatrix}+\underbrace{\left[{}^R\omega'_{J,\hat{\omega}'_L}\right]}_{\text{zero}}\begin{Bmatrix}\hat{\omega}'_{L1}\\\hat{\omega}'_{L2}\\\hat{\omega}'_{L3}\end{Bmatrix}$$

$$\left\{\omega'_K\right\}=\left[{}^JR_K\right]\left\{\omega'_J\right\}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\left\{\hat{\omega}'_K\right\}\triangleq\left[{}^R\omega'_{K,\omega'_J}\right]\left\{\omega'_J\right\}+\left[{}^R\omega'_{K,\hat{\omega}'_K}\right]\left\{\hat{\omega}'_K\right\}+\underbrace{\left[{}^R\omega'_{K,\hat{\omega}'_L}\right]}_{\text{zero}}\left\{\hat{\omega}'_L\right\}$$

$$\left\{\omega'_L\right\}=\left[{}^KR_L\right]\left[{}^JR_K\right]\left\{\omega'_J\right\}+\left[{}^KR_L\right]\left\{\hat{\omega}'_K\right\}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\left\{\hat{\omega}'_L\right\}$$
$$\triangleq\left[{}^R\omega'_{K,\omega'_J}\right]\left\{\omega'_J\right\}+\left[{}^R\omega'_{K,\hat{\omega}'_K}\right]\left\{\hat{\omega}'_K\right\}+\left[{}^R\omega'_{K,\hat{\omega}'_L}\right]\left\{\hat{\omega}'_L\right\}$$

a) Find $\left\{\alpha_J\right\}$ the ***fixed frame components*** and $\left\{\alpha'_J\right\}$ the ***body frame components*** of the ***angular acceleration*** of body J in R.

b) Find $\left\{\alpha_K\right\}$ the ***fixed frame components*** and $\left\{\alpha'_K\right\}$ the ***body frame components*** of the ***angular acceleration*** of body K in R.

c) Find $\left\{\alpha_L\right\}$ the ***fixed frame components*** and $\left\{\alpha'_L\right\}$ the ***body frame components*** of the ***angular acceleration*** of body L in R.

Answers:

a) $$\left\{\alpha_J\right\}=\left\{\dot{\omega}_J\right\}=\begin{Bmatrix}\dot{\omega}_{J1}\\\dot{\omega}_{J2}\\\dot{\omega}_{J3}\end{Bmatrix}\qquad\left\{\alpha'_J\right\}=\left\{\dot{\omega}'_J\right\}=\begin{Bmatrix}\dot{\omega}'_{J1}\\\dot{\omega}'_{J2}\\\dot{\omega}'_{J3}\end{Bmatrix}$$

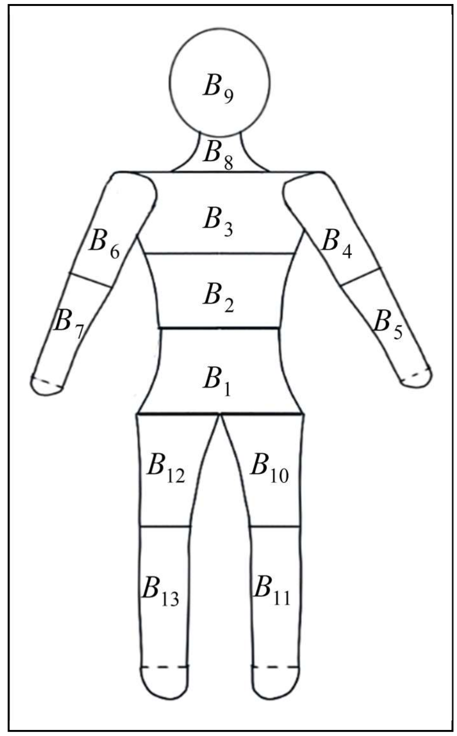
b) $$\left\{\alpha_K\right\}=\left\{\dot{\omega}_J\right\}+\left[R_J\right]^T\left\{\dot{\hat{\omega}}_K\right\}+\left[\tilde{\omega}_J\right]\left[R_J\right]^T\left\{\hat{\omega}_K\right\}$$

$$\left\{\alpha'_K\right\}=\left[{}^JR_K\right]\left\{\dot{\omega}'_J\right\}+\left[{}^J\tilde{\omega}'_K\right]^T\left[{}^JR_K\right]\left\{\omega'_J\right\}+\left\{\dot{\hat{\omega}}'_K\right\}$$

c) $\{\alpha_L\} = \{\dot{\omega}_J\} + [R_J]^T \{\dot{\hat{\omega}}_K\} + [\tilde{\omega}_J][R_J]^T \{\hat{\omega}_K\} + [R_K]^T \{\dot{\hat{\omega}}_L\} + [\tilde{\omega}_K][R_K]^T \{\hat{\omega}_L\}$

$$\{\alpha_L'\} = \left[{}^J R_L\right]\{\dot{\omega}_J'\} + \left[{}^K R_L\right]\left[{}^J \tilde{\omega}_K'\right]^T \left[{}^J R_K\right]\{\omega_J'\} + \left[{}^K R_L\right]\left\{\dot{\hat{\omega}}_K'\right\} + \left[{}^K \tilde{\omega}_L'\right]^T \left[{}^J R_L\right]\{\omega_J'\}$$
$$+ \left[{}^K \tilde{\omega}_L'\right]^T \left[{}^K R_L\right]\{\hat{\omega}_K'\} + \left\{\dot{\hat{\omega}}_L'\right\}$$

3.9 The figure shows a thirteen-body model of the human body numbered using the numbering scheme presented in Unit 1. Body 1 is the lower torso, and it is the system reference body. The rest of the bodies are numbered in ascending progression outward along the branches. As structured, the lower-numbered body array for the system is as follows.



$$\mathcal{L}(1,2,\ldots 12,13) = (0,1,2,3,4,3,6,3,8,1,10,1,12)$$

The orientation of body 1 is defined relative to the fixed frame $R:(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3)$, and the orientations of all the other bodies are defined relative to their adjacent, lower-numbered bodies. Using ***base frame components*** of the ***relative angular velocities*** of the bodies as ***generalized speeds***, complete the following. The 3×1 vectors $\{\omega_K\}$ $(K=1,\ldots,13)$ contain the ***fixed frame components*** of the angular velocities of the bodies. The 3×1 vectors $\{\hat{\omega}_K\}$ $(K=1,\ldots,13)$ are of the ***base frame components*** of the angular velocities of the bodies ***relative*** to their ***base frames*** (lower-numbered bodies). In Exercise 2.9, the ***fixed frame components*** of the ***angular velocities*** of the bodies were written as follows.

$$\{\omega_K\}_{3\times 1} = \left[{}^R\omega_{K,\omega}\right]_{3\times 39}\{\omega\}_{39\times 1} \quad (K=1,\ldots,13)$$

The matrix of ***angular velocity components*** $\{\omega\}_{39\times 1}$ were defined as follows. Note that the ***base frame*** and ***fixed frame components*** of the angular velocity of body 1 are the ***same***.

$$\{\omega\}_{39\times 1} = \left[(\hat{\omega}_1)_1 \quad (\hat{\omega}_1)_2 \quad (\hat{\omega}_1)_3 \quad (\hat{\omega}_2)_1 \quad (\hat{\omega}_2)_2 \quad (\hat{\omega}_2)_3 \quad \ldots \quad (\hat{\omega}_{13})_1 \quad (\hat{\omega}_{13})_2 \quad (\hat{\omega}_{13})_3\right]^T$$

The ***partial angular velocity matrices*** $\left[{}^R\omega_{K,\omega}\right]_{3\times 39}$ $(K=1,\ldots,13)$ were defined as follows.

$$
\begin{array}{c}
K=1 \rightarrow \\
K=2 \rightarrow \\
K=3 \rightarrow \\
K=4 \rightarrow \\
K=5 \rightarrow \\
K=6 \rightarrow \\
K=7 \rightarrow \\
K=8 \rightarrow \\
K=9 \rightarrow \\
K=10 \rightarrow \\
K=11 \rightarrow \\
K=12 \rightarrow \\
K=13 \rightarrow
\end{array}
\begin{bmatrix}
[I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[I] & [R_1]^T & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[I] & [R_1]^T & [R_2]^T & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[I] & [R_1]^T & [R_2]^T & [R_3]^T & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[I] & [R_1]^T & [R_2]^T & [R_3]^T & [R_4]^T & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[I] & [R_1]^T & [R_2]^T & [0] & [0] & [R_3]^T & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[I] & [R_1]^T & [R_2]^T & [0] & [0] & [R_3]^T & [R_6]^T & [0] & [0] & [0] & [0] & [0] & [0] \\
[I] & [R_1]^T & [R_2]^T & [0] & [0] & [0] & [0] & [R_3]^T & [0] & [0] & [0] & [0] & [0] \\
[I] & [R_1]^T & [R_2]^T & [0] & [0] & [0] & [0] & [R_3]^T & [R_8]^T & [0] & [0] & [0] & [0] \\
[I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [R_1]^T & [0] & [0] & [0] \\
[I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [R_1]^T & [R_{10}]^T & [0] & [0] \\
[I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [R_1]^T & [0] \\
[I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [R_1]^T & [R_{12}]^T
\end{bmatrix}
$$

Calculate $\{\alpha_K\}$ $(K=1,\ldots,13)$ the ***fixed frame components*** of the ***angular accelerations*** of all the bodies using ***two approaches***. In the first approach, use Equation (40) relating the angular acceleration of a body to the angular acceleration of its lower-numbered body. In the second, differentiate Equation (46).

Answers:

Body 1: $$\{\alpha_1\} = \{\dot{\omega}_1\} = \{\dot{\hat{\omega}}_1\} = \begin{bmatrix} \dot{\hat{\omega}}_{11} & \dot{\hat{\omega}}_{12} & \dot{\hat{\omega}}_{13} \end{bmatrix}^T$$

Body 2: $$\{\alpha_2\} = \{\dot{\hat{\omega}}_1\} + [R_1]^T\{\dot{\hat{\omega}}_2\} + [\tilde{\omega}_1][R_1]^T\{\hat{\omega}_2\}$$

Body 3: $$\{\alpha_3\} = \{\dot{\hat{\omega}}_1\} + [R_1]^T\{\dot{\hat{\omega}}_2\} + [\tilde{\omega}_1][R_1]^T\{\hat{\omega}_2\} + [R_2]^T\{\dot{\hat{\omega}}_3\} + [\tilde{\omega}_2][R_2]^T\{\hat{\omega}_3\}$$

Body 4: $$\begin{aligned}\{\alpha_4\} = {} & \{\dot{\hat{\omega}}_1\} + [R_1]^T\{\dot{\hat{\omega}}_2\} + [\tilde{\omega}_1][R_1]^T\{\hat{\omega}_2\} + [R_2]^T\{\dot{\hat{\omega}}_3\} + [\tilde{\omega}_2][R_2]^T\{\hat{\omega}_3\} \\ & + [R_3]^T\{\dot{\hat{\omega}}_4\} + [\tilde{\omega}_3][R_3]^T\{\hat{\omega}_4\}\end{aligned}$$

Body 5: $$\begin{aligned}\{\alpha_5\} = {} & \{\dot{\hat{\omega}}_1\} + [R_1]^T\{\dot{\hat{\omega}}_2\} + [\tilde{\omega}_1][R_1]^T\{\hat{\omega}_2\} + [R_2]^T\{\dot{\hat{\omega}}_3\} + [\tilde{\omega}_2][R_2]^T\{\hat{\omega}_3\} \\ & + [R_3]^T\{\dot{\hat{\omega}}_4\} + [\tilde{\omega}_3][R_3]^T\{\hat{\omega}_4\} + [R_4]^T\{\dot{\hat{\omega}}_5\} + [\tilde{\omega}_4][R_4]^T\{\hat{\omega}_5\}\end{aligned}$$

Body 6: $$\begin{aligned}\{\alpha_6\} = {} & \{\dot{\hat{\omega}}_1\} + [R_1]^T\{\dot{\hat{\omega}}_2\} + [\tilde{\omega}_1][R_1]^T\{\hat{\omega}_2\} + [R_2]^T\{\dot{\hat{\omega}}_3\} + [\tilde{\omega}_2][R_2]^T\{\hat{\omega}_3\} \\ & + [R_3]^T\{\dot{\hat{\omega}}_6\} + [\tilde{\omega}_3][R_3]^T\{\hat{\omega}_6\}\end{aligned}$$

Body 7: $$\begin{aligned}\{\alpha_7\} = {} & \{\dot{\hat{\omega}}_1\} + [R_1]^T\{\dot{\hat{\omega}}_2\} + [\tilde{\omega}_1][R_1]^T\{\hat{\omega}_2\} + [R_2]^T\{\dot{\hat{\omega}}_3\} + [\tilde{\omega}_2][R_2]^T\{\hat{\omega}_3\} \\ & + [R_3]^T\{\dot{\hat{\omega}}_6\} + [\tilde{\omega}_3][R_3]^T\{\hat{\omega}_6\} + [R_6]^T\{\dot{\hat{\omega}}_7\} + [\tilde{\omega}_6][R_6]^T\{\hat{\omega}_7\}\end{aligned}$$

Body 8:

$$\{\alpha_8\} = \{\dot{\hat{\omega}}_1\} + [R_1]^T\{\dot{\hat{\omega}}_2\} + [\tilde{\omega}_1][R_1]^T\{\hat{\omega}_2\} + [R_2]^T\{\dot{\hat{\omega}}_3\} + [\tilde{\omega}_2][R_2]^T\{\hat{\omega}_3\} + [R_3]^T\{\dot{\hat{\omega}}_8\} + [\tilde{\omega}_3][R_3]^T\{\hat{\omega}_8\}$$

Body 9:

$$\{\alpha_9\} = \{\dot{\hat{\omega}}_1\} + [R_1]^T\{\dot{\hat{\omega}}_2\} + [\tilde{\omega}_1][R_1]^T\{\hat{\omega}_2\} + [R_2]^T\{\dot{\hat{\omega}}_3\} + [\tilde{\omega}_2][R_2]^T\{\hat{\omega}_3\} + [R_3]^T\{\dot{\hat{\omega}}_8\} + [\tilde{\omega}_3][R_3]^T\{\hat{\omega}_8\} + [R_8]^T\{\dot{\hat{\omega}}_9\} + [\tilde{\omega}_8][R_8]^T\{\hat{\omega}_9\}$$

Body 10:

$$\{\alpha_{10}\} = \{\dot{\hat{\omega}}_1\} + [R_1]^T\{\dot{\hat{\omega}}_{10}\} + [\tilde{\omega}_1][R_1]^T\{\hat{\omega}_{10}\}$$

Body 11:

$$\{\alpha_{11}\} = \{\dot{\hat{\omega}}_1\} + [R_1]^T\{\dot{\hat{\omega}}_{10}\} + [\tilde{\omega}_1][R_1]^T\{\hat{\omega}_{10}\} + [R_{10}]^T\{\dot{\hat{\omega}}_{11}\} + [\tilde{\omega}_{10}][R_{10}]^T\{\hat{\omega}_{11}\}$$

Body 12:

$$\{\alpha_{12}\} = \{\dot{\hat{\omega}}_1\} + [R_1]^T\{\dot{\hat{\omega}}_{12}\} + [\tilde{\omega}_1][R_1]^T\{\hat{\omega}_{12}\}$$

Body 13:

$$\{\alpha_{13}\} = \{\dot{\hat{\omega}}_1\} + [R_1]^T\{\dot{\hat{\omega}}_{12}\} + [\tilde{\omega}_1][R_1]^T\{\hat{\omega}_{12}\} + [R_{12}]^T\{\dot{\hat{\omega}}_{13}\} + [\tilde{\omega}_{12}][R_{12}]^T\{\hat{\omega}_{13}\}$$

3.10 Consider again the thirteen-body model of the human body of Exercise 3.9. In Exercise 2.10, the ***body frame components*** of the ***angular velocities*** of the bodies were written as follows.

$$\{\omega'_K\}_{3\times 1} = [{}^R\omega'_{K,\omega'}]_{3\times 39}\{\omega'\}_{39\times 1} \quad (K = 1,\ldots,13)$$

The matrix of ***angular velocity components*** $\{\omega'\}_{39\times 1}$ and the ***partial angular velocity matrices*** $[{}^R\omega'_{K,\omega'}]_{3\times 39}$ $(K = 1,\ldots,13)$ were defined as follows.

$$\{\omega'\}_{39\times 1} = \left[(\hat{\omega}'_1)_1 \;\; (\hat{\omega}'_1)_2 \;\; (\hat{\omega}'_1)_3 \;\; (\hat{\omega}'_2)_1 \;\; (\hat{\omega}'_2)_2 \;\; (\hat{\omega}'_2)_3 \;\; \ldots \;\; (\hat{\omega}'_{13})_1 \;\; (\hat{\omega}'_{13})_2 \;\; (\hat{\omega}'_{13})_3 \right]^T$$

$$\begin{array}{l} K=1 \rightarrow \\ K=2 \rightarrow \\ K=3 \rightarrow \\ K=4 \rightarrow \\ K=5 \rightarrow \\ K=6 \rightarrow \\ K=7 \rightarrow \\ K=8 \rightarrow \\ K=9 \rightarrow \\ K=10 \rightarrow \\ K=11 \rightarrow \\ K=12 \rightarrow \\ K=13 \rightarrow \end{array}
\begin{bmatrix}
[I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[{}^1R_2] & [I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[{}^1R_3] & [{}^2R_3] & [I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[{}^1R_4] & [{}^2R_4] & [{}^3R_4] & [I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[{}^1R_5] & [{}^2R_5] & [{}^3R_5] & [{}^4R_5] & [I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[{}^1R_6] & [{}^2R_6] & [{}^3R_6] & [0] & [0] & [I] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\
[{}^1R_7] & [{}^2R_7] & [{}^3R_7] & [0] & [0] & [{}^6R_7] & [I] & [0] & [0] & [0] & [0] & [0] & [0] \\
[{}^1R_8] & [{}^2R_8] & [{}^3R_8] & [0] & [0] & [0] & [0] & [I] & [0] & [0] & [0] & [0] & [0] \\
[{}^1R_9] & [{}^2R_9] & [{}^3R_9] & [0] & [0] & [0] & [0] & [{}^8R_9] & [I] & [0] & [0] & [0] & [0] \\
[{}^1R_{10}] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [I] & [0] & [0] & [0] \\
[{}^1R_{11}] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [{}^{10}R_{11}] & [I] & [0] & [0] \\
[{}^1R_{12}] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [I] & [0] \\
[{}^1R_{13}] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [0] & [{}^{12}R_{13}] & [I]
\end{bmatrix}$$

Calculate $\{\alpha'_K\}$ $(K=1,\ldots,13)$ the ***body frame components*** of the ***angular accelerations*** of all the bodies using two approaches. In the first approach, use Equation (41) relating the angular acceleration of a body to the angular acceleration of its lower-numbered body. In the second, differentiate Equation (51).

Answers:

Body 1: $$\{\alpha'_1\}=\{\dot{\hat{\omega}}'_1\}$$

Body 2: $$\{\alpha'_2\}=\left[{}^1R_2\right]\{\dot{\hat{\omega}}_1\}+\{\dot{\hat{\omega}}_2\}+\left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]\{\hat{\omega}_1\}$$

Body 3: $$\begin{aligned}\{\alpha'_3\}=&\left[{}^1R_3\right]\{\dot{\hat{\omega}}'_1\}+\left[{}^2R_3\right]\{\dot{\hat{\omega}}'_2\}+\{\dot{\hat{\omega}}'_3\}+\left(\left[{}^2R_3\right]\left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]+\left[{}^2\tilde{\omega}'_3\right]^T\left[{}^2R_3\right]\left[{}^1R_2\right]\right)\{\hat{\omega}'_1\}\\&+\left[{}^2\tilde{\omega}'_3\right]^T\left[{}^2R_3\right]\{\hat{\omega}'_2\}\end{aligned}$$

Body 4: $$\begin{aligned}\{\alpha'_4\}=&\left[{}^1R_4\right]\{\dot{\hat{\omega}}'_1\}+\left[{}^2R_4\right]\{\dot{\hat{\omega}}'_2\}+\left[{}^3R_4\right]\{\dot{\hat{\omega}}'_3\}+\{\dot{\hat{\omega}}'_4\}\\&+\left(\left[{}^2R_4\right]\left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]+\left[{}^3R_4\right]\left[{}^2\tilde{\omega}'_3\right]^T\left[{}^1R_3\right]+\left[{}^3\tilde{\omega}'_4\right]^T\left[{}^1R_4\right]\right)\{\hat{\omega}'_1\}\\&+\left(\left[{}^3R_4\right]\left[{}^2\tilde{\omega}'_3\right]^T\left[{}^2R_3\right]+\left[{}^3\tilde{\omega}'_4\right]^T\left[{}^2R_4\right]\right)\{\hat{\omega}'_2\}+\left[{}^3\tilde{\omega}'_4\right]^T\left[{}^3R_4\right]\{\hat{\omega}'_3\}\end{aligned}$$

Body 5: $$\begin{aligned}\{\alpha'_5\}=&\left[{}^1R_5\right]\{\dot{\hat{\omega}}'_1\}+\left[{}^2R_5\right]\{\dot{\hat{\omega}}'_2\}+\left[{}^3R_5\right]\{\dot{\hat{\omega}}'_3\}+\left[{}^4R_5\right]\{\dot{\hat{\omega}}'_4\}+\{\dot{\hat{\omega}}'_5\}\\&+\left(\left[{}^2R_5\right]\left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]+\left[{}^3R_5\right]\left[{}^2\tilde{\omega}'_3\right]^T\left[{}^1R_3\right]+\left[{}^4R_5\right]\left[{}^3\tilde{\omega}'_4\right]^T\left[{}^1R_4\right]+\left[{}^4\tilde{\omega}'_5\right]^T\left[{}^1R_5\right]\right)\{\hat{\omega}'_1\}\\&+\left(\left[{}^3R_5\right]\left[{}^2\tilde{\omega}'_3\right]^T\left[{}^2R_3\right]+\left[{}^4R_5\right]\left[{}^3\tilde{\omega}'_4\right]^T\left[{}^2R_4\right]+\left[{}^4\tilde{\omega}'_5\right]^T\left[{}^2R_5\right]\right)\{\hat{\omega}'_2\}\\&+\left(\left[{}^4R_5\right]\left[{}^3\tilde{\omega}'_4\right]^T\left[{}^3R_4\right]+\left[{}^4\tilde{\omega}'_5\right]^T\left[{}^3R_5\right]\right)\{\hat{\omega}'_3\}+\left[{}^4\tilde{\omega}'_5\right]^T\left[{}^4R_5\right]\{\hat{\omega}'_4\}\end{aligned}$$

Body 6: $$\begin{aligned}\{\alpha'_6\}=&\left[{}^1R_6\right]\{\dot{\hat{\omega}}'_1\}+\left[{}^2R_6\right]\{\dot{\hat{\omega}}'_2\}+\left[{}^3R_6\right]\{\dot{\hat{\omega}}'_3\}+\{\dot{\hat{\omega}}'_6\}\\&+\left(\left[{}^2R_6\right]\left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]+\left[{}^3R_6\right]\left[{}^2\tilde{\omega}'_3\right]^T\left[{}^1R_3\right]+\left[{}^3\tilde{\omega}'_6\right]^T\left[{}^1R_6\right]\right)\{\hat{\omega}'_1\}\\&+\left(\left[{}^3R_6\right]\left[{}^2\tilde{\omega}'_3\right]^T\left[{}^2R_3\right]+\left[{}^3\tilde{\omega}'_6\right]^T\left[{}^2R_6\right]\right)\{\hat{\omega}'_2\}+\left[{}^3\tilde{\omega}'_6\right]^T\left[{}^3R_6\right]\{\hat{\omega}'_3\}\end{aligned}$$

Body 7: $$\begin{aligned}\{\alpha'_7\}=&\left[{}^1R_7\right]\{\dot{\hat{\omega}}'_1\}+\left[{}^2R_7\right]\{\dot{\hat{\omega}}'_2\}+\left[{}^3R_7\right]\{\dot{\hat{\omega}}'_3\}+\left[{}^6R_7\right]\{\dot{\hat{\omega}}'_6\}+\{\dot{\hat{\omega}}'_7\}\\&+\left(\left[{}^2R_7\right]\left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right]+\left[{}^3R_7\right]\left[{}^2\tilde{\omega}'_3\right]^T\left[{}^1R_3\right]+\left[{}^6R_7\right]\left[{}^3\tilde{\omega}'_6\right]^T\left[{}^1R_6\right]+\left[{}^6\tilde{\omega}'_7\right]^T\left[{}^1R_7\right]\right)\{\hat{\omega}'_1\}\\&+\left(\left[{}^3R_7\right]\left[{}^2\tilde{\omega}'_3\right]^T\left[{}^2R_3\right]+\left[{}^6R_7\right]\left[{}^3\tilde{\omega}'_6\right]^T\left[{}^2R_6\right]+\left[{}^6\tilde{\omega}'_7\right]^T\left[{}^2R_7\right]\right)\{\hat{\omega}'_2\}\\&+\left(\left[{}^6R_7\right]\left[{}^3\tilde{\omega}'_6\right]^T\left[{}^3R_6\right]+\left[{}^6\tilde{\omega}'_7\right]^T\left[{}^3R_7\right]\right)\{\hat{\omega}'_3\}+\left[{}^6\tilde{\omega}'_7\right]^T\left[{}^6R_7\right]\{\hat{\omega}'_6\}\end{aligned}$$

Body 8:
$$\begin{aligned}\left\{\alpha'_8\right\} &= \left[{}^1R_8\right]\left\{\dot{\hat{\omega}}'_1\right\} + \left[{}^2R_8\right]\left\{\dot{\hat{\omega}}'_2\right\} + \left[{}^3R_8\right]\left\{\dot{\hat{\omega}}'_3\right\} + \left\{\dot{\hat{\omega}}'_8\right\} \\ &+ \left(\left[{}^2R_8\right]\left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right] + \left[{}^3R_8\right]\left[{}^2\tilde{\omega}'_3\right]^T\left[{}^1R_3\right] + \left[{}^3\tilde{\omega}'_8\right]^T\left[{}^1R_8\right]\right)\left\{\hat{\omega}'_1\right\} \\ &+ \left(\left[{}^3R_8\right]\left[{}^2\tilde{\omega}'_3\right]^T\left[{}^2R_3\right] + \left[{}^3\tilde{\omega}'_8\right]^T\left[{}^2R_8\right]\right)\left\{\hat{\omega}'_2\right\} + \left[{}^3\tilde{\omega}'_8\right]^T\left[{}^3R_8\right]\left\{\hat{\omega}'_3\right\}\end{aligned}$$

Body 9:
$$\begin{aligned}\left\{\alpha'_9\right\} &= \left[{}^1R_9\right]\left\{\dot{\hat{\omega}}'_1\right\} + \left[{}^2R_9\right]\left\{\dot{\hat{\omega}}'_2\right\} + \left[{}^3R_9\right]\left\{\dot{\hat{\omega}}'_3\right\} + \left[{}^8R_9\right]\left\{\dot{\hat{\omega}}'_8\right\} + \left\{\dot{\hat{\omega}}'_9\right\} \\ &+ \left(\left[{}^2R_9\right]\left[{}^1\tilde{\omega}'_2\right]^T\left[{}^1R_2\right] + \left[{}^3R_9\right]\left[{}^2\tilde{\omega}'_3\right]^T\left[{}^1R_3\right] + \left[{}^8R_9\right]\left[{}^3\tilde{\omega}'_8\right]^T\left[{}^1R_8\right] + \left[{}^8\tilde{\omega}'_9\right]^T\left[{}^1R_9\right]\right)\left\{\hat{\omega}'_1\right\} \\ &+ \left(\left[{}^3R_9\right]\left[{}^2\tilde{\omega}'_3\right]^T\left[{}^2R_3\right] + \left[{}^8R_9\right]\left[{}^3\tilde{\omega}'_8\right]^T\left[{}^2R_8\right] + \left[{}^8\tilde{\omega}'_9\right]^T\left[{}^2R_9\right]\right)\left\{\hat{\omega}'_2\right\} \\ &+ \left(\left[{}^8R_9\right]\left[{}^3\tilde{\omega}'_8\right]^T\left[{}^3R_8\right] + \left[{}^8\tilde{\omega}'_9\right]^T\left[{}^3R_9\right]\right)\left\{\hat{\omega}'_3\right\} + \left[{}^8\tilde{\omega}'_9\right]^T\left[{}^8R_9\right]\left\{\hat{\omega}'_8\right\}\end{aligned}$$

Body 10:
$$\left\{\alpha'_{10}\right\} = \left[{}^1R_{10}\right]\left\{\dot{\hat{\omega}}'_1\right\} + \left\{\dot{\hat{\omega}}'_{10}\right\} + \left[{}^1\tilde{\omega}'_{10}\right]^T\left[{}^1R_{10}\right]\left\{\hat{\omega}'_1\right\}$$

Body 11:
$$\begin{aligned}\left\{\alpha'_{11}\right\} &= \left[{}^1R_{11}\right]\left\{\dot{\hat{\omega}}'_1\right\} + \left[{}^{10}R_{11}\right]\left\{\dot{\hat{\omega}}'_{10}\right\} + \left\{\dot{\hat{\omega}}'_{11}\right\} \\ &+ \left(\left[{}^{10}R_{11}\right]\left[{}^1\tilde{\omega}'_{10}\right]^T\left[{}^1R_{10}\right] + \left[{}^{10}\tilde{\omega}'_{11}\right]^T\left[{}^1R_{11}\right]\right)\left\{\hat{\omega}'_1\right\} + \left[{}^{10}\tilde{\omega}'_{11}\right]^T\left[{}^{10}R_{11}\right]\left\{\hat{\omega}'_{10}\right\}\end{aligned}$$

Body 12:
$$\left\{\alpha'_{12}\right\} = \left[{}^1R_{12}\right]\left\{\dot{\hat{\omega}}'_1\right\} + \left\{\dot{\hat{\omega}}'_{12}\right\} + \left[{}^1\tilde{\omega}'_{12}\right]^T\left[{}^1R_{12}\right]\left\{\hat{\omega}'_1\right\}$$

Body 13:
$$\begin{aligned}\left\{\alpha'_{13}\right\} &= \left[{}^1R_{13}\right]\left\{\dot{\hat{\omega}}'_1\right\} + \left[{}^{12}R_{13}\right]\left\{\dot{\hat{\omega}}'_{12}\right\} + \left\{\dot{\hat{\omega}}'_{13}\right\} \\ &+ \left(\left[{}^{12}R_{13}\right]\left[{}^1\tilde{\omega}'_{12}\right]^T\left[{}^1R_{12}\right] + \left[{}^{12}\tilde{\omega}'_{13}\right]^T\left[{}^1R_{13}\right]\right)\left\{\hat{\omega}'_1\right\} + \left[{}^{12}\tilde{\omega}'_{13}\right]^T\left[{}^{12}R_{13}\right]\left\{\hat{\omega}'_{12}\right\}\end{aligned}$$

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