An Introduction to
Three-Dimensional, Rigid Body Dynamics

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Volume III: Introduction to Multibody Kinematics

Unit 4

Velocity and Partial Velocity

Summary

This unit focuses on the ***matrix-based*** calculation of components of ***velocity vectors*** and ***partial velocity matrices***. The calculations are based on ***absolute*** and ***relative coordinates***. Both ***orientation angle derivatives*** and ***angular velocity components*** can be used as ***generalized speeds***. Algorithms are developed for the efficient calculation of these quantities for multibody systems.

Explicit results are generated for some ***examples*** with the ***purpose*** of being ***clear*** about ***how*** the calculations are done. However, keep in mind that the ***goal*** of developing such procedures is to ***implement*** them into ***computer algorithms***.

Results for calculating the ***derivatives*** of ***transformation matrices*** presented in Units 1 and 3 of this volume are ***repeated*** in this unit for convenience of the reader.

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Time Derivatives of Coordinate Transformation Matrices – Summary

The ***time derivatives*** of ***absolute*** and ***relative coordinate transformation matrices*** are derived in Units 1 and 3 of this volume. For the convenience of the reader, those results are repeated here as they are used repeatedly in the calculation of velocities of points within multibody systems when using relative coordinates. Continuing with the notation used in previous units, $\left[R_K\right]$ represents a matrix that ***transforms vector components*** from a ***fixed frame*** into the ***frame*** of ***body*** K, and $\left[{}^J R_K\right]$ represents a matrix that ***transforms vector components*** from the ***frame*** of ***body*** J to the ***frame*** of ***body*** K. The transposes of these matrices perform the inverses of these transformations.

The ***time derivative*** of the ***coordinate transformation matrix*** $\left[R_K\right]$ can be computed using either ***fixed frame*** or ***body frame components*** of ${}^R\underset{\sim}{\omega}_K$ the ***angular velocity*** of body K relative to the fixed frame R.

$$\boxed{\left[\dot{R}_K\right] \triangleq \frac{d}{dt}\left[R_K\right] = \left[R_K\right]\left[\tilde{\omega}_K\right]^T} \qquad \text{(using } \textit{\textbf{fixed frame components}} \text{ of } {}^R\underset{\sim}{\omega}_K\text{)} \qquad (1)$$

$$\boxed{\left[\dot{R}_K\right] \triangleq \frac{d}{dt}\left[R_K\right] = \left[\tilde{\omega}'_K\right]^T\left[R_K\right]} \qquad \text{(using } \textit{\textbf{body frame components}} \text{ of } {}^R\underset{\sim}{\omega}_K\text{)} \qquad (2)$$

Here, $\left[\tilde{\omega}_K\right]$ is the 3×3 ***skew-symmetric matrix*** constructed using the ***fixed frame components*** $\omega_{Ki}\;\left(i=1,2,3\right)$, and $\left[\tilde{\omega}'_K\right]$ is a 3×3 ***skew-symmetric matrix*** constructed using the ***body frame components*** $\omega'_{Ki}\;\left(i=1,2,3\right)$. Specifically,

$$\boxed{\left[\tilde{\omega}_K\right] = \begin{bmatrix} 0 & -\omega_{K3} & \omega_{K2} \\ \omega_{K3} & 0 & -\omega_{K1} \\ -\omega_{K2} & \omega_{K1} & 0 \end{bmatrix}} \quad \text{and} \quad \boxed{\left[\tilde{\omega}'_K\right] = \begin{bmatrix} 0 & -\omega'_{K3} & \omega'_{K2} \\ \omega'_{K3} & 0 & -\omega'_{K1} \\ -\omega'_{K2} & \omega'_{K1} & 0 \end{bmatrix}} \qquad (3)$$

These skew-symmetric matrices are used to calculate ***vector cross products*** as described in Unit 1 of this volume.

The ***time derivative*** of the ***relative coordinate transformation matrix*** $\left[{}^J R_K\right]$ can be computed using either ***fixed frame*** or ***body frame components*** of ${}^J\underset{\sim}{\omega}_K$ the ***angular velocity*** of body K ***relative*** to ***body*** J.

$$\boxed{\left[{}^J\dot{R}_K\right] = \left[{}^J R_K\right]\left[{}^J\tilde{\omega}_K\right]^T} \qquad \text{(using } \textit{\textbf{fixed frame components}} \text{ of } {}^J\underset{\sim}{\omega}_K\text{)} \qquad (4)$$

$$\boxed{\left[{}^J\dot{R}_K\right] = \left[{}^J\tilde{\omega}'_K\right]^T\left[{}^J R_K\right]} \qquad \text{(using } \textit{\textbf{body frame components}} \text{ of } {}^J\underset{\sim}{\omega}_K\text{)} \qquad (5)$$

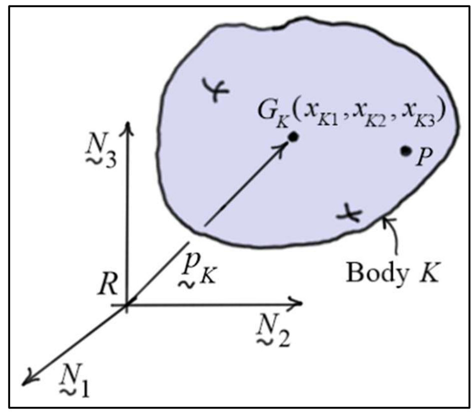
Here, $\left[{}^J\tilde{\omega}_K\right]$ is a 3×3 ***skew-symmetric matrix*** constructed using the ***fixed frame components*** $\hat{\omega}_{Ki}\;\left(i=1,2,3\right)$, and $\left[{}^J\tilde{\omega}'_K\right]$ is a 3×3 ***skew-symmetric matrix*** constructed using the ***body frame components*** $\hat{\omega}'_{Ki}\;\left(i=1,2,3\right)$. That is,

$$\left[{}^J\tilde{\omega}_K \right] = \begin{bmatrix} 0 & -\hat{\omega}_{K3} & \hat{\omega}_{K2} \\ \hat{\omega}_{K3} & 0 & -\hat{\omega}_{K1} \\ -\hat{\omega}_{K2} & \hat{\omega}_{K1} & 0 \end{bmatrix} \quad \text{and} \quad \left[{}^J\tilde{\omega}'_K \right] = \begin{bmatrix} 0 & -\hat{\omega}'_{K3} & \hat{\omega}'_{K2} \\ \hat{\omega}'_{K3} & 0 & -\hat{\omega}'_{K1} \\ -\hat{\omega}'_{K2} & \hat{\omega}'_{K1} & 0 \end{bmatrix} \tag{6}$$

Clearly, the results presented in Equations (4) and (5) are the ***same*** as those presented in Equations (1) and (2) when ${}^J\underset{\sim}{\omega}_K$ the angular velocity of body K ***relative*** to body J is used instead of ${}^R\underset{\sim}{\omega}_K$ the angular velocity of body K ***relative*** to a ***fixed frame***.

Mass Center Positions and Velocities Using Absolute Coordinates

The diagram shows a rigid body K whose orientation is given by a set of three ***orientation angles*** $\theta_{Ki}\ (i=1,2,3)$ and whose mass center G_K has ***Cartesian coordinates*** $x_{Ki}\ (i=1,2,3)$. Both the orientation angles and the Cartesian coordinates are measured ***relative*** to the ***fixed frame*** R. The components of $\underset{\sim}{p}_K$ the position vector of G_K and ${}^R\underset{\sim}{v}_G$ the velocity of can be written in matrix form as follows.



$$\{p_K\} = \begin{Bmatrix} x_{K1} \\ x_{K2} \\ x_{K3} \end{Bmatrix} \qquad \{v_K\} = \begin{Bmatrix} \dot{x}_{K1} \\ \dot{x}_{K2} \\ \dot{x}_{K3} \end{Bmatrix} \tag{7}$$

Defining $\{\theta_K\}$ as a 3×1 matrix whose elements are the orientation angles $\theta_{Ki}\ (i=1,2,3)$, and $\{x_K\}$ as a 3×1 matrix whose elements are the mass center position coordinates $x_{Ki}\ (i=1,2,3)$, then the velocity component matrix $\{v_K\}$ can be written as

$$\{v_K\} = \left[{}^R v_{K,\dot{\theta}_K} \right]\{\dot{\theta}_K\} + \left[{}^R v_{K,\dot{x}_K} \right]\{\dot{x}_K\} \tag{8}$$

Here, $\left[{}^R v_{K,\dot{\theta}_K} \right]$ and $\left[{}^R v_{K,\dot{x}_K} \right]$ are the ***partial velocity matrices*** associated with the ***time derivatives*** of the ***orientation angles*** and the ***mass center position coordinates***, respectively. These matrices are easily identified by comparing Equations (7) and (8) to be

$$\left[{}^R v_{K,\dot{\theta}_K} \right] = [0]_{3\times3} \qquad \left[{}^R v_{K,\dot{x}_K} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

As used in Unit 2, the subscripts "$,\dot{\theta}_K$" and "$,\dot{x}_K$" in Equations (8) and (9) indicate ***partial derivatives*** with respect the ***time derivatives*** of the ***orientation angles*** and the ***position coordinates***, respectively.

Velocities of Other Points in the Body

The ***fixed frame components*** of the ***velocities*** of ***other points*** in body *K*, such as point *P* in the diagram above, can be written in terms of the fixed frame components of ${}^R\underset{\sim}{v}_G$ the velocity of mass center *G* as follows.

$$\{v_P\} = \{v_G\} + [\tilde{\omega}_K][R_K]^T\{p'_{P/G}\} \quad (10)$$

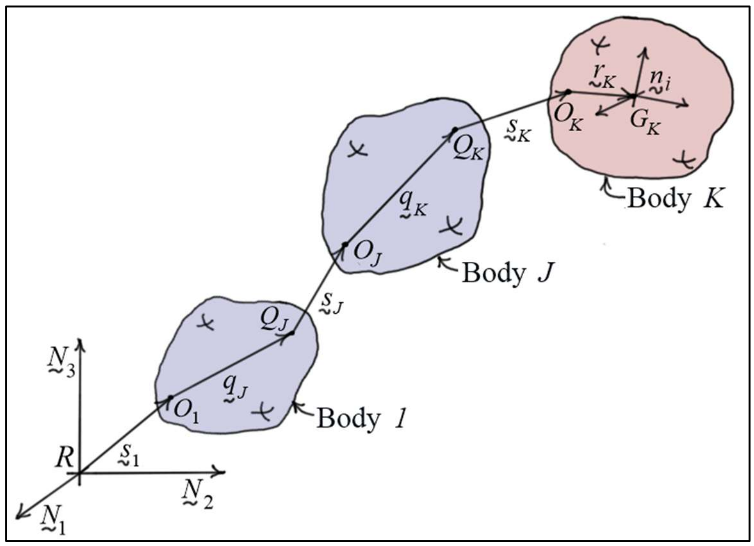
$$\{v_P\} = \{v_G\} + [R_K]^T[\tilde{\omega}'_K]\{p'_{P/G}\} \quad (11)$$

Equation (10) is written using the ***fixed frame components*** of ${}^R\underset{\sim}{\omega}_K$ the angular velocity of body *K* relative to the fixed frame, and Equation (11) is written using the ***body frame components*** of ${}^R\underset{\sim}{\omega}_K$. The 3×1 matrix $\{p'_{P/G}\}$ contains the ***body*** *K* ***components*** of $\underset{\sim}{r}_{P/G}$ the position vector of point *P* relative to the mass center *G*.

Mass Center Positions and Velocities Using Relative Coordinates

Basic Organization

To calculate the position and velocity of G_K the mass center of body *K* using ***relative coordinates***, consider three bodies that form a branch of a multibody system as shown in the diagram. Body *1* is the system's ***reference body*** and bodies *J* and *K* form a ***branch*** off body *1*. The ***motion*** of body *K* is measured ***relative*** to body *J*, the ***motion*** of body *J* is measured ***relative*** to body *1*, and the ***motion*** of body *1* is measured ***relative*** to the ***fixed frame*** *R*.



Consistent with the presentation of ***body-connection arrays*** in Unit 1, body *J* is the ***lower numbered body*** of body *K*, and body *1* is the ***lower numbered body*** of body *J*. So, the ***lower numbered body array*** is as follows.

$$\mathcal{L}(B_i = 1, J, K) = (\mathcal{L}(1), \mathcal{L}(J), \mathcal{L}(K)) = (0, 1, J)$$

When using ***relative coordinates***, it is ***convenient*** (but not necessary) to use the setup described below to ***organize*** (or structure) the ***development*** of the ***equations*** of ***motion***. This setup is ***not unique***, but for clarity, it will be used exclusively in the units that follow.

Organizational Setup:

1. For ***each body*** of the system an ***origin*** and a ***mass center*** are defined in the body. In the diagram, the points O_1, O_J, and O_K represent the ***origins*** for the three bodies. The point G_K represents the ***mass center*** of body *K*. The mass centers of bodies *1* and *J* are not shown.

2. For ***each body*** in the system, a ***reference point*** is also defined. A body's reference point is ***fixed in*** its ***lower numbered body*** and represents the ***stepping off point*** for that body. In the diagram, the points Q_J and Q_K are the ***reference points*** for bodies *J* and *K*, and they are fixed in the lower numbered bodies. The ***reference point*** for body 1 is taken as the ***origin*** of the ***fixed frame*** *R*.
3. The ***base frame*** for each body is the ***reference frame*** of its ***lower numbered body*** as indicated by the lower numbered body array.

Positions and Velocities of the Origins of Bodies in Fixed Frame Components

The ***fixed frame components*** $\{p_{O_i}\}$ of the ***positions*** of the origins of the bodies can be written as follows.

$$\boxed{\{p_{O_1}\}=\{s_1'\}=[s_{11}' \quad s_{12}' \quad s_{13}']^T}$$

$$\boxed{\{p_{O_J}\}=\{p_{O_1}\}+\{p_{O_J/O_1}\}=\{p_{O_1}\}+[R_1]^T\left(\{q_J'\}+\{s_J'\}\right)}$$

$$\boxed{\{p_{O_K}\}=\{p_{O_J}\}+\{p_{O_K/O_J}\}=\{p_{O_J}\}+[R_J]^T\left(\{q_K'\}+\{s_K'\}\right)}$$

Or, more generally for a body in a ***multibody system***, the ***fixed frame components*** of the position vectors of the ***origins*** of the bodies can be written as follows.

$$\boxed{\{p_{O_K}\}=\{p_{O_{\mathfrak{L}(K)}}\}+[R_{\mathfrak{L}(K)}]^T\left(\{q_K'\}+\{s_K'\}\right)} \tag{12}$$

The ***starting condition*** for this ***recursive relationship*** is the position vector for the origin of body 1, the reference body of the system. Specifically, as stated above,

$$\boxed{\{p_{O_1}\}=\{s_1'\}=[s_{11}' \quad s_{12}' \quad s_{13}']^T} \tag{13}$$

Here, $\{q_K'\}$ and $\{s_K'\}$ represent the components of the vectors $\underset{\sim}{q}_K$ and $\underset{\sim}{s}_K$ resolved in body $J=\mathfrak{L}(K)$.

Using the ***fixed frame components*** of the ***angular velocities*** of the bodies, the ***fixed frame components*** of the ***velocities*** of the ***origins*** of the ***bodies*** can be found by differentiating Equations (12) and (13) as follows.

$$\begin{aligned}\{v_{O_K}\}=\{\dot{p}_{O_K}\}&=\{\dot{p}_{O_{\mathfrak{L}(K)}}\}+[\dot{R}_{\mathfrak{L}(K)}]^T\left(\{q_K'\}+\{s_K'\}\right)+[R_{\mathfrak{L}(K)}]^T\{\dot{s}_K'\}\\&=\{v_{O_{\mathfrak{L}(K)}}\}+[\tilde{\omega}_{\mathfrak{L}(K)}][R_{\mathfrak{L}(K)}]^T\left(\{q_K'\}+\{s_K'\}\right)+[R_{\mathfrak{L}(K)}]^T\{\dot{s}_K'\}\\&=\{v_{O_{\mathfrak{L}(K)}}\}+[\tilde{\omega}_{\mathfrak{L}(K)}]\left(\{q_K\}+\{s_K\}\right)+[R_{\mathfrak{L}(K)}]^T\{\dot{s}_K'\}\end{aligned}$$

Using the vector identity ${}^R\underset{\sim}{\omega}_{\mathfrak{L}(K)}\times\left(\underset{\sim}{q}+\underset{\sim}{s}\right)=-\left(\underset{\sim}{q}+\underset{\sim}{s}\right)\times{}^R\underset{\sim}{\omega}_{\mathfrak{L}(K)}$ gives the following result.

$$\boxed{\{v_{O_K}\}=\{v_{O_{\mathfrak{L}(K)}}\}-\left([\tilde{q}_K]+[\tilde{s}_K]\right)\{\omega_{\mathfrak{L}(K)}\}+[R_{\mathfrak{L}(K)}]^T\{\dot{s}_K'\}} \tag{14}$$

The starting condition for this result is as follows.

$$\boxed{\left\{v_{O_1}\right\}=\left\{\dot{s}'_1\right\}=\left[\dot{s}'_{11} \quad \dot{s}'_{12} \quad \dot{s}'_{13}\right]^T} \tag{15}$$

Equation (14) represents a ***recursive relationship*** for finding the velocities of the origins of the bodies in terms of the velocities of the origins of their lower numbered bodies. Here, $\left\{v_{O_K}\right\}$ and $\left\{v_{O_{\mathfrak{L}(K)}}\right\}$ represent the ***fixed frame components*** of the ***velocities*** of the ***origins*** of body K and its lower numbered body, $\left[\tilde{q}_K\right]$ and $\left[\tilde{s}_K\right]$ represent 3×3 ***skew symmetric matrices*** built with the ***fixed frame components*** of position vectors $\underset{\sim}{q}_K$ and $\underset{\sim}{s}_K$, and $\left\{\omega_{\mathfrak{L}(K)}\right\}$ represents the ***fixed frame components*** of ${}^R\underset{\sim}{\omega}_{\mathfrak{L}(K)}$ the angular velocity of body $\mathfrak{L}(K)$.

Using the ***body frame components*** of the ***angular velocities*** of the bodies, the ***fixed frame components*** of the ***velocities*** of the ***origins*** of the ***bodies*** can be found by ***differentiating*** Equations (12) and (13) as follows.

$$\begin{aligned}\left\{v_{O_K}\right\}=\left\{\dot{p}_{O_K}\right\}&=\left\{\dot{p}_{O_{\mathfrak{L}(K)}}\right\}+\left[\dot{R}_{\mathfrak{L}(K)}\right]^T\left(\left\{q'_K\right\}+\left\{s'_K\right\}\right)+\left[R_{\mathfrak{L}(K)}\right]^T\left\{\dot{s}'_K\right\}\\&=\left\{v_{O_{\mathfrak{L}(K)}}\right\}+\left[R_{\mathfrak{L}(K)}\right]^T\left[\tilde{\omega}'_{\mathfrak{L}(K)}\right]\left(\left\{q'_K\right\}+\left\{s'_K\right\}\right)+\left[R_{\mathfrak{L}(K)}\right]^T\left\{\dot{s}'_K\right\}\end{aligned}$$

Using the vector identity ${}^R\underset{\sim}{\omega}_{\mathfrak{L}(K)}\times\left(\underset{\sim}{q}+\underset{\sim}{s}\right)=-\left(\underset{\sim}{q}+\underset{\sim}{s}\right)\times{}^R\underset{\sim}{\omega}_{\mathfrak{L}(K)}$ gives the following result.

$$\boxed{\left\{v_{O_K}\right\}=\left\{v_{O_{\mathfrak{L}(K)}}\right\}-\left[R_{\mathfrak{L}(K)}\right]^T\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\left\{\omega'_{\mathfrak{L}(K)}\right\}+\left[R_{\mathfrak{L}(K)}\right]^T\left\{\dot{s}'_K\right\}} \tag{16}$$

As before, the starting condition is as follows.

$$\boxed{\left\{v_{O_1}\right\}=\left\{\dot{s}'_1\right\}=\left[\dot{s}'_{11} \quad \dot{s}'_{12} \quad \dot{s}'_{13}\right]^T} \tag{17}$$

Like Equation (14), Equation (16) represents a ***recursive relationship*** for finding the velocities of the origins of the bodies in terms of the velocities of the origins of the lower numbered bodies. The ***first*** and ***third terms*** on the right side of Equation (16) are the ***same*** as those in Equation (14). The ***second term***, however, uses $\left\{\omega'_{\mathfrak{L}(K)}\right\}$ the ***same body components*** of ${}^R\underset{\sim}{\omega}_{\mathfrak{L}(K)}$, and the 3×3 skew symmetric matrices $\left[\tilde{q}'_K\right]$ and $\left[\tilde{s}'_K\right]$ are built using the body $\mathfrak{L}(K)$ components of the vectors $\underset{\sim}{q}_K$ and $\underset{\sim}{s}_K$.

<u>Mass Center Positions and Velocities in Fixed Frame Components</u>

Given the positions and velocities of the ***body origins***, the ***positions*** and ***velocities*** of the ***body mass centers*** can be calculated as follows. First, $\left\{p_K\right\}$ the ***fixed frame components*** of $\underset{\sim}{p}_K$ the position vector of G_K the mass center of body K can be written as follows.

$$\boxed{\left\{p_K\right\}=\left\{p_{O_K}\right\}+\left\{p_{G_K/O_K}\right\}=\left\{p_{O_K}\right\}+\left[R_K\right]^T\left\{r'_K\right\}} \tag{18}$$

Using the ***fixed frame components*** of the ***angular velocities*** of the bodies, the ***fixed frame components*** of ${}^R\underset{\sim}{v}_K$ the ***velocity*** of G_K the mass center of body *K* can be found by differentiating Equation (18) as follows.

$$\begin{aligned}\{v_K\} = \{\dot{p}_K\} &= \{\dot{p}_{O_K}\} + \left[\dot{R}_K\right]^T \{r'_K\} \\ &= \{v_{O_K}\} + [\tilde{\omega}_K][R_K]^T \{r'_K\} \\ &= \{v_{O_K}\} + [\tilde{\omega}_K]\{r_K\}\end{aligned}$$

Finally, using the vector identity ${}^R\underset{\sim}{\omega}_K \times \underset{\sim}{r}_K = -\underset{\sim}{r}_K \times {}^R\underset{\sim}{\omega}_K$ gives the following result.

$$\boxed{\{v_K\} = \{v_{O_K}\} - [\tilde{r}_K]\{\omega_K\}} \qquad (19)$$

In this result, the terms $\{v_K\}$, $\{v_{O_K}\}$, and $\{\omega_K\}$ all represent ***fixed frame components***, and the 3×3 skew symmetric matrix $[\tilde{r}_K]$ is built using ***fixed frame components*** of the position vector $\underset{\sim}{r}_K$.

Using the ***body frame components*** of the ***angular velocities*** of the bodies, the ***fixed frame components*** of ${}^R\underset{\sim}{v}_K$ the ***velocity*** of G_K the mass center of body *K* can be found by ***differentiating*** Equation (18) as follows.

$$\{v_K\} = \{\dot{p}_K\} = \{\dot{p}_{O_K}\} + \left[\dot{R}_K\right]^T \{r'_K\} = \{v_{O_K}\} + [R_K]^T [\tilde{\omega}'_K]\{r'_K\}$$

Finally, using the vector identity ${}^R\underset{\sim}{\omega}_K \times \underset{\sim}{r}_K = -\underset{\sim}{r}_K \times {}^R\underset{\sim}{\omega}_K$ gives the following result.

$$\boxed{\{v_K\} = \{v_{O_K}\} - [R_K]^T [\tilde{r}'_K]\{\omega'_K\}} \qquad (20)$$

In this result, the terms $\{v_K\}$ and $\{v_{O_K}\}$ represent ***fixed frame components*** of ${}^R\underset{\sim}{v}_K$ and ${}^R\underset{\sim}{v}_{O_K}$, the term $\{\omega'_K\}$ represents the ***body K frame components*** of ${}^R\underset{\sim}{\omega}_K$, and the 3×3 skew symmetric matrix $[\tilde{r}'_K]$ is built using ***body K frame components*** of the position vector $\underset{\sim}{r}_K$.

<u>Positions and Velocities of the Origins of Bodies in Body Frame Components</u>

The ***body frame components*** $\{p'_{O_i}\}$ of the ***positions*** of the origins of the bodies can be written as follows.

$$\boxed{\{p'_{O_1}\} = [R_1]\{s'_1\} = [R_1]\begin{Bmatrix} s'_{11} \\ s'_{12} \\ s'_{13} \end{Bmatrix}}$$

$$\boxed{\{p'_{O_J}\} = \left[{}^1R_J\right]\left(\{p'_{O_1}\} + \{q'_J\} + \{s'_J\}\right)}$$

$$\boxed{\{p'_{O_K}\} = \left[{}^JR_K\right]\left(\{p'_{O_J}\} + \{q'_K\} + \{s'_K\}\right)}$$

Or, more generally for a body in a ***multibody system***, the ***body frame components*** of the position vectors of the ***origins*** of the bodies can be written as follows.

$$\left\{ p'_{O_K} \right\} = \left[{}^{\mathfrak{L}(K)}R_K \right] \left(\left\{ p'_{O_{\mathfrak{L}(K)}} \right\} + \left\{ q'_K \right\} + \left\{ s'_K \right\} \right) \tag{21}$$

The ***starting condition*** for this ***recursive relationship*** is the position vector for the origin of body 1, the reference body of the system. Specifically, as stated above,

$$\left\{ p'_{O_1} \right\} = \left[R_1 \right] \left\{ s'_1 \right\} \tag{22}$$

The ***velocity*** of O_K the origin of body K can be found by ***differentiating*** using the ***derivative rule***. Using vector notation, write

$$\begin{aligned} {}^R\underset{\sim}{v}_{O_K} &= \frac{{}^R d}{dt}\left(\underset{\sim}{p}_{O_K}\right) = \frac{{}^R d}{dt}\left(\underset{\sim}{p}_{O_J} + \underset{\sim}{q}_K + \underset{\sim}{s}_K\right) = \frac{{}^R d}{dt}\left(\underset{\sim}{p}_{O_J}\right) + \frac{{}^R d}{dt}\left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right) \\ &= {}^R\underset{\sim}{v}_{O_J} + \frac{{}^J d}{dt}\left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right) + {}^R\underset{\sim}{\omega}_J \times \left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right) \\ &= {}^R\underset{\sim}{v}_{O_J} + \frac{{}^J d}{dt}\left(\underset{\sim}{s}_K\right) + {}^R\underset{\sim}{\omega}_J \times \left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right) \end{aligned}$$

$$ {}^R\underset{\sim}{v}_{O_K} = {}^R\underset{\sim}{v}_{O_J} + \frac{{}^J d}{dt}\left(\underset{\sim}{s}_K\right) + {}^R\underset{\sim}{\omega}_J \times \left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right) \tag{23}$$

Using ***body frame components*** of the ***angular velocities*** of the bodies, the ***body frame components*** of the ***velocity*** of O_K the origin of body K can be written as follows.

$$\left\{ v'_{O_K} \right\} = \left[{}^J R_K \right] \left(\left\{ v'_{O_J} \right\} + \left\{ \dot{s}'_K \right\} + \left[\tilde{\omega}'_J \right] \left(\left\{ q'_K \right\} + \left\{ s'_K \right\} \right) \right)$$

Or,

$$\left\{ v'_{O_K} \right\} = \left[{}^{\mathfrak{L}(K)}R_K \right] \left(\left\{ v'_{O_{\mathfrak{L}(K)}} \right\} + \left\{ \dot{s}'_K \right\} + \left[\tilde{\omega}'_{\mathfrak{L}(K)} \right] \left(\left\{ q'_K \right\} + \left\{ s'_K \right\} \right) \right) \tag{24}$$

Like Equations (14) and (16), Equation (24) is a ***recursive relationship*** for finding the velocities of the origins of the bodies in terms of the velocities of the origins of their lower numbered bodies. The starting condition is

$$\left\{ v'_{O_1} \right\} = \left[R_1 \right] \left\{ \dot{s}'_1 \right\} \tag{25}$$

Here, $\left\{ v'_{O_K} \right\}$ represents the ***body*** K ***components*** of ${}^R\underset{\sim}{v}_{O_K}$, $\left\{ v'_{O_{\mathfrak{L}(K)}} \right\}$ represents the ***body*** $\mathfrak{L}(K)$ ***components*** of ${}^R\underset{\sim}{v}_{O_{\mathfrak{L}(K)}}$, and the skew symmetric matrix $\left[\tilde{\omega}'_{\mathfrak{L}(K)} \right]$ is built using the body frame components of ${}^R\underset{\sim}{\omega}_{\mathfrak{L}(K)}$. The coefficient matrix $\left[{}^{\mathfrak{L}(K)}R_K \right]$ converts body $\mathfrak{L}(K)$ components into body K components.

Mass Center Positions and Velocities in Body Frame Components

Given the ***positions*** and ***velocities*** of the ***body origins***, the ***positions*** and ***velocities*** of the ***body mass centers*** can be calculated as follows. First, $\{p'_K\}$ the ***body frame components*** of $\underset{\sim}{p}_K$ the position vector of G_K the ***mass center*** of body *K* can be written as follows.

$$\boxed{\{p'_K\} = \left\{p'_{O_K}\right\} + \left\{p'_{G_K/O_K}\right\} = \left[{}^{\mathfrak{L}(K)}R_K\right]\left(\left\{p'_{O_{\mathfrak{L}(K)}}\right\} + \{q'_K\} + \{s'_K\}\right) + \{r'_K\}} \qquad (26)$$

Using the ***body frame components*** of the ***angular velocities*** of the bodies, the ***body frame components*** of ${}^R\underset{\sim}{v}_K$ the ***velocity*** of G_K the mass center of body *K* can be found as follows.

$$\boxed{\{v'_K\} = \left\{v'_{O_K}\right\} + \left\{v'_{G_K/O_K}\right\} = \left[{}^{\mathfrak{L}(K)}R_K\right]\left(\left\{v'_{O_{\mathfrak{L}(K)}}\right\} + \{\dot{s}'_K\} + \left[\tilde{\omega}'_{\mathfrak{L}(K)}\right]\left(\{q'_K\} + \{s'_K\}\right)\right) + \left[\tilde{\omega}'_K\right]\left[r'_K\right]}$$

Finally, using the vector identity ${}^R\underset{\sim}{\omega}_K \times \underset{\sim}{r}_K = -\underset{\sim}{r}_K \times {}^R\underset{\sim}{\omega}_K$ gives the following result.

$$\boxed{\{v'_K\} = \left\{v'_{O_K}\right\} + \left\{v'_{G_K/O_K}\right\} = \left[{}^{\mathfrak{L}(K)}R_K\right]\left(\left\{v'_{O_{\mathfrak{L}(K)}}\right\} + \{\dot{s}'_K\} + \left[\tilde{\omega}'_{\mathfrak{L}(K)}\right]\left(\{q'_K\} + \{s'_K\}\right)\right) - \left[\tilde{r}'_K\right]\{\omega'_K\}} \qquad (27)$$

Here, $\{v'_K\}$ represents the body *K* components of ${}^R\underset{\sim}{v}_K$, $\left\{v'_{O_K}\right\}$ represents the body *K* components of ${}^R\underset{\sim}{v}_{O_K}$, $\{\omega'_K\}$ represents the body *K* components of ${}^R\underset{\sim}{\omega}_K$, and the skew symmetric matrix $\left[\tilde{r}'_K\right]$ is built using the body *K* components of $\underset{\sim}{r}_K$.

Partial Velocities of the Body Origins and Mass Centers

Equations (14), (16), (19), (20), (24), and (27) can be used to find the ***partial velocities*** of the ***body origins*** and ***mass centers***. In a multibody system, these partial velocities ***depend*** on the ***interconnective structure*** of the bodies of the system and the ***choice*** of ***generalized speeds***. In this section, three choices of generalized speeds are considered. In each case, the ***system generalized speed column matrix*** for a system of *N* bodies is ***defined*** and ***partitioned*** as follows.

$$\boxed{\{y\}_{6N\times 1} \triangleq \begin{Bmatrix} \{y_1\}_{3N\times 1} \\ \{y_2\}_{3N\times 1} \end{Bmatrix}} \qquad (28)$$

In each case, the column matrix $\{y_2\}_{3N\times 1}$ is defined to be

$$\boxed{\{y_2\}_{3N\times 1} \triangleq \left[\dot{s}'_{11} \quad \dot{s}'_{12} \quad \dot{s}'_{13} \quad \cdots \quad \dot{s}'_{J1} \quad \dot{s}'_{J2} \quad \dot{s}'_{J3} \quad \cdots \quad \dot{s}'_{K1} \quad \dot{s}'_{K2} \quad \dot{s}'_{K3} \quad \cdots \quad \dot{s}'_{N1} \quad \dot{s}'_{N2} \quad \dot{s}'_{N3}\right]^T} \qquad (29)$$

Here, $\dot{s}'_{Bi}\ (i = 1,2,3)$ are the ***time derivatives*** of the ***body*** $\mathfrak{L}(B)$ ***components*** of the ***position vector*** of the ***origin*** of body *B* ***relative*** to its ***reference point*** fixed in body $\mathfrak{L}(B)$. As described below, three cases are considered for the column matrix $\{y_1\}_{3N\times 1}$.

The first case (***Case 1***) uses the ***lower body*** (base frame) ***components*** of the ***relative angular velocity vectors*** of the bodies. In this case, $\{y_1\}_{3N\times1}$ is defined as follows.

$$\{y_1\}_{3N\times1} \triangleq \left[\hat{\omega}_{11} \quad \hat{\omega}_{12} \quad \hat{\omega}_{13} \quad \cdots \quad \hat{\omega}_{J1} \quad \hat{\omega}_{J2} \quad \hat{\omega}_{J3} \quad \cdots \quad \hat{\omega}_{K1} \quad \hat{\omega}_{K2} \quad \hat{\omega}_{K3} \quad \cdots \quad \hat{\omega}_{N1} \quad \hat{\omega}_{N2} \quad \hat{\omega}_{N3}\right]^T \tag{30}$$

Here, $\hat{\omega}_{Bi}\ (i=1,2,3)$ are the ***base frame components*** of the ***angular velocity*** of body B ***relative*** to its ***lower numbered body*** $\mathfrak{L}(B)$.

The second case (***Case 2***) uses the ***body frame components*** of the ***relative angular velocity vectors***. In this case, $\{y_1\}_{3N\times1}$ is defined as follows.

$$\{y_1\}_{3N\times1} \triangleq \left[\hat{\omega}'_{11} \quad \hat{\omega}'_{12} \quad \hat{\omega}'_{13} \quad \cdots \quad \hat{\omega}'_{J1} \quad \hat{\omega}'_{J2} \quad \hat{\omega}'_{J3} \quad \cdots \quad \hat{\omega}'_{K1} \quad \hat{\omega}'_{K2} \quad \hat{\omega}'_{K3} \quad \cdots \quad \hat{\omega}'_{N1} \quad \hat{\omega}'_{N2} \quad \hat{\omega}'_{N3}\right]^T \tag{31}$$

Here, $\hat{\omega}'_{Bi}\ (i=1,2,3)$ are the ***body*** B ***frame components*** of the ***angular velocity*** of body B ***relative*** to $\mathfrak{L}(B)$ its ***lower numbered body***.

The third case (***Case 3***) assumes that ***angles*** are used to define the ***orientations*** of the bodies ***relative*** to their ***lower numbered bodies***, and it uses the ***time derivatives*** of those ***angles*** as ***generalized speeds*** for the system. In this case, $\{y_1\}_{3N\times1}$ is defined as follows.

$$\{y_1\}_{3N\times1} \triangleq \left[\dot{\hat{\theta}}_{11} \quad \dot{\hat{\theta}}_{12} \quad \dot{\hat{\theta}}_{13} \quad \cdots \quad \dot{\hat{\theta}}_{J1} \quad \dot{\hat{\theta}}_{J2} \quad \dot{\hat{\theta}}_{J3} \quad \cdots \quad \dot{\hat{\theta}}_{K1} \quad \dot{\hat{\theta}}_{K2} \quad \dot{\hat{\theta}}_{K3} \quad \cdots \quad \dot{\hat{\theta}}_{N1} \quad \dot{\hat{\theta}}_{N2} \quad \dot{\hat{\theta}}_{N3}\right]^T \tag{32}$$

Here, $\dot{\hat{\theta}}_{Bi}\ (B=1,\ldots,N;\ i=1,2,3)$ are the ***time derivatives*** of the ***angles*** that define the ***orientations*** of each of the bodies ***relative*** to their ***lower numbered bodies***. In this case, ***either base frame*** components or ***body frame*** components of the relative angular velocity vectors can be used.

Partial Velocities of the Body Origins in Fixed Frame Components

Using Equation (14), the ***fixed frame components*** of the ***partial velocity matrices*** of the ***origins*** of the bodies can be written in terms of the ***fixed frame components*** of the ***angular velocities*** of the bodies. First, rewrite $\{v_{O_K}\}$ the ***fixed frame components*** of the ***velocity*** of O_K the ***origin*** of body K as follows.

$$\begin{aligned}\{v_{O_K}\} &= \{v_{O_{\mathfrak{L}(K)}}\} - \left([\tilde{q}_K]+[\tilde{s}_K]\right)\{\omega_{\mathfrak{L}(K)}\} + \left[R_{\mathfrak{L}(K)}\right]^T\{\dot{s}'_K\} \\ &= \left[{}^R v_{O_{\mathfrak{L}(K)},y}\right]\{y\} - \left([\tilde{q}_K]+[\tilde{s}_K]\right)\left[{}^R\omega_{\mathfrak{L}(K),y}\right]\{y\} + \left[R_{\mathfrak{L}(K)}\right]^T\{\dot{s}'_K\} \\ &= \left[{}^R v_{O_K,y}\right]\{y\}\end{aligned} \tag{33}$$

Note here that $\left\{v_{O_{\mathfrak{L}(K)}}\right\}$ and $\left\{\omega_{\mathfrak{L}(K)}\right\}$ are the ***fixed frame components*** of the ***velocity*** of $O_{\mathfrak{L}(K)}$ and the ***fixed frame components*** of the ***angular velocity*** of body $\mathfrak{L}(K)$. They do not depend on $\left\{\dot{s}'_K\right\}$, so $\left[{}^R v_{O_K,y}\right]$ the ***partial velocity matrix*** of O_K the ***origin*** of body K using ***fixed frame components*** can be built as follows.

1. First, set

$$\left[{}^R v_{O_K,y}\right]_{3\times 6N} = \left[{}^R v_{O_{\mathfrak{L}(K)},y}\right]_{3\times 6N} - \left(\left[\tilde{q}_K\right]+\left[\tilde{s}_K\right]\right)\left[{}^R \omega_{\mathfrak{L}(K),y}\right]_{3\times 6N} \tag{34}$$

2. Then, set the ***three columns*** associated with $\dot{s}'_{Ki}$ $(i=1,2,3)$ as follows.

$$\left[{}^R v_{O_K,y}\right]_{ik} = \left[R_{\mathfrak{L}(K)}\right]^T_{ij} \quad \left(i=1,2,3;\ j=1,2,3;\ k=3N+(3K-3+j)\right) \tag{35}$$

For body 1, only Equation (35) applies. All entries are ***zero*** except for the three columns associated with $\left\{\dot{s}'_1\right\}$ giving the following result.

$$\left[{}^R v_{O_K,y}\right]_{ik} = \left[I\right]_{ij} \quad \left(i=1,2,3;\ j=1,2,3;\ k=3N+j;\ K=1\right) \tag{36}$$

Note in Equation (34) that $\left[{}^R \omega_{\mathfrak{L}(K),y}\right]$ the ***partial angular velocity matrix*** of body $\mathfrak{L}(K)$ can be partitioned as follows.

$$\left[{}^R \omega_{\mathfrak{L}(K),y}\right]_{3\times 6N} = \left[\left[{}^R \omega_{\mathfrak{L}(K),y_1}\right]_{3\times 3N} \quad \left[{}^R \omega_{\mathfrak{L}(K),y_2}\right]_{3\times 3N}\right]_{3\times 6N} \tag{37}$$

Because ${}^R\underset{\sim}{\omega}_{\mathfrak{L}(K)}$ does not depend on s'_{Bi} $(B=1,\ldots N;\ i=1,2,3)$ and their time derivatives, the entire ***latter half*** of $\left[{}^R \omega_{\mathfrak{L}(K),y}\right]_{3\times 6N}$ is ***zero***. That is,

$$\left[{}^R \omega_{\mathfrak{L}(K),y_2}\right]_{3\times 3N} = \left[0\right]_{3\times 3N} \tag{38}$$

Using Equation (16), the ***fixed frame components*** of the ***partial velocity matrices*** of the ***origins*** of the bodies can be written in terms of the ***body frame components*** of the ***angular velocities*** of the bodies. First, rewrite $\left\{v_{O_K}\right\}$ the ***fixed frame components*** of the ***velocity*** of O_K the origin of body K as follows.

$$\begin{aligned}\left\{v_{O_K}\right\} &= \left\{v_{O_{\mathfrak{L}(K)}}\right\} - \left[R_{\mathfrak{L}(K)}\right]^T\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\left\{\omega'_{\mathfrak{L}(K)}\right\} + \left[R_{\mathfrak{L}(K)}\right]^T\left\{\dot{s}'_K\right\} \\ &= \left[{}^R v_{O_{\mathfrak{L}(K)},y}\right]\left\{y\right\} - \left[R_{\mathfrak{L}(K)}\right]^T\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\left[{}^R \omega'_{\mathfrak{L}(K),y}\right]\left\{y\right\} + \left[R_{\mathfrak{L}(K)}\right]^T\left\{\dot{s}'_K\right\} \\ &= \left[{}^R v_{O_K,y}\right]\left\{y\right\}\end{aligned} \tag{39}$$

Note here that $\left\{v_{O_{\mathfrak{L}(K)}}\right\}$ are the ***fixed frame components*** of the ***velocity*** of $O_{\mathfrak{L}(K)}$ the ***origin*** of body $\mathfrak{L}(K)$ and $\left\{\omega'_{\mathfrak{L}(K)}\right\}$ are the ***body frame components*** of the ***angular velocity*** of body $\mathfrak{L}(K)$. They do not depend on $\left\{\dot{s}'_K\right\}$, so $\left[{}^{R}v_{O_K,y}\right]$ the ***partial velocity matrix*** of O_K the ***origin*** of body K using ***fixed frame components*** can be built as follows.

1. First, set

$$\left[{}^{R}v_{O_K,y}\right]_{3\times 6N} = \left[{}^{R}v_{O_{\mathfrak{L}(K)},y}\right]_{3\times 6N} - \left[R_{\mathfrak{L}(K)}\right]^T \left(\left[\tilde{q}'_K\right] + \left[\tilde{s}'_K\right]\right)\left[{}^{R}\omega'_{\mathfrak{L}(K),y}\right]_{3\times 6N} \tag{40}$$

2. Then, set the ***three columns*** associated with $\dot{s}'_{Ki}$ $(i = 1,2,3)$ as follows.

$$\left[{}^{R}v_{O_K,y}\right]_{ik} = \left[R_{\mathfrak{L}(K)}\right]^T_{ij} \quad \left(i = 1,2,3;\ j = 1,2,3;\ k = 3N + (3K - 3 + j)\right) \tag{41}$$

For body 1, only Equation (41) applies. All entries are ***zero*** except for the three columns associated with $\{\dot{s}'_1\}$ giving the following result.

$$\left[{}^{R}v_{O_K,y}\right]_{ik} = [I]_{ij} \quad \left(i = 1,2,3;\ j = 1,2,3;\ k = 3N + j;\ K = 1\right) \tag{42}$$

As noted in Equations (37) and (38) for $\left[{}^{R}\omega_{\mathfrak{L}(K),y}\right]$, the ***partial angular velocity matrix*** $\left[{}^{R}\omega'_{\mathfrak{L}(K),y}\right]$ of body $\mathfrak{L}(K)$ can be partitioned as follows.

$$\left[{}^{R}\omega'_{\mathfrak{L}(K),y}\right]_{3\times 6N} = \left[\left[{}^{R}\omega'_{\mathfrak{L}(K),y_1}\right]_{3\times 3N} \quad \left[{}^{R}\omega'_{\mathfrak{L}(K),y_2}\right]_{3\times 3N}\right]_{3\times 6N} = \left[\left[{}^{R}\omega'_{\mathfrak{L}(K),y_1}\right]_{3\times 3N} \quad [0]_{3\times 3N}\right]_{3\times 6N} \tag{43}$$

<u>Partial Velocities of the Body Mass Centers in Fixed Frame Components</u>

Using Equation (19), the ***fixed frame components*** of the ***partial velocity matrices*** of the ***mass centers*** of the bodies can be developed using the ***base frame components*** of the ***relative angular velocity vectors***. First, rewrite $\{v_K\}$ the ***fixed frame components*** of the ***velocity*** of G_K the mass center of body K as follows.

$$\begin{aligned} \{v_K\} &= \{v_{O_K}\} - [\tilde{r}_K]\{\omega_K\} \\ &= \left[{}^{R}v_{O_K,y}\right]\{y\} - [\tilde{r}_K]\left[{}^{R}\omega_{K,y}\right]\{y\} \\ &= \left[{}^{R}v_{K,y}\right]\{y\} \end{aligned} \tag{44}$$

Comparing the last two lines in Equation (44) gives the ***fixed frame components*** of $\left[{}^{R}v_{K,y}\right]_{3\times 6N}$ in terms of the ***fixed frame components*** of $\left[{}^{R}v_{O_K,y}\right]_{3\times 6N}$ and $\left[{}^{R}\omega_{K,y}\right]_{3\times 6N}$. That is,

$$\left[{}^{R}v_{K,y}\right]_{3\times 6N} = \left[{}^{R}v_{O_K,y}\right]_{3\times 6N} - [\tilde{r}_K]\left[{}^{R}\omega_{K,y}\right]_{3\times 6N} \tag{45}$$

As noted in Equation (37) above, the ***partial angular velocity matrix*** can be ***partitioned*** as follows.

$$\left[{}^{R}\omega_{K,y}\right]_{3\times 6N} = \left[\left[{}^{R}\omega_{K,y_1}\right]_{3\times 3N} \quad \left[{}^{R}\omega_{K,y_2}\right]_{3\times 3N}\right]_{3\times 6N} = \left[\left[{}^{R}\omega_{K,y_1}\right]_{3\times 3N} \quad [0]_{3\times 3N}\right]_{3\times 6N} \tag{46}$$

Using Equation (20), the ***fixed frame components*** of the ***partial velocity matrices*** of the ***mass centers*** of the bodies can be developed using the ***body frame components*** of the ***relative angular velocity*** vectors. First, rewrite $\{v_K\}$ the ***fixed frame components*** of the ***velocity*** of G_K the mass center of body K as follows.

$$\begin{aligned}\{v_K\} &= \{v_{O_K}\} - [R_K]^T[\tilde{r}'_K]\{\omega'_K\} \\ &= \left[{}^R v_{O_K,y}\right]\{y\} - [R_K]^T[\tilde{r}'_K]\left[{}^R\omega'_{K,y}\right]\{y\} \\ &= \left[{}^R v_{K,y}\right]\{y\}\end{aligned} \tag{47}$$

Comparing the last two lines of Equation (47) gives the ***fixed frame components*** of $\left[{}^R v_{K,y}\right]_{3\times 6N}$ in terms of the ***fixed frame components*** of $\left[{}^R v_{O_K,y}\right]_{3\times 6N}$ and the ***body frame components*** of $\left[{}^R\omega'_{K,y}\right]_{3\times 6N}$. That is,

$$\left[{}^R v_{K,y}\right]_{3\times 6N} = \left[{}^R v_{O_K,y}\right]_{3\times 6N} - [R_K]^T[\tilde{r}'_K]\left[{}^R\omega'_{K,y}\right]_{3\times 6N} \tag{48}$$

As noted in Equation (43) above, the ***partial angular velocity matrix*** can be ***partitioned*** as follows.

$$\left[{}^R\omega'_{K,y}\right]_{3\times 6N} = \left[\left[{}^R\omega'_{K,y_1}\right]_{3\times 3N} \quad \left[{}^R\omega'_{K,y_2}\right]_{3\times 3N}\right]_{3\times 6N} = \left[\left[{}^R\omega'_{K,y_1}\right]_{3\times 3N} \quad [0]_{3\times 3N}\right]_{3\times 6N} \tag{49}$$

<u>Partial Velocities of the Body Origins in Body Frame Components</u>

Using Equation (24), the ***body frame components*** of the ***partial velocity matrices*** of the ***origins*** of the bodies can be written in terms of the ***body frame components*** of the ***angular velocities*** of the bodies. First, rewrite $\{v'_{O_K}\}$ the ***body frame components*** of the ***velocity*** of O_K the origin of body K as follows.

$$\begin{aligned}\{v'_{O_K}\} &= \left[{}^{\mathfrak{L}(K)}R_K\right]\left(\{v'_{O_{\mathfrak{L}(K)}}\} + \{\dot{s}'_K\} + \left[\tilde{\omega}'_{\mathfrak{L}(K)}\right]\left(\{q'_K\}+\{s'_K\}\right)\right) \\ &= \left[{}^{\mathfrak{L}(K)}R_K\right]\{v'_{O_{\mathfrak{L}(K)}}\} + \left[{}^{\mathfrak{L}(K)}R_K\right]\{\dot{s}'_K\} + \left[{}^{\mathfrak{L}(K)}R_K\right]\left[\tilde{\omega}'_{\mathfrak{L}(K)}\right]\left(\{q'_K\}+\{s'_K\}\right) \\ &= \left[{}^{\mathfrak{L}(K)}R_K\right]\left[{}^R v'_{O_{\mathfrak{L}(K)},y}\right]\{y\} + \left[{}^{\mathfrak{L}(K)}R_K\right]\{\dot{s}'_K\} - \left[{}^{\mathfrak{L}(K)}R_K\right]\left([\tilde{q}'_K]+[\tilde{s}'_K]\right)\{\omega'_{\mathfrak{L}(K)}\} \\ &= \left[{}^{\mathfrak{L}(K)}R_K\right]\left[{}^R v'_{O_{\mathfrak{L}(K)},y}\right]\{y\} + \left[{}^{\mathfrak{L}(K)}R_K\right]\{\dot{s}'_K\} - \left[{}^{\mathfrak{L}(K)}R_K\right]\left([\tilde{q}'_K]+[\tilde{s}'_K]\right)\left[{}^R\omega'_{\mathfrak{L}(K),y}\right]\{y\}\end{aligned} \tag{50}$$

Note here that $\{v'_{O_{\mathfrak{L}(K)}}\}$ are the ***body frame components*** of the ***velocity*** of $O_{\mathfrak{L}(K)}$ the ***origin*** of body $\mathfrak{L}(K)$ and $\{\omega'_{\mathfrak{L}(K)}\}$ are the ***body frame components*** of the ***angular velocity*** of body $\mathfrak{L}(K)$. They do not depend on $\{\dot{s}'_K\}$, so $\left[{}^R v'_{O_K,y}\right]$ the ***partial velocity matrix*** of O_K the ***origin*** of body K using ***body frame components*** can be built as follows.

1. First, set

$$\left[{}^{R}v'_{O_K,y}\right]_{3\times 6N} = \left[{}^{\mathfrak{L}(K)}R_K\right]\left(\left[{}^{R}v'_{O_{\mathfrak{L}(K)},y}\right]_{3\times 6N} - \left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\left[{}^{R}\omega'_{\mathfrak{L}(K),y}\right]_{3\times 6N}\right) \tag{51}$$

2. Then, set the ***three columns*** associated with $\dot{s}'_{Ki}\ (i=1,2,3)$ as follows.

$$\left[{}^{R}v'_{O_K,y}\right]_{ik} = \left[{}^{\mathfrak{L}(K)}R_K\right]_{ij} \quad \left(i=1,2,3;\ j=1,2,3;\ k=3N+(3K-3+j)\right) \tag{52}$$

For body 1, only Equation (52) applies. All entries are ***zero*** except for the three columns associated with $\{\dot{s}'_1\}$ giving the following result.

$$\left[{}^{R}v'_{O_K,y}\right]_{ik} = \left[R_1\right]_{ij} \quad \left(i=1,2,3;\ j=1,2,3;\ k=3N+j;\ K=1\right) \tag{53}$$

Partial Velocities of the Body Mass Centers in Body Frame Components

Using Equation (27), the ***body frame components*** of the ***partial velocity matrices*** of the ***mass centers*** of the bodies can be developed using the ***body frame components*** of the ***relative angular velocity vectors*** as follows. First, rewrite $\{v'_K\}$ the ***fixed frame components*** of the ***velocity*** of G_K the mass center of body K as follows.

$$\begin{aligned}\{v'_K\} &= \{v'_{O_K}\} - [\tilde{r}'_K]\{\omega'_K\}\\ &= \left[{}^{R}v'_{O_K,y}\right]\{y\} - [\tilde{r}'_K]\left[{}^{R}\omega'_{K,y}\right]\{y\}\\ &= \left[{}^{R}v'_{K,y}\right]\{y\}\end{aligned} \tag{54}$$

Comparing the last two lines in Equation (54) gives $\left[{}^{R}v'_{K,y}\right]_{3\times 6N}$ in ***body frame components*** in terms of $\left[{}^{R}v'_{O_K,y}\right]_{3\times 6N}$ and $\left[{}^{R}\omega'_{K,y}\right]_{3\times 6N}$. That is,

$$\left[{}^{R}v'_{K,y}\right]_{3\times 6N} = \left[{}^{R}v'_{O_K,y}\right]_{3\times 6N} - [\tilde{r}'_K]\left[{}^{R}\omega'_{K,y}\right]_{3\times 6N} \tag{55}$$

Choice of Generalized Speeds

The equations developed above can be used to find the ***velocities*** and ***partial velocity matrices*** of the ***origins*** and ***mass centers*** of the bodies for ***each*** of the ***three choices*** of ***generalized speeds*** listed above.

Case 1:

Base frame (lower body) ***components*** of the ***relative angular velocity vectors*** can be used to find the ***fixed frame components*** of the ***angular velocities*** and ***partial angular velocities*** of the bodies. These, in turn, are used to find the ***fixed frame components*** of the ***velocities*** and ***partial velocities*** of the ***origins*** and ***mass centers*** of the bodies.

Case 2:

Body frame components of the ***relative angular velocity vectors*** can be used to find the ***body frame components*** of the ***angular velocities*** and ***partial angular velocities*** of the bodies. These, in turn, are used to find the ***fixed frame*** or ***body frame components*** of the ***velocities*** and ***partial velocities*** of the ***origins*** and ***mass centers*** of the bodies.

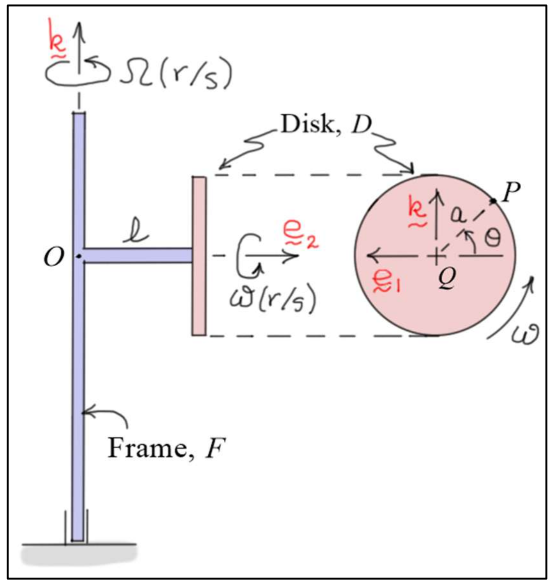
Case 3:

Angles can be used to define the ***orientations*** of bodies ***relative*** to their lower numbered bodies. The ***time derivatives*** of these angles can be used to find ***either*** the ***fixed frame*** or the ***body frame components*** of the ***angular velocities*** and ***partial angular velocities*** of the bodies. These, in turn, are used to find the ***fixed frame*** or ***body frame components*** of the ***velocities*** and ***partial velocities*** of the ***origins*** and ***mass centers*** of the bodies.

Examples

Example 1

The diagram shows a two-body system with frame F and disk D. Frame F rotates at a rate of $\dot{\phi} = \Omega$ (rad/s) about the fixed vertical direction (annotated by the unit vector $\underset{\sim}{k}$). Disk D is affixed to and rotates relative to F at a rate of $\dot{\theta} = \omega$ (rad/s) about the horizontal arm of F (direction annotated by the rotating unit vector $\underset{\sim}{e}_2$).



Reference frames: (all frames align when $\phi = \theta = 0$)

$R: (\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k})$ (fixed frame)

$F: (\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{k})$ (rotating with frame F)

$D: (\underset{\sim}{n}_1, \underset{\sim}{e}_2, \underset{\sim}{n}_3)$ (rotating with disk D)

Let $\{\hat{\omega}_F\}$ and $\{\hat{\omega}_D\}$ be the ***base frame components*** and let $\{\hat{\omega}'_F\}$ and $\{\hat{\omega}'_D\}$ be the ***body frame components*** of the ***relative angular velocities*** ${}^R\underset{\sim}{\omega}_F$ and ${}^F\underset{\sim}{\omega}_D$. Find the ***fixed frame components*** of the ***velocity*** of point P and the ***partial velocity matrix*** associated with the relative angular velocity components using a) ***base frame*** components, and b) ***body frame*** components. (c) Find the ***body frame components*** of the ***velocity*** of P and the ***partial velocity matrix*** associated with the ***body frame components*** of the ***relative angular velocities***.

Solution:

(a) The ***angular velocities*** of the two bodies can be calculated as follows.

$$\boxed{{}^R\underset{\sim}{\omega}_F = \Omega \underset{\sim}{k}} \quad \boxed{{}^R\underset{\sim}{\omega}_D = {}^R\underset{\sim}{\omega}_F + {}^F\underset{\sim}{\omega}_D}$$

The ***base frame components*** of the ***relative angular velocities*** are

$$\{\hat{\omega}_F\} = [0 \quad 0 \quad \Omega]^T \qquad \{\hat{\omega}_D\} = [0 \quad \omega \quad 0]^T$$

Using these results, the ***fixed frame components*** of the ***angular velocities*** of the bodies can be written as

$$\{\omega_F\} = \{\hat{\omega}_F\} \qquad \{\omega_D\} = \{\hat{\omega}_F\} + [R_F]^T \{\hat{\omega}_D\}$$

Defining the column vector of ***generalized speeds*** to be $\{y\} \triangleq \begin{Bmatrix} \{\hat{\omega}_F\} \\ \{\hat{\omega}_D\} \end{Bmatrix}$, the ***fixed frame components*** of the ***partial angular velocity matrices*** are observed to be

$$\left[{}^R\omega_{F,y}\right]_{3\times 6} = \left[\left[{}^R\omega_{F,\hat{\omega}_F}\right] \quad \left[{}^R\omega_{F,\hat{\omega}_D}\right]\right] = \left[[I]_{3\times 3} \quad [0]_{3\times 3}\right]$$

$$\left[{}^R\omega_{D,y}\right]_{3\times 6} = \left[\left[{}^R\omega_{D,\hat{\omega}_F}\right] \quad \left[{}^R\omega_{D,\hat{\omega}_D}\right]\right] = \left[[I]_{3\times 3} \quad [R_F]^T\right]$$

Using the concept of ***relative velocity***, ${}^R\underset{\sim}{v}_P$ the velocity of point P in R can be written as follows.

$${}^R\underset{\sim}{v}_P = {}^R\underset{\sim}{v}_{Q/O} + {}^R\underset{\sim}{v}_{P/Q} = \left({}^R\underset{\sim}{\omega}_F \times \underset{\sim}{r}_{Q/O}\right) + \left({}^R\underset{\sim}{\omega}_D \times \underset{\sim}{r}_{P/Q}\right) = -\left(\underset{\sim}{r}_{Q/O} \times {}^R\underset{\sim}{\omega}_F\right) - \left(\underset{\sim}{r}_{P/Q} \times {}^R\underset{\sim}{\omega}_D\right)$$

Using this result, the ***fixed frame components*** of ${}^R\underset{\sim}{v}_P$ can be written as follows.

$$\begin{aligned}
\{v_P\} &= -\left[\tilde{r}_{Q/O}\right]\{\omega_F\} - \left[\tilde{r}_{P/Q}\right]\{\omega_D\} \\
&= -\left[\tilde{r}_{Q/O}\right]\left[{}^R\omega_{F,y}\right]\{y\} - \left[\tilde{r}_{P/Q}\right]\left[{}^R\omega_{D,y}\right]\{y\} \\
&= -\left[\tilde{r}_{Q/O}\right]\{\hat{\omega}_F\} - \left[\tilde{r}_{P/Q}\right]\left(\{\hat{\omega}_F\} + \left([R_F]^T\{\hat{\omega}_D\}\right)\right)
\end{aligned}$$

$$\{v_P\} = -\left(\left[\tilde{r}_{Q/O}\right] + \left[\tilde{r}_{P/Q}\right]\right)\{\hat{\omega}_F\} - \left[\tilde{r}_{P/Q}\right][R_F]^T\{\hat{\omega}_D\}$$

Observation of this result gives the following ***partial velocity matrices*** in ***fixed frame components***.

$$\left[{}^R v_{P,\hat{\omega}_F}\right] = -\left(\left[\tilde{r}_{Q/O}\right] + \left[\tilde{r}_{P/Q}\right]\right) \qquad \left[{}^R v_{P,\hat{\omega}_D}\right] = -\left[\tilde{r}_{P/Q}\right][R_F]^T$$

Recall that the ***skew symmetric matrices*** are built using ***fixed frame components*** of the indicated position vectors. The following equations can be used to find these components.

$$[R_F]^T = \begin{bmatrix} C_\phi & S_\phi & 0 \\ -S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \{r_{Q/O}\} = [R_F]^T \begin{Bmatrix} 0 \\ \ell \\ 0 \end{Bmatrix} = \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ \ell \\ 0 \end{Bmatrix} = \begin{Bmatrix} -\ell S_\phi \\ \ell C_\phi \\ 0 \end{Bmatrix}$$

$$[R_D]^T = \left(\left[{}^F R_D\right][R_F]\right)^T = \left(\begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix}\begin{bmatrix} C_\phi & S_\phi & 0 \\ -S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)^T = \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} C_\theta & 0 & S_\theta \\ 0 & 1 & 0 \\ -S_\theta & 0 & C_\theta \end{bmatrix}$$

$$\Rightarrow \left[R_D\right]^T = \begin{bmatrix} C_\phi C_\theta & -S_\phi & C_\phi S_\theta \\ S_\phi C_\theta & C_\phi & S_\phi S_\theta \\ -S_\theta & 0 & C_\theta \end{bmatrix} \qquad \left\{r_{P/Q}\right\} = \left[R_D\right]^T \begin{Bmatrix} -a \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} C_\phi C_\theta & -S_\phi & C_\phi S_\theta \\ S_\phi C_\theta & C_\phi & S_\phi S_\theta \\ -S_\theta & 0 & C_\theta \end{bmatrix} \begin{Bmatrix} -a \\ 0 \\ 0 \end{Bmatrix} = -a \begin{Bmatrix} C_\phi C_\theta \\ S_\phi C_\theta \\ -S_\theta \end{Bmatrix}$$

Using these results, ***explicit equations*** for the ***fixed frame components*** of the ***partial velocity matrices*** and ${}^R\underset{\sim}{v}_P$ the ***velocity*** of P can be calculated as follows.

$$\left[{}^R v_{P,\hat{\omega}_F}\right] = -\left(\left[\tilde{r}_{Q/O}\right] + \left[\tilde{r}_{P/Q}\right]\right) = -\left(\begin{bmatrix} 0 & 0 & \ell C_\phi \\ 0 & 0 & \ell S_\phi \\ -\ell C_\phi & -\ell S_\phi & 0 \end{bmatrix} + \begin{bmatrix} 0 & -aS_\theta & -aS_\phi C_\theta \\ aS_\theta & 0 & aC_\phi C_\theta \\ aS_\phi C_\theta & -aC_\phi C_\theta & 0 \end{bmatrix} \right)$$

$$\Rightarrow \left[{}^R v_{P,\hat{\omega}_F}\right] = -\begin{bmatrix} 0 & -aS_\theta & \ell C_\phi - aS_\phi C_\theta \\ aS_\theta & 0 & \ell S_\phi + aC_\phi C_\theta \\ -\ell C_\phi + aS_\phi C_\theta & -\ell S_\phi - aC_\phi C_\theta & 0 \end{bmatrix}$$

$$\left[{}^R v_{P,\hat{\omega}_D}\right] = -\left[\tilde{r}_{P/Q}\right]\left[R_F\right]^T = -\begin{bmatrix} 0 & -aS_\theta & -aS_\phi C_\theta \\ aS_\theta & 0 & aC_\phi C_\theta \\ aS_\phi C_\theta & -aC_\phi C_\theta & 0 \end{bmatrix} \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \left[{}^R v_{P,\hat{\omega}_D}\right] = -\begin{bmatrix} -aS_\phi S_\theta & -aC_\phi S_\theta & -aS_\phi C_\theta \\ aC_\phi S_\theta & -aS_\phi S_\theta & aC_\phi C_\theta \\ 0 & -aC_\theta & 0 \end{bmatrix}$$

$$\begin{aligned} \left\{v_P\right\} &= \left[{}^R v_{P,\hat{\omega}_F}\right]\left\{\hat{\omega}_F\right\} + \left[{}^R v_{P,\hat{\omega}_D}\right]\left\{\hat{\omega}_D\right\} \\ &= -\begin{bmatrix} 0 & -aS_\theta & \ell C_\phi - aS_\phi C_\theta \\ aS_\theta & 0 & \ell S_\phi + aC_\phi C_\theta \\ -\ell C_\phi + aS_\phi C_\theta & -\ell S_\phi - aC_\phi C_\theta & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix} - \begin{bmatrix} -aS_\phi S_\theta & -aC_\phi S_\theta & -aS_\phi C_\theta \\ aC_\phi S_\theta & -aS_\phi S_\theta & aC_\phi C_\theta \\ 0 & -aC_\theta & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ \omega \\ 0 \end{Bmatrix} \\ &= -\begin{Bmatrix} \left(\ell C_\phi - aS_\phi C_\theta\right)\Omega + \left(-aC_\phi S_\theta\right)\omega \\ \left(\ell S_\phi + aC_\phi C_\theta\right)\Omega + \left(-aS_\phi S_\theta\right)\omega \\ -aC_\theta \omega \end{Bmatrix} \end{aligned}$$

$$\Rightarrow \left\{v_P\right\} = \begin{Bmatrix} \left(-\ell C_\phi + aS_\phi C_\theta\right)\Omega + \left(aC_\phi S_\theta\right)\omega \\ -\left(\ell S_\phi + aC_\phi C_\theta\right)\Omega + \left(aS_\phi S_\theta\right)\omega \\ \left(aC_\theta\right)\omega \end{Bmatrix} \quad \text{(fixed frame components)}$$

b) The ***joints*** connecting the two bodies are ***simple revolute joints***, and the bodies share a ***common unit vector*** along the axis of the joint. Consequently, the ***body frame components*** of the relative angular velocities are the ***same*** as the ***base frame components***. That is,

$$\boxed{\{\hat{\omega}'_F\} = [0 \quad 0 \quad \Omega]^T} \quad \boxed{\{\hat{\omega}'_D\} = [0 \quad \omega \quad 0]^T}$$

Using these results, the ***body frame components*** of the ***angular velocities*** of the bodies can be written as

$$\boxed{\{\omega'_F\} = \{\hat{\omega}'_F\}} \quad \boxed{\{\omega'_D\} = \left[{}^F R_D\right]\{\hat{\omega}'_F\} + \{\hat{\omega}'_D\}}$$

Defining the column vector of ***generalized speeds*** to be $\boxed{\{y\} \triangleq \begin{Bmatrix} \{\hat{\omega}'_F\} \\ \{\hat{\omega}'_D\} \end{Bmatrix}}$, the ***body frame components*** of the ***partial angular velocity matrices*** are observed to be

$$\boxed{\left[{}^R\omega'_{F,y}\right]_{3\times 6} = \left[\left[{}^R\omega'_{F,\hat{\omega}_F}\right] \quad \left[{}^R\omega'_{F,\hat{\omega}_D}\right]\right] = \left[[I]_{3\times 3} \quad [0]_{3\times 3}\right]}$$

$$\boxed{\left[{}^R\omega'_{D,y}\right]_{3\times 6} = \left[\left[{}^R\omega'_{D,\hat{\omega}_F}\right] \quad \left[{}^R\omega'_{D,\hat{\omega}_D}\right]\right] = \left[\left[{}^F R_D\right] \quad [I]_{3\times 3}\right]}$$

As noted above, using the concept of ***relative velocity***, ${}^R\underset{\sim}{v}_P$ the velocity of point P in R can be written as follows.

$$\boxed{{}^R\underset{\sim}{v}_P = {}^R\underset{\sim}{v}_{Q/O} + {}^R\underset{\sim}{v}_{P/Q} = \left({}^R\underset{\sim}{\omega}_F \times \underset{\sim}{r}_{Q/O}\right) + \left({}^R\underset{\sim}{\omega}_D \times \underset{\sim}{r}_{P/Q}\right) = -\left(\underset{\sim}{r}_{Q/O} \times {}^R\underset{\sim}{\omega}_F\right) - \left(\underset{\sim}{r}_{P/Q} \times {}^R\underset{\sim}{\omega}_D\right)} \tag{56}$$

Using this result, the ***fixed frame components*** of ${}^R\underset{\sim}{v}_P$ can be written as follows.

$$\begin{aligned}\{v_P\} &= -[R_F]^T\left[\tilde{r}'_{Q/O}\right]\{\omega'_F\} - [R_D]^T\left[\tilde{r}'_{P/Q}\right]\{\omega'_D\} \\ &= -[R_F]^T\left[\tilde{r}'_{Q/O}\right]\{\hat{\omega}'_F\} - [R_D]^T\left[\tilde{r}'_{P/Q}\right]\left(\left(\left[{}^F R_D\right]\{\hat{\omega}'_F\}\right) + \{\hat{\omega}'_D\}\right)\end{aligned}$$

$$\boxed{\{v_P\} = -\left([R_F]^T\left[\tilde{r}'_{Q/O}\right] + [R_D]^T\left[\tilde{r}'_{P/Q}\right]\left[{}^F R_D\right]\right)\{\hat{\omega}'_F\} - [R_D]^T\left[\tilde{r}'_{P/Q}\right]\{\hat{\omega}'_D\}}$$

Observation of this result gives the following ***partial velocity matrices*** in ***fixed frame components***.

$$\boxed{\left[{}^R v_{P,\hat{\omega}'_F}\right] = -\left([R_F]^T\left[\tilde{r}'_{Q/O}\right] + [R_D]^T\left[\tilde{r}'_{P/Q}\right]\left[{}^F R_D\right]\right)} \quad \boxed{\left[{}^R v_{P,\hat{\omega}'_D}\right] = -[R_D]^T\left[\tilde{r}'_{P/Q}\right]}$$

The skew-symmetric matrix $\left[\tilde{r}'_{Q/O}\right]$ is built with position vector components in the frame F, and the skew-symmetric matrix $\left[\tilde{r}'_{P/Q}\right]$ is built with position vector components in the D frame. That is,

$$\boxed{\underset{\sim}{r}_{Q/O} = \ell\,\underset{\sim}{e}_2} \quad \Rightarrow \boxed{\left[\tilde{r}'_{Q/O}\right] = \begin{bmatrix} 0 & 0 & \ell \\ 0 & 0 & 0 \\ -\ell & 0 & 0 \end{bmatrix}} \quad \text{and} \quad \boxed{\underset{\sim}{r}_{P/Q} = -a\,\underset{\sim}{n}_1} \quad \Rightarrow \boxed{\left[\tilde{r}'_{P/Q}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{bmatrix}}$$

Using these results, ***explicit equations*** for the ***fixed frame components*** of the ***partial velocity matrices*** and ${}^R\underset{\sim}{v}_P$ the ***velocity*** of P can be calculated as follows.

$$\left[{}^{R}v_{P,\hat{\omega}'_F}\right] = -\left[R_F\right]^T\left[\tilde{r}'_{Q/O}\right] - \left[R_D\right]^T\left[\tilde{r}'_{P/Q}\right]\left[{}^{F}R_D\right]$$

$$= -\begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & \ell \\ 0 & 0 & 0 \\ -\ell & 0 & 0 \end{bmatrix} - \begin{bmatrix} C_\phi C_\theta & -S_\phi & C_\phi S_\theta \\ S_\phi C_\theta & C_\phi & S_\phi S_\theta \\ -S_\theta & 0 & C_\theta \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{bmatrix}\begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -\ell C_\phi \\ 0 & 0 & -\ell S_\phi \\ \ell & 0 & 0 \end{bmatrix} - \begin{bmatrix} C_\phi C_\theta & -S_\phi & C_\phi S_\theta \\ S_\phi C_\theta & C_\phi & S_\phi S_\theta \\ -S_\theta & 0 & C_\theta \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ aS_\theta & 0 & aC_\theta \\ 0 & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -\ell C_\phi \\ 0 & 0 & -\ell S_\phi \\ \ell & 0 & 0 \end{bmatrix} + \begin{bmatrix} aS_\phi S_\theta & aC_\phi S_\theta & aS_\phi C_\theta \\ -aC_\phi S_\theta & aS_\phi S_\theta & -aC_\phi C_\theta \\ 0 & aC_\theta & 0 \end{bmatrix}$$

$$\Rightarrow \boxed{\left[{}^{R}v_{P,\hat{\omega}'_F}\right] = \begin{bmatrix} aS_\phi S_\theta & aC_\phi S_\theta & -\ell C_\phi + aS_\phi C_\theta \\ -aC_\phi S_\theta & aS_\phi S_\theta & -\ell S_\phi - aC_\phi C_\theta \\ \ell & aC_\theta & 0 \end{bmatrix}}$$

$$\boxed{\left[{}^{R}v_{P,\hat{\omega}'_D}\right] = -\left[R_D\right]^T\left[\tilde{r}'_{P/Q}\right] = -\begin{bmatrix} C_\phi C_\theta & -S_\phi & C_\phi S_\theta \\ S_\phi C_\theta & C_\phi & S_\phi S_\theta \\ -S_\theta & 0 & C_\theta \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{bmatrix} = \begin{bmatrix} 0 & aC_\phi S_\theta & aS_\phi \\ 0 & aS_\phi S_\theta & -aC_\phi \\ 0 & aC_\theta & 0 \end{bmatrix}}$$

$$\{v_P\} = \left[{}^{R}v_{P,\hat{\omega}'_F}\right]\{\hat{\omega}'_F\} + \left[{}^{R}v_{P,\hat{\omega}'_D}\right]\{\hat{\omega}'_D\}$$

$$= \begin{bmatrix} aS_\phi S_\theta & aC_\phi S_\theta & -\ell C_\phi + aS_\phi C_\theta \\ -aC_\phi S_\theta & aS_\phi S_\theta & -\ell S_\phi - aC_\phi C_\theta \\ \ell & aC_\theta & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix} + \begin{bmatrix} 0 & aC_\phi S_\theta & aS_\phi \\ 0 & aS_\phi S_\theta & -aC_\phi \\ 0 & aC_\theta & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ \omega \\ 0 \end{Bmatrix}$$

$$= \Omega\begin{Bmatrix} -\ell C_\phi + aS_\phi C_\theta \\ -\ell S_\phi - aC_\phi C_\theta \\ 0 \end{Bmatrix} + \omega\begin{Bmatrix} aC_\phi S_\theta \\ aS_\phi S_\theta \\ aC_\theta \end{Bmatrix}$$

$$\Rightarrow \boxed{\{v_P\} = \begin{Bmatrix} \left(-\ell C_\phi + aS_\phi C_\theta\right)\Omega + \left(aC_\phi S_\theta\right)\omega \\ -\left(\ell S_\phi + aC_\phi C_\theta\right)\Omega + \left(aS_\phi S_\theta\right)\omega \\ \left(aC_\theta\right)\omega \end{Bmatrix}} \quad \text{(fixed frame components)}$$

These results are ***identical*** to those found in part (a).

c) Using Equation (56), the ***body frame components*** of ${}^R\underset{\sim}{v}_P$ can be written as follows.

$$\begin{aligned}\{v'_P\} &= -\left[{}^F R_D\right]\left[\tilde{r}'_{Q/O}\right]\{\omega'_F\} - \left[\tilde{r}'_{P/Q}\right]\{\omega'_D\}\\ &= -\left[{}^F R_D\right]\left[\tilde{r}'_{Q/O}\right]\{\hat{\omega}'_F\} - \left[\tilde{r}'_{P/Q}\right]\left(\left[{}^F R_D\right]\{\hat{\omega}'_F\} + \{\hat{\omega}'_D\}\right)\\ &= -\left(\left[{}^F R_D\right]\left[\tilde{r}'_{Q/O}\right] + \left[\tilde{r}'_{P/Q}\right]\left[{}^F R_D\right]\right)\{\hat{\omega}'_F\} - \left[\tilde{r}'_{P/Q}\right]\{\hat{\omega}'_D\}\end{aligned}$$

Observation of this result gives the following ***partial velocity matrices*** in ***body frame components***.

$$\boxed{\left[{}^R v'_{P,\hat{\omega}'_F}\right] = -\left(\left[{}^F R_D\right]\left[\tilde{r}'_{Q/O}\right] + \left[\tilde{r}'_{P/Q}\right]\left[{}^F R_D\right]\right)} \quad \boxed{\left[{}^R v'_{P,\hat{\omega}'_D}\right] = -\left[\tilde{r}'_{P/Q}\right]}$$

As noted above,

$$\boxed{\left[\tilde{r}'_{Q/O}\right] = \begin{bmatrix} 0 & 0 & \ell \\ 0 & 0 & 0 \\ -\ell & 0 & 0 \end{bmatrix}} \quad \boxed{\left[\tilde{r}'_{P/Q}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{bmatrix}} \quad \boxed{\left[{}^F R_D\right] = \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix}}$$

Using these results, ***explicit equations*** for the ***body frame components*** of the ***partial velocity matrices*** and ${}^R\underset{\sim}{v}_P$ the ***velocity*** of P can be calculated as follows.

$$\begin{aligned}\left[{}^R v'_{P,\hat{\omega}'_F}\right] &= -\left(\left[{}^F R_D\right]\left[\tilde{r}'_{Q/O}\right] + \left[\tilde{r}'_{P/Q}\right]\left[{}^F R_D\right]\right)\\ &= -\begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix}\begin{bmatrix} 0 & 0 & \ell \\ 0 & 0 & 0 \\ -\ell & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{bmatrix}\begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix}\\ &= -\begin{bmatrix} \ell S_\theta & 0 & \ell C_\theta \\ 0 & 0 & 0 \\ -\ell C_\theta & 0 & \ell S_\theta \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ a S_\theta & 0 & a C_\theta \\ 0 & -a & 0 \end{bmatrix}\end{aligned}$$

$$\Rightarrow \boxed{\left[{}^R v'_{P,\hat{\omega}'_F}\right] = \begin{bmatrix} -\ell S_\theta & 0 & -\ell C_\theta \\ -a S_\theta & 0 & -a C_\theta \\ \ell C_\theta & a & -\ell S_\theta \end{bmatrix}}$$

$$\boxed{\left[{}^R v'_{P,\hat{\omega}'_D}\right] = -\left[\tilde{r}'_{P/Q}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & a & 0 \end{bmatrix}}$$

$$\begin{aligned}\{v_P\} &= \left[{}^R v'_{P,\hat{\omega}'_F}\right]\{\hat{\omega}'_F\} + \left[{}^R v'_{P,\hat{\omega}'_D}\right]\{\hat{\omega}'_D\}\\ &= \begin{bmatrix} -\ell S_\theta & 0 & -\ell C_\theta \\ -a S_\theta & 0 & -a C_\theta \\ \ell C_\theta & a & -\ell S_\theta \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & a & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ \omega \\ 0 \end{Bmatrix}\end{aligned}$$

$$\Rightarrow \boxed{\left\{v'_P\right\} = \begin{Bmatrix} -\ell C_\theta \Omega \\ -a C_\theta \Omega \\ -\ell S_\theta \Omega + a\omega \end{Bmatrix}}$$

Comparison with Previous Results: (Volume I, Examples in Units 2-4)

In Units 2-4 of Volume I, ${}^R\underset{\sim}{v}_P$ the velocity of P in R was expressed in frame F as follows.

$$\boxed{{}^R\underset{\sim}{v}_P = \left(a\,\omega\, S_\theta - \ell\,\Omega\right)\underset{\sim}{e}_1 - \left(a\,\Omega\, C_\theta\right)\underset{\sim}{e}_2 + \left(a\,\omega\, C_\theta\right)\underset{\sim}{k}}$$

This result can be converted into ***fixed frame components*** as follows.

$$\left\{v_P\right\} = \left[R_F\right]^T \left\{v_P\right\}_F = \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} a\,\omega\, S_\theta - \ell\,\Omega \\ -a\,\Omega\, C_\theta \\ a\,\omega\, C_\theta \end{Bmatrix} = \begin{Bmatrix} \left(a\,\omega\, S_\theta - \ell\,\Omega\right) C_\phi + \left(a\,\Omega\, C_\theta\right) S_\phi \\ \left(a\,\omega\, S_\theta - \ell\,\Omega\right) S_\phi - \left(a\,\Omega\, C_\theta\right) C_\phi \\ a\,\omega\, C_\theta \end{Bmatrix}$$

$$\Rightarrow \boxed{\left\{v_P\right\} = \begin{Bmatrix} \left(-\ell C_\phi + a\, C_\theta S_\phi\right)\Omega + \left(a\, S_\theta C_\phi\right)\omega \\ -\left(\ell S_\phi + a\, C_\theta C_\phi\right)\Omega + \left(a\, S_\theta S_\phi\right)\omega \\ \left(a\, C_\theta\right)\omega \end{Bmatrix}} \quad \checkmark$$

These results are ***identical*** the those found in parts (a) and (b).

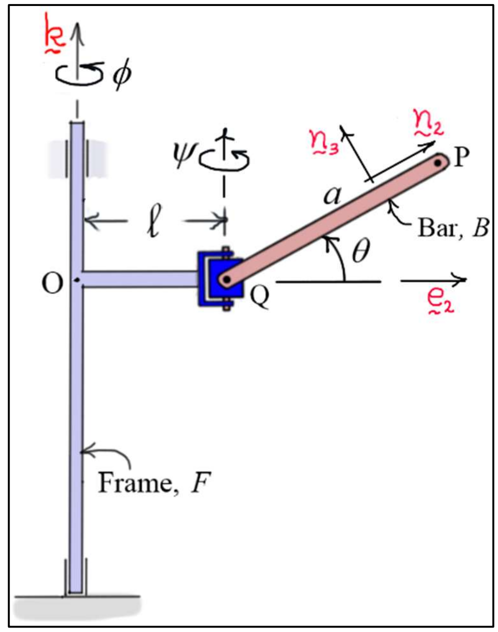
This result can be converted into ***body frame components*** as follows.

$$\left\{v'_P\right\} = \left[{}^F R_D\right] \left\{v_P\right\}_F = \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix} \begin{Bmatrix} a\,\omega\, S_\theta - \ell\,\Omega \\ -a\,\Omega\, C_\theta \\ a\,\omega\, C_\theta \end{Bmatrix} = \begin{Bmatrix} C_\theta \left(a\,\omega\, S_\theta - \ell\,\Omega\right) - S_\theta a\,\omega\, C_\theta \\ -a\,\Omega\, C_\theta \\ S_\theta \left(a\,\omega\, S_\theta - \ell\,\Omega\right) + C_\theta a\,\omega\, C_\theta \end{Bmatrix}$$

$$\Rightarrow \boxed{\left\{v'_P\right\} = \begin{Bmatrix} -\ell\, C_\theta \Omega \\ -a\, C_\theta \Omega \\ -\ell S_\theta \Omega + a\omega \end{Bmatrix}} \quad \checkmark$$

These results are ***identical*** the those found in part (c).

Example 2



The diagram shows a ***two-body*** system with ***frame*** F and ***bar*** B. Frame F rotates ($\dot{\phi}=\Omega$) about a fixed vertical direction annotated by the unit vector $\underset{\sim}{k}$, and bar B is attached to F with a joint allowing rotations about ***two axes***. Rotation about the first axis is measured by the angle ψ, and rotation about the second axis is measured by the angle θ. The angles ϕ and ψ are both ***zero*** in the position shown. B is oriented with respect to F using a 3-1 rotation sequence. Three reference frames are defined for the system as follows.

$R:\left(\underset{\sim}{i},\underset{\sim}{j},\underset{\sim}{k}\right)$ fixed frame

$F:\left(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{k}\right)$ rotating with Frame, F

$B:\left(\underset{\sim}{n}_1,\underset{\sim}{n}_2,\underset{\sim}{n}_3\right)$ rotating with Bar, B

Let $\{\hat{\omega}_F\}$ and $\{\hat{\omega}_B\}$ be the ***base frame components*** and let $\{\hat{\omega}'_F\}$ and $\{\hat{\omega}'_B\}$ be the ***body frame components*** of the ***relative angular velocities*** ${}^R\underset{\sim}{\omega}_F$ and ${}^F\underset{\sim}{\omega}_B$. Find the ***fixed frame components*** of the ***velocity*** of point P and the ***partial velocity matrix*** associated with the relative angular velocity components using a) ***base frame*** components, and b) ***body frame*** components. (c) Find the ***body frame components*** of the ***velocity*** of P and the ***partial velocity matrix*** associated with the ***body frame components*** of the ***relative angular velocities***.

Solution:

(a) The ***angular velocities*** of the two bodies can be calculated as follows.

$${}^R\underset{\sim}{\omega}_F=\dot{\phi}\,\underset{\sim}{k} \qquad {}^R\underset{\sim}{\omega}_D={}^R\underset{\sim}{\omega}_F+{}^F\underset{\sim}{\omega}_B$$

The ***base frame components*** of the ***relative angular velocities*** can be written as follows.

$$\{\hat{\omega}_F\}=\begin{bmatrix}0 & 0 & \dot{\phi}\end{bmatrix}^T$$

$${}^F\underset{\sim}{\omega}_B=\dot{\theta}\,\underset{\sim}{n}_1+\dot{\psi}\,\underset{\sim}{k}=\dot{\theta}\left(C_\psi\underset{\sim}{e}_1+S_\psi\underset{\sim}{e}_2\right)+\dot{\psi}\,\underset{\sim}{k} \quad\Rightarrow\quad \{\hat{\omega}_B\}=\begin{bmatrix}\dot{\theta}C_\psi & \dot{\theta}S_\psi & \dot{\psi}\end{bmatrix}^T$$

Using these results, the ***fixed frame components*** of the ***angular velocities*** of the bodies can be written as

$$\{\omega_F\}=\{\hat{\omega}_F\} \qquad \{\omega_B\}=\{\hat{\omega}_F\}+[R_F]^T\{\hat{\omega}_B\}$$

Defining the column vector of ***generalized speeds*** to be $\{y\}\triangleq\begin{Bmatrix}\{\hat{\omega}_F\}\\ \{\hat{\omega}_B\}\end{Bmatrix}$, the ***fixed frame components*** of the ***partial angular velocity matrices*** are observed to be

$$\left[{}^R\omega_{F,y}\right]_{3\times 6} = \left[\left[{}^R\omega_{F,\hat{\omega}_F}\right] \quad \left[{}^R\omega_{F,\hat{\omega}_B}\right]\right] = \left[[I]_{3\times 3} \quad [0]_{3\times 3}\right]$$

$$\left[{}^R\omega_{B,y}\right]_{3\times 6} = \left[\left[{}^R\omega_{B,\hat{\omega}_F}\right] \quad \left[{}^R\omega_{B,\hat{\omega}_B}\right]\right] = \left[[I]_{3\times 3} \quad [R_F]^T\right]$$

Using the concept of ***relative velocity***, ${}^R\underset{\sim}{v}_P$ the velocity of point P in R can be written as follows.

$${}^R\underset{\sim}{v}_P = {}^R\underset{\sim}{v}_{Q/O} + {}^R\underset{\sim}{v}_{P/Q} = \left({}^R\underset{\sim}{\omega}_F \times \underset{\sim}{r}_{Q/O}\right) + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/Q}\right) = -\left(\underset{\sim}{r}_{Q/O} \times {}^R\underset{\sim}{\omega}_F\right) - \left(\underset{\sim}{r}_{P/Q} \times {}^R\underset{\sim}{\omega}_B\right)$$

Using this result, the ***fixed frame components*** of ${}^R\underset{\sim}{v}_P$ can be written as follows.

$$\begin{aligned}
\{v_P\} &= -\left[\tilde{r}_{Q/O}\right]\{\omega_F\} - \left[\tilde{r}_{P/Q}\right]\{\omega_B\} \\
&= -\left[\tilde{r}_{Q/O}\right]\left[{}^R\omega_{F,y}\right]\{y\} - \left[\tilde{r}_{P/Q}\right]\left[{}^R\omega_{B,y}\right]\{y\} \\
&= -\left[\tilde{r}_{Q/O}\right]\{\hat{\omega}_F\} - \left[\tilde{r}_{P/Q}\right]\left(\{\hat{\omega}_F\} + \left([R_F]^T\{\hat{\omega}_B\}\right)\right)
\end{aligned}$$

$$\{v_P\} = -\left(\left[\tilde{r}_{Q/O}\right] + \left[\tilde{r}_{P/Q}\right]\right)\{\hat{\omega}_F\} - \left[\tilde{r}_{P/Q}\right][R_F]^T\{\hat{\omega}_B\}$$

Observation of this result gives the following ***partial velocity matrices*** in ***fixed frame components***.

$$\left[{}^R v_{P,\hat{\omega}_F}\right] = -\left[\tilde{r}_{Q/O}\right] - \left[\tilde{r}_{P/Q}\right] \qquad \left[{}^R v_{P,\hat{\omega}_B}\right] = -\left[\tilde{r}_{P/Q}\right][R_F]^T$$

Recall that the ***skew symmetric matrices*** are built using ***fixed frame components*** of the indicated position vectors. The following equations can be used to find these components.

$$[R_F]^T = \begin{bmatrix} C_\phi & S_\phi & 0 \\ -S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\{r_{Q/O}\} = [R_F]^T \begin{Bmatrix} 0 \\ \ell \\ 0 \end{Bmatrix} = \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ \ell \\ 0 \end{Bmatrix} = \begin{Bmatrix} -\ell S_\phi \\ \ell C_\phi \\ 0 \end{Bmatrix}$$

$$\left[{}^F R_B\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & S_\theta \\ 0 & -S_\theta & C_\theta \end{bmatrix} \begin{bmatrix} C_\psi & S_\psi & 0 \\ -S_\psi & C_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_\psi & S_\psi & 0 \\ -S_\psi C_\theta & C_\psi C_\theta & S_\theta \\ S_\psi S_\theta & -C_\psi S_\theta & C_\theta \end{bmatrix}$$

The angles ϕ and ψ are measured along the ***same direction*** so B can be oriented using a 3-1 rotation sequence. The first angle is simply the sum $\phi + \psi$.

$$[R_B] = \begin{bmatrix} C_{\phi+\psi} & S_{\phi+\psi} & 0 \\ -S_{\phi+\psi} C_\theta & C_{\phi+\psi} C_\theta & S_\theta \\ S_{\phi+\psi} S_\theta & -C_{\phi+\psi} S_\theta & C_\theta \end{bmatrix}$$

$$\left\{ r_{P/Q} \right\} = \left[R_B \right]^T \begin{Bmatrix} 0 \\ a \\ 0 \end{Bmatrix} = \begin{bmatrix} C_{\phi+\psi} & -S_{\phi+\psi} C_\theta & S_{\phi+\psi} S_\theta \\ S_{\phi+\psi} & C_{\phi+\psi} C_\theta & -C_{\phi+\psi} S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \begin{Bmatrix} 0 \\ a \\ 0 \end{Bmatrix} = \begin{Bmatrix} -a S_{\phi+\psi} C_\theta \\ a C_{\phi+\psi} C_\theta \\ a S_\theta \end{Bmatrix}$$

Using these results, ***explicit equations*** for the ***fixed frame components*** of the ***partial velocity matrices*** and ${}^R\underset{\sim}{v}_P$ the ***velocity*** of P can be calculated as follows.

$$\left[{}^R v_{P,\hat{\omega}_F} \right] = -\left[\tilde{r}_{Q/O} \right] - \left[\tilde{r}_{P/Q} \right] = -\begin{bmatrix} 0 & 0 & \ell C_\phi \\ 0 & 0 & \ell S_\phi \\ -\ell C_\phi & -\ell S_\phi & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a S_\theta & a C_{\phi+\psi} C_\theta \\ a S_\theta & 0 & a S_{\phi+\psi} C_\theta \\ -a C_{\phi+\psi} C_\theta & -a S_{\phi+\psi} C_\theta & 0 \end{bmatrix}$$

$$\Rightarrow \boxed{\left[{}^R v_{P,\hat{\omega}_F} \right] = \begin{bmatrix} 0 & a S_\theta & -\left(\ell C_\phi + a C_{\phi+\psi} C_\theta \right) \\ -a S_\theta & 0 & -\left(\ell S_\phi + a S_{\phi+\psi} C_\theta \right) \\ \ell C_\phi + a C_{\phi+\psi} C_\theta & \ell S_\phi + a S_{\phi+\psi} C_\theta & 0 \end{bmatrix}}$$

$$\left[{}^R v_{P,\hat{\omega}_B} \right] = -\left[\tilde{r}_{P/Q} \right] \left[R_F \right]^T = \begin{bmatrix} 0 & a S_\theta & -a C_{\phi+\psi} C_\theta \\ -a S_\theta & 0 & -a S_{\phi+\psi} C_\theta \\ a C_{\phi+\psi} C_\theta & a S_{\phi+\psi} C_\theta & 0 \end{bmatrix} \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \boxed{\left[{}^R v_{P,\hat{\omega}_B} \right] = \begin{bmatrix} a S_\phi S_\theta & a C_\phi S_\theta & -a C_{\phi+\psi} C_\theta \\ -a C_\phi S_\theta & a S_\phi S_\theta & -a S_{\phi+\psi} C_\theta \\ a C_\theta \left(C_\phi C_{\phi+\psi} + S_\phi S_{\phi+\psi} \right) & -a C_\theta \left(S_\phi C_{\phi+\psi} - C_\phi S_{\phi+\psi} \right) & 0 \end{bmatrix}}$$

This result can be ***simplified*** using the following ***trigonometric identities***.

$$\boxed{\cos(\phi)\cos(\phi+\psi) + \sin(\phi)\sin(\phi+\psi) = \cos\left(\phi - (\phi+\psi)\right) = \cos(-\psi) = \cos(\psi)}$$

$$\boxed{\sin(\phi)\cos(\phi+\psi) - \cos(\phi)\sin(\phi+\psi) = \sin\left(\phi - (\phi+\psi)\right) = \sin(-\psi) = -\sin(\psi)}$$

$$\Rightarrow \boxed{\left[{}^R v_{P,\hat{\omega}_B} \right] = \begin{bmatrix} a S_\phi S_\theta & a C_\phi S_\theta & -a C_{\phi+\psi} C_\theta \\ -a C_\phi S_\theta & a S_\phi S_\theta & -a S_{\phi+\psi} C_\theta \\ a C_\theta C_\psi & a C_\theta S_\psi & 0 \end{bmatrix}}$$

$$\{v_P\} = \left[{}^R v_{P,\hat{\omega}_F}\right]\{\hat{\omega}_F\} + \left[{}^R v_{P,\hat{\omega}_B}\right]\{\hat{\omega}_B\}$$

$$= \begin{bmatrix} 0 & aS_\theta & -\left(\ell C_\phi + aC_{\phi+\psi}C_\theta\right) \\ -aS_\theta & 0 & -\left(\ell S_\phi + aS_{\phi+\psi}C_\theta\right) \\ \ell C_\phi + aC_{\phi+\psi}C_\theta & \ell S_\phi + aS_{\phi+\psi}C_\theta & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\phi} \end{Bmatrix}$$

$$+ \begin{bmatrix} aS_\phi S_\theta & aC_\phi S_\theta & -aC_{\phi+\psi}C_\theta \\ -aC_\phi S_\theta & aS_\phi S_\theta & -aS_{\phi+\psi}C_\theta \\ aC_\theta C_\psi & aC_\theta S_\psi & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta}C_\psi \\ \dot{\theta}S_\psi \\ \dot{\psi} \end{Bmatrix}$$

$$= \begin{Bmatrix} -\dot{\phi}\left(\ell C_\phi + aC_{\phi+\psi}C_\theta\right) \\ -\dot{\phi}\left(\ell S_\phi + aS_{\phi+\psi}C_\theta\right) \\ 0 \end{Bmatrix} + \begin{Bmatrix} a\dot{\theta}\left(S_\phi C_\psi + C_\phi S_\psi\right)S_\theta - a\dot{\psi}C_{\phi+\psi}C_\theta \\ -a\dot{\theta}\left(C_\psi C_\phi - S_\psi S_\phi\right)S_\theta - a\dot{\psi}S_{\phi+\psi}C_\theta \\ a\dot{\theta}C_\theta\left(C_\psi^2 + S_\psi^2\right) \end{Bmatrix}$$

This result can be ***simplified*** using the following ***trigonometric identities***.

$$\boxed{\sin(\phi)\cos(\psi) + \cos(\phi)\sin(\psi) = \sin(\phi+\psi)} \quad \boxed{\cos(\psi)\cos(\phi) - \sin(\psi)\sin(\phi) = \cos(\phi+\psi)}$$

$$\{v_P\} = \begin{Bmatrix} -\dot{\phi}\left(\ell C_\phi + aC_{\phi+\psi}C_\theta\right) \\ -\dot{\phi}\left(\ell S_\phi + aS_{\phi+\psi}C_\theta\right) \\ 0 \end{Bmatrix} + \begin{Bmatrix} a\dot{\theta}S_{\phi+\psi}S_\theta - a\dot{\psi}C_{\phi+\psi}C_\theta \\ -a\dot{\theta}C_{\phi+\psi}S_\theta - a\dot{\psi}S_{\phi+\psi}C_\theta \\ a\dot{\theta}C_\theta \end{Bmatrix}$$

$$\Rightarrow \boxed{\{v_P\} = \begin{Bmatrix} -\dot{\phi}\left(\ell C_\phi + aC_{\phi+\psi}C_\theta\right) + a\dot{\theta}S_{\phi+\psi}S_\theta - a\dot{\psi}C_{\phi+\psi}C_\theta \\ -\dot{\phi}\left(\ell S_\phi + aS_{\phi+\psi}C_\theta\right) - a\dot{\theta}C_{\phi+\psi}S_\theta - a\dot{\psi}S_{\phi+\psi}C_\theta \\ a\dot{\theta}C_\theta \end{Bmatrix} = \begin{Bmatrix} -\ell\dot{\phi}C_\phi + a\dot{\theta}S_{\phi+\psi}S_\theta - \left(\dot{\phi}+\dot{\psi}\right)aC_{\phi+\psi}C_\theta \\ -\ell\dot{\phi}S_\phi - a\dot{\theta}C_{\phi+\psi}S_\theta - \left(\dot{\phi}+\dot{\psi}\right)aS_{\phi+\psi}C_\theta \\ a\dot{\theta}C_\theta \end{Bmatrix}}$$

(b) The ***angular velocities*** of the two bodies can be calculated as follows.

$$\boxed{{}^R\underset{\sim}{\omega}_F = \dot{\phi}\,\underset{\sim}{k}} \quad \boxed{{}^R\underset{\sim}{\omega}_B = {}^R\underset{\sim}{\omega}_F + {}^F\underset{\sim}{\omega}_B}$$

As in the previous example, the ***base frame*** and ***body frame components*** of ${}^R\underset{\sim}{\omega}_F$ are ***identical***.

$$\boxed{\{\hat{\omega}_F'\} = \{\hat{\omega}_F\} = \begin{bmatrix} 0 & 0 & \dot{\phi} \end{bmatrix}^T}$$

The angular velocity of B relative to F (3-1 rotation sequence) can be written in the ***body frame*** as follows.

$$\boxed{{}^F\underset{\sim}{\omega}_B = \dot{\theta}\,\underset{\sim}{n}_1 + \dot{\psi}\,\underset{\sim}{k} = \dot{\theta}\,\underset{\sim}{n}_1 + \dot{\psi}\left(S_\theta\underset{\sim}{n}_2 + C_\theta\underset{\sim}{n}_3\right)} \quad \Rightarrow \boxed{\{\hat{\omega}_B'\} = \begin{bmatrix} \dot{\theta} & \dot{\psi}S_\theta & \dot{\psi}C_\theta \end{bmatrix}^T}$$

Using these results, the ***body frame components*** of the ***angular velocities*** of the bodies can be written as

$$\boxed{\{\omega_F'\} = \{\hat{\omega}_F'\}} \quad \boxed{\{\omega_B'\} = \left[{}^F R_B\right]\{\hat{\omega}_F'\} + \{\hat{\omega}_B'\}}$$

Defining the column vector of ***generalized speeds*** to be $\boxed{\{y\} \triangleq \begin{Bmatrix} \{\hat{\omega}'_F\} \\ \{\hat{\omega}'_B\} \end{Bmatrix}}$, the ***body frame components*** of the ***partial angular velocity matrices*** are observed to be

$$\boxed{\left[{}^R\omega'_{F,y}\right]_{3\times 6} = \left[\left[{}^R\omega'_{F,\hat{\omega}'_F}\right] \quad \left[{}^R\omega'_{F,\hat{\omega}'_B}\right]\right] = \left[[I]_{3\times 3} \quad [0]_{3\times 3}\right]}$$

$$\boxed{\left[{}^R\omega'_{B,y}\right]_{3\times 6} = \left[\left[{}^R\omega'_{B,\hat{\omega}'_F}\right] \quad \left[{}^R\omega'_{B,\hat{\omega}'_B}\right]\right] = \left[\left[{}^F R_B\right] \quad [I]_{3\times 3}\right]}$$

As noted above, using the concept of ***relative velocity***, ${}^R\underset{\sim}{v}_P$ the velocity of point P in R can be written as follows.

$$\boxed{{}^R\underset{\sim}{v}_P = {}^R\underset{\sim}{v}_{Q/O} + {}^R\underset{\sim}{v}_{P/Q} = \left({}^R\underset{\sim}{\omega}_F \times \underset{\sim}{r}_{Q/O}\right) + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/Q}\right) = -\left(\underset{\sim}{r}_{Q/O} \times {}^R\underset{\sim}{\omega}_F\right) - \left(\underset{\sim}{r}_{P/Q} \times {}^R\underset{\sim}{\omega}_B\right)} \qquad (57)$$

Using this result, the ***fixed frame components*** of ${}^R\underset{\sim}{v}_P$ can be written as follows.

$$\begin{aligned} \{v_P\} &= -[R_F]^T\left[\tilde{r}'_{Q/O}\right]\{\omega'_F\} - [R_B]^T\left[\tilde{r}'_{P/Q}\right]\{\omega'_B\} \\ &= -[R_F]^T\left[\tilde{r}'_{Q/O}\right]\{\hat{\omega}'_F\} - [R_B]^T\left[\tilde{r}'_{P/Q}\right]\left(\left(\left[{}^F R_B\right]\{\hat{\omega}'_F\}\right) + \{\hat{\omega}'_B\}\right) \end{aligned}$$

$$\boxed{\{v_P\} = -\left([R_F]^T\left[\tilde{r}'_{Q/O}\right] + [R_B]^T\left[\tilde{r}'_{P/Q}\right]\left[{}^F R_B\right]\right)\{\hat{\omega}'_F\} - [R_B]^T\left[\tilde{r}'_{P/Q}\right]\{\hat{\omega}'_B\}}$$

Observation of this result gives the following ***partial velocity matrices*** in ***fixed frame components***.

$$\boxed{\left[{}^R v_{P,\hat{\omega}'_F}\right] = -[R_F]^T\left[\tilde{r}'_{Q/O}\right] - [R_B]^T\left[\tilde{r}'_{P/Q}\right]\left[{}^F R_B\right]} \qquad \boxed{\left[{}^R v_{P,\hat{\omega}'_B}\right] = -[R_B]^T\left[\tilde{r}'_{P/Q}\right]}$$

The skew-symmetric matrix $\left[\tilde{r}'_{Q/O}\right]$ is built with position vector components in frame F, and the skew-symmetric matrix $\left[\tilde{r}'_{P/Q}\right]$ is built with position vector components in frame B. That is,

$$\boxed{\underset{\sim}{r}_{Q/O} = \ell\,\underset{\sim}{e}_2} \quad \Rightarrow \boxed{\left[\tilde{r}'_{Q/O}\right] = \begin{bmatrix} 0 & 0 & \ell \\ 0 & 0 & 0 \\ -\ell & 0 & 0 \end{bmatrix}} \quad \text{and} \quad \boxed{\underset{\sim}{r}_{P/Q} = a\,\underset{\sim}{n}_2} \quad \Rightarrow \boxed{\left[\tilde{r}'_{P/Q}\right] = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ -a & 0 & 0 \end{bmatrix}}$$

Using these results, ***explicit equations*** for the ***fixed frame components*** of the ***partial velocity matrices*** and ${}^R\underset{\sim}{v}_P$ the ***velocity*** of P can be calculated as follows.

$$\left[{}^{R}v_{P,\hat{\omega}'_F}\right] = -\left[R_F\right]^T\left[\tilde{r}'_{Q/O}\right] - \left[R_B\right]^T\left[\tilde{r}'_{P/Q}\right]\left[{}^{F}R_B\right]$$

$$= -\begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & \ell \\ 0 & 0 & 0 \\ -\ell & 0 & 0 \end{bmatrix}$$

$$-\begin{bmatrix} C_{\phi+\psi} & -S_{\phi+\psi}C_\theta & S_{\phi+\psi}S_\theta \\ S_{\phi+\psi} & C_{\phi+\psi}C_\theta & -C_{\phi+\psi}S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix}\begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ -a & 0 & 0 \end{bmatrix}\begin{bmatrix} C_\psi & S_\psi & 0 \\ -S_\psi C_\theta & C_\psi C_\theta & S_\theta \\ S_\psi S_\theta & -C_\psi S_\theta & C_\theta \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & 0 & \ell C_\phi \\ 0 & 0 & \ell S_\phi \\ -\ell & 0 & 0 \end{bmatrix} - \begin{bmatrix} C_{\phi+\psi} & -S_{\phi+\psi}C_\theta & S_{\phi+\psi}S_\theta \\ S_{\phi+\psi} & C_{\phi+\psi}C_\theta & -C_{\phi+\psi}S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix}\begin{bmatrix} aS_\psi S_\theta & -aC_\psi S_\theta & aC_\theta \\ 0 & 0 & 0 \\ -aC_\psi & -aS_\psi & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & 0 & \ell C_\phi \\ 0 & 0 & \ell S_\phi \\ -\ell & 0 & 0 \end{bmatrix} - \begin{bmatrix} aS_\theta\left(S_\psi C_{\phi+\psi} - C_\psi S_{\phi+\psi}\right) & -aS_\theta\left(C_\psi C_{\phi+\psi} + S_\psi S_{\phi+\psi}\right) & aC_\theta C_{\phi+\psi} \\ aS_\theta\left(S_\psi S_{\phi+\psi} + C_\psi C_{\phi+\psi}\right) & aS_\theta\left(-C_\psi S_{\phi+\psi} + S_\psi C_{\phi+\psi}\right) & aC_\theta S_{\phi+\psi} \\ -aC_\psi C_\theta & -aS_\psi C_\theta & 0 \end{bmatrix}$$

This result can be ***simplified*** using the following ***trigonometric identities***.

$$\boxed{\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)} \quad \boxed{\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)}$$

$$\boxed{\sin(-\alpha) = -\sin(\alpha)} \quad \boxed{\cos(-\alpha) = \cos(\alpha)}$$

$$\left[{}^{R}v_{P,\hat{\omega}'_F}\right] = \begin{bmatrix} 0 & 0 & -\ell C_\phi \\ 0 & 0 & -\ell S_\phi \\ \ell & 0 & 0 \end{bmatrix} + \begin{bmatrix} aS_\theta S_\phi & aS_\theta C_\phi & -aC_\theta C_{\phi+\psi} \\ -aS_\theta C_\phi & aS_\theta S_\phi & -aC_\theta S_{\phi+\psi} \\ aC_\psi C_\theta & aS_\psi C_\theta & 0 \end{bmatrix}$$

$$\Rightarrow \boxed{\left[{}^{R}v_{P,\hat{\omega}'_F}\right] = \begin{bmatrix} aS_\theta S_\phi & aS_\theta C_\phi & -\left(\ell C_\phi + aC_\theta C_{\phi+\psi}\right) \\ -aS_\theta C_\phi & aS_\theta S_\phi & -\left(\ell S_\phi + aC_\theta S_{\phi+\psi}\right) \\ \ell + aC_\psi C_\theta & aS_\psi C_\theta & 0 \end{bmatrix}}$$

$$\left[{}^{R}v_{P,\hat{\omega}'_B}\right] = -\left[R_B\right]^T\left[\tilde{r}'_{P/Q}\right] = -\begin{bmatrix} C_{\phi+\psi} & -S_{\phi+\psi}C_\theta & S_{\phi+\psi}S_\theta \\ S_{\phi+\psi} & C_{\phi+\psi}C_\theta & -C_{\phi+\psi}S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix}\begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ -a & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \boxed{\left[{}^{R}v_{P,\hat{\omega}'_B}\right] = \begin{bmatrix} aS_{\phi+\psi}S_\theta & 0 & -aC_{\phi+\psi} \\ -aC_{\phi+\psi}S_\theta & 0 & -aS_{\phi+\psi} \\ aC_\theta & 0 & 0 \end{bmatrix}}$$

$$\{v_P\} = \left[{}^R v_{P,\hat{\omega}'_F}\right]\{\hat{\omega}'_F\} + \left[{}^R v_{P,\hat{\omega}'_B}\right]\{\hat{\omega}'_B\}$$

$$= \begin{bmatrix} aS_\theta S_\phi & aS_\theta C_\phi & -\left(\ell C_\phi + aC_\theta C_{\phi+\psi}\right) \\ -aS_\theta C_\phi & aS_\theta S_\phi & -\left(\ell S_\phi + aC_\theta S_{\phi+\psi}\right) \\ \ell + aC_\psi C_\theta & aS_\psi C_\theta & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\phi} \end{Bmatrix} + \begin{bmatrix} aS_{\phi+\psi} S_\theta & 0 & -aC_{\phi+\psi} \\ -aC_{\phi+\psi} S_\theta & 0 & -aS_{\phi+\psi} \\ aC_\theta & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ \dot{\psi} S_\theta \\ \dot{\psi} C_\theta \end{Bmatrix}$$

$$= \begin{Bmatrix} -\left(\ell C_\phi + aC_\theta C_{\phi+\psi}\right)\dot{\phi} \\ -\left(\ell S_\phi + aC_\theta S_{\phi+\psi}\right)\dot{\phi} \\ 0 \end{Bmatrix} + \begin{Bmatrix} a\dot{\theta} S_{\phi+\psi} S_\theta - a\dot{\psi} C_\theta C_{\phi+\psi} \\ -a\dot{\theta} C_{\phi+\psi} S_\theta - a\dot{\psi} C_\theta S_{\phi+\psi} \\ a\dot{\theta} C_\theta \end{Bmatrix}$$

$$\Rightarrow \boxed{\{v_P\} = \begin{Bmatrix} -\left(\ell C_\phi + aC_\theta C_{\phi+\psi}\right)\dot{\phi} + a\dot{\theta} S_{\phi+\psi} S_\theta - a\dot{\psi} C_\theta C_{\phi+\psi} \\ -\left(\ell S_\phi + aC_\theta S_{\phi+\psi}\right)\dot{\phi} - a\dot{\theta} C_{\phi+\psi} S_\theta - a\dot{\psi} C_\theta S_{\phi+\psi} \\ a\dot{\theta} C_\theta \end{Bmatrix} = \begin{Bmatrix} -\ell\dot{\phi} C_\phi + a\dot{\theta} S_{\phi+\psi} S_\theta - \left(\dot{\phi} + \dot{\psi}\right) aC_{\phi+\psi} C_\theta \\ -\ell\dot{\phi} S_\phi - a\dot{\theta} C_{\phi+\psi} S_\theta - \left(\dot{\phi} + \dot{\psi}\right) aS_{\phi+\psi} C_\theta \\ a\dot{\theta} C_\theta \end{Bmatrix}}$$

As expected, these results are ***identical*** to those found in part (a).

c) Using Equation (57), the ***body frame components*** of ${}^R \underset{\sim}{v}_P$ can be written as follows.

$$\{v'_P\} = -\left[{}^F R_B\right]\left[\tilde{r}'_{Q/O}\right]\{\omega'_F\} - \left[\tilde{r}'_{P/Q}\right]\{\omega'_B\}$$

$$= -\left[{}^F R_B\right]\left[\tilde{r}'_{Q/O}\right]\{\omega'_F\} - \left[\tilde{r}'_{P/Q}\right]\left(\left[{}^F R_B\right]\{\hat{\omega}'_F\} + \{\hat{\omega}'_B\}\right)$$

$$\boxed{\{v'_P\} = -\left(\left[{}^F R_B\right]\left[\tilde{r}'_{Q/O}\right] + \left[\tilde{r}'_{P/Q}\right]\left[{}^F R_B\right]\right)\{\hat{\omega}'_F\} - \left[\tilde{r}'_{P/Q}\right]\{\hat{\omega}'_B\}}$$

Observation of this result gives the following ***partial velocity matrices*** in ***body frame components***.

$$\boxed{\left[v'_{P,\hat{\omega}'_F}\right] = -\left[{}^F R_B\right]\left[\tilde{r}'_{Q/O}\right] - \left[\tilde{r}'_{P/Q}\right]\left[{}^F R_B\right]} \quad \boxed{\left[{}^R v'_{P,\hat{\omega}'_B}\right] = -\left[\tilde{r}'_{P/Q}\right]}$$

As noted above,

$$\boxed{\left[\tilde{r}'_{Q/O}\right] = \begin{bmatrix} 0 & 0 & \ell \\ 0 & 0 & 0 \\ -\ell & 0 & 0 \end{bmatrix}} \quad \boxed{\left[\tilde{r}'_{P/Q}\right] = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ -a & 0 & 0 \end{bmatrix}} \quad \boxed{\left[{}^F R_B\right] = \begin{bmatrix} C_\psi & S_\psi & 0 \\ -S_\psi C_\theta & C_\psi C_\theta & S_\theta \\ S_\psi S_\theta & -C_\psi S_\theta & C_\theta \end{bmatrix}}$$

Using these results, the ***explicit forms*** of the partial velocity matrices and velocity vector in ***body frame components*** are as follows.

$$\left[v'_{P,\hat{\omega}'_F}\right] = -\left[{}^F R_B\right]\left[\tilde{r}'_{Q/O}\right] - \left[\tilde{r}'_{P/Q}\right]\left[{}^F R_B\right]$$

$$= -\begin{bmatrix} C_\psi & S_\psi & 0 \\ -S_\psi C_\theta & C_\psi C_\theta & S_\theta \\ S_\psi S_\theta & -C_\psi S_\theta & C_\theta \end{bmatrix}\begin{bmatrix} 0 & 0 & \ell \\ 0 & 0 & 0 \\ -\ell & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ -a & 0 & 0 \end{bmatrix}\begin{bmatrix} C_\psi & S_\psi & 0 \\ -S_\psi C_\theta & C_\psi C_\theta & S_\theta \\ S_\psi S_\theta & -C_\psi S_\theta & C_\theta \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & 0 & \ell C_\psi \\ -\ell S_\theta & 0 & -\ell S_\psi C_\theta \\ -\ell C_\theta & 0 & \ell S_\psi S_\theta \end{bmatrix} - \begin{bmatrix} aS_\psi S_\theta & -aC_\psi S_\theta & aC_\theta \\ 0 & 0 & 0 \\ -aC_\psi & -aS_\psi & 0 \end{bmatrix}$$

$$\boxed{\left[v'_{P,\hat{\omega}'_F}\right] = \begin{bmatrix} -aS_\psi S_\theta & aC_\psi S_\theta & -\left(\ell C_\psi + aC_\theta\right) \\ \ell S_\theta & 0 & \ell S_\psi C_\theta \\ \ell C_\theta + aC_\psi & aS_\psi & -\ell S_\psi S_\theta \end{bmatrix}}$$

$$\boxed{\left[{}^R v'_{P,\hat{\omega}'_B}\right] = -\left[\tilde{r}'_{P/Q}\right] = \begin{bmatrix} 0 & 0 & -a \\ 0 & 0 & 0 \\ a & 0 & 0 \end{bmatrix}}$$

$$\left\{v'_P\right\} = \left[v'_{P,\hat{\omega}'_F}\right]\left\{\hat{\omega}'_F\right\} + \left[v'_{P,\hat{\omega}'_B}\right]\left\{\hat{\omega}'_B\right\}$$

$$= \begin{bmatrix} -aS_\psi S_\theta & aC_\psi S_\theta & -\left(\ell C_\psi + aC_\theta\right) \\ \ell S_\theta & 0 & \ell S_\psi C_\theta \\ \ell C_\theta + aC_\psi & aS_\psi & -\ell S_\psi S_\theta \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \dot{\phi} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & -a \\ 0 & 0 & 0 \\ a & 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{\theta} \\ \dot{\psi} S_\theta \\ \dot{\psi} C_\theta \end{Bmatrix}$$

$$\boxed{\left\{v'_P\right\} = \begin{Bmatrix} -\left(\ell C_\psi + aC_\theta\right)\dot{\phi} - aC_\theta \dot{\psi} \\ \ell S_\psi C_\theta \dot{\phi} \\ -\ell S_\psi S_\theta \dot{\phi} + a\dot{\theta} \end{Bmatrix}}$$

Check:

These body frame components can be ***converted*** to the fixed frame components as follows.

$$\left\{v_P\right\} = \left[R_B\right]^T \left\{v'_P\right\} = \begin{bmatrix} C_{\phi+\psi} & -S_{\phi+\psi} C_\theta & S_{\phi+\psi} S_\theta \\ S_{\phi+\psi} & C_{\phi+\psi} C_\theta & -C_{\phi+\psi} S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix}\begin{Bmatrix} -\left(\ell C_\psi + aC_\theta\right)\dot{\phi} - aC_\theta \dot{\psi} \\ \ell S_\psi C_\theta \dot{\phi} \\ -\ell S_\psi S_\theta \dot{\phi} + a\dot{\theta} \end{Bmatrix}$$

$$= \begin{Bmatrix} -\left(\ell C_\psi + aC_\theta\right)C_{\phi+\psi}\dot{\phi} - aC_{\phi+\psi}C_\theta\dot{\psi} - \ell S_{\phi+\psi}C_\theta S_\psi C_\theta\dot{\phi} - \ell S_{\phi+\psi}S_\theta S_\psi S_\theta\dot{\phi} + aS_{\phi+\psi}S_\theta\dot{\theta} \\ -\left(\ell C_\psi + aC_\theta\right)S_{\phi+\psi}\dot{\phi} - aS_{\phi+\psi}C_\theta\dot{\psi} + \ell C_{\phi+\psi}C_\theta S_\psi C_\theta\dot{\phi} + \ell C_{\phi+\psi}S_\theta S_\psi S_\theta\dot{\phi} - aC_{\phi+\psi}S_\theta\dot{\theta} \\ \cancel{\ell S_\psi S_\theta C_\theta\dot{\phi}} - \cancel{\ell S_\psi S_\theta C_\theta\dot{\phi}} + aC_\theta\dot{\theta} \end{Bmatrix}$$

$\vdots$

$$
= \begin{Bmatrix} -\left(\left(\ell C_\psi + aC_\theta \right) C_{\phi+\psi} + \ell S_{\phi+\psi} S_\psi \left(C_\theta^2 + S_\theta^2 \right) \right) \dot{\phi} - aC_{\phi+\psi} C_\theta \dot{\psi} + aS_{\phi+\psi} S_\theta \dot{\theta} \\ \left(-\left(\ell C_\psi + aC_\theta \right) S_{\phi+\psi} + \ell C_{\phi+\psi} S_\psi \left(C_\theta^2 + S_\theta^2 \right) \right) \dot{\phi} - aS_{\phi+\psi} C_\theta \dot{\psi} - aC_{\phi+\psi} S_\theta \dot{\theta} \\ aC_\theta \dot{\theta} \end{Bmatrix}
$$

$$
= \begin{Bmatrix} -\left(aC_{\phi+\psi} C_\theta + \ell \left(C_{\phi+\psi} C_\psi + S_{\phi+\psi} S_\psi \right) \right) \dot{\phi} - aC_{\phi+\psi} C_\theta \dot{\psi} + aS_{\phi+\psi} S_\theta \dot{\theta} \\ \left(-aS_{\phi+\psi} C_\theta - \ell \left(S_{\phi+\psi} C_\psi - C_{\phi+\psi} S_\psi \right) \right) \dot{\phi} - aS_{\phi+\psi} C_\theta \dot{\psi} - aC_{\phi+\psi} S_\theta \dot{\theta} \\ aC_\theta \dot{\theta} \end{Bmatrix}
$$

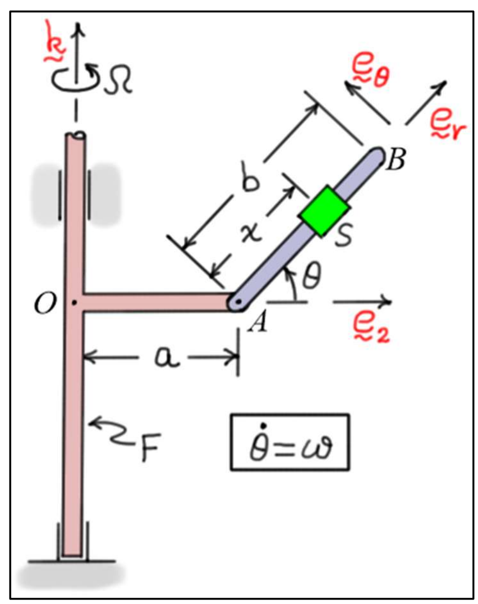
Using trigonometric identities:

$$
\boxed{C_{\phi+\psi} C_\psi + S_{\phi+\psi} S_\psi = C_{\phi+\psi-\psi} = C_\phi} \qquad \boxed{S_{\phi+\psi} C_\psi - C_{\phi+\psi} S_\psi = S_{\phi+\psi-\psi} = S_\phi}
$$

$$
\boxed{\{v_P\} = \begin{Bmatrix} -\left(aC_{\phi+\psi} C_\theta + \ell C_\phi \right) \dot{\phi} - aC_{\phi+\psi} C_\theta \dot{\psi} + aS_{\phi+\psi} S_\theta \dot{\theta} \\ \left(-aS_{\phi+\psi} C_\theta - \ell S_\phi \right) \dot{\phi} - aS_{\phi+\psi} C_\theta \dot{\psi} - aC_{\phi+\psi} S_\theta \dot{\theta} \\ aC_\theta \dot{\theta} \end{Bmatrix}}
$$

These results ***match*** with the results of parts (a) and (b).

Example 3



The figure shows a ***three-body*** system with ***frame*** F, ***arm*** AB, and ***slider*** S. Frame F rotates about the ***fixed vertical direction*** with rate $\dot{\phi} = \Omega$, and the arm AB rotates ***relative*** to F with rate $\dot{\theta} = \omega$. Slider S is connected to bar AB with a ***prismatic joint***, meaning it is free to ***slide along*** AB but ***does not rotate*** relative to it. Its ***position*** relative to point A is given by the ***variable distance*** x.

Reference frames: (all frames align when $\phi = \theta = 0$)

$R : (\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k})$	(fixed frame)
$F : (\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{k})$	(rotating with frame F)
$AB : (\underset{\sim}{e}_1, \underset{\sim}{e}_r, \underset{\sim}{e}_\theta)$	(rotating with bar AB)

Let $\{\hat{\omega}_F\}$ and $\{\hat{\omega}_{AB}\}$ be the ***base frame components*** and let $\{\hat{\omega}'_F\}$ and $\{\hat{\omega}'_{AB}\}$ be the ***body frame components*** of the ***relative angular velocities*** ${}^R\underset{\sim}{\omega}_F$ and ${}^F\underset{\sim}{\omega}_{AB}$. Find the ***fixed frame components*** of the ***velocity*** of slider S and the ***partial velocity matrix*** associated with the relative angular velocity components using a) ***base frame*** components, and b) ***body frame*** components. (c) Find the ***body frame components*** of the ***velocity*** of S and the ***partial velocity matrix*** associated with the ***body frame components*** of the ***relative angular velocities***.

Solution:

a) The ***angular velocities*** of the three bodies can be calculated as follows.

$$\boxed{{}^R\underset{\sim}{\omega}_F = \Omega \underset{\sim}{k}} \quad \boxed{{}^R\underset{\sim}{\omega}_{AB} = {}^R\underset{\sim}{\omega}_F + {}^F\underset{\sim}{\omega}_{AB}} \quad \boxed{{}^R\underset{\sim}{\omega}_S = {}^R\underset{\sim}{\omega}_{AB}}$$

The ***base frame components*** of the ***relative angular velocities*** are

$$\boxed{\{\hat{\omega}_F\} = [0 \quad 0 \quad \Omega]^T} \quad \boxed{\{\hat{\omega}_{AB}\} = \{\hat{\omega}_S\} = [\omega \quad 0 \quad 0]^T}$$

Using these results, the ***fixed frame components*** of the ***angular velocities*** of the bodies can be written as

$$\boxed{\{\omega_F\} = \{\hat{\omega}_F\}} \quad \boxed{\{\omega_S\} = \{\omega_{AB}\} = \{\hat{\omega}_F\} + [R_F]^T \{\hat{\omega}_S\}}$$

Defining the column vector of ***generalized speeds*** to be $\boxed{\{y\} \triangleq \begin{Bmatrix} \{\hat{\omega}_F\} \\ \{\hat{\omega}_S\} \\ \{\dot{x}'\} \end{Bmatrix}}$, the ***fixed frame components*** of the ***partial angular velocity matrices*** are observed to be

$$\boxed{\left[{}^R\omega_{F,y}\right]_{3\times 9} = \left[\left[{}^R\omega_{F,\hat{\omega}_F}\right] \quad \left[{}^R\omega_{F,\hat{\omega}_S}\right] \quad \left[{}^R\omega_{F,\dot{x}'}\right]\right] = \left[[I]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3}\right]}$$

$$\boxed{\left[{}^R\omega_{S,y}\right]_{3\times 9} = \left[\left[{}^R\omega_{S,\hat{\omega}_F}\right] \quad \left[{}^R\omega_{S,\hat{\omega}_S}\right] \quad \left[{}^R\omega_{S,\dot{x}'}\right]\right] = \left[[I]_{3\times 3} \quad [R_F]^T \quad [0]_{3\times 3}\right]}$$

The ***body frame components*** of the ***translation vector*** $\{x'\}$ are as follows.

$$\boxed{\{x'\} = [0 \quad x \quad 0]^T}$$

Using the concept of ***points moving on bodies***, ${}^R\underset{\sim}{v}_S$ the velocity of point P in R can be written as follows.

$${}^R\underset{\sim}{v}_S = {}^R\underset{\sim}{v}_{\hat{S}} + {}^{AB}\underset{\sim}{v}_S = {}^R\underset{\sim}{v}_A + {}^R\underset{\sim}{v}_{\hat{S}/A} + {}^{AB}\underset{\sim}{v}_S = \left({}^R\underset{\sim}{\omega}_F \times \underset{\sim}{r}_{A/O}\right) + \left({}^R\underset{\sim}{\omega}_S \times \underset{\sim}{r}_{S/A}\right) + \left(\dot{x}\,\underset{\sim}{e}_r\right)$$

$$\Rightarrow \boxed{{}^R\underset{\sim}{v}_S = -\left(\underset{\sim}{r}_{A/O} \times {}^R\underset{\sim}{\omega}_F\right) - \left(\underset{\sim}{r}_{S/A} \times {}^R\underset{\sim}{\omega}_S\right) + \left(\dot{x}\,\underset{\sim}{e}_r\right)} \qquad (58)$$

Using this result, the ***fixed frame components*** of ${}^R\underset{\sim}{v}_S$ can be written as follows.

$$\begin{aligned}\{v_S\} &= -\left[\tilde{r}_{A/O}\right]\{\omega_F\} - \left[\tilde{r}_{S/A}\right]\{\omega_S\} + [R_{AB}]^T \{\dot{x}'\} \\ &= -\left[\tilde{r}_{A/O}\right]\left[{}^R\omega_{F,y}\right]\{y\} - \left[\tilde{r}_{S/A}\right]\left[{}^R\omega_{S,y}\right]\{y\} + [R_{AB}]^T \{\dot{x}'\} \\ &= -\left[\tilde{r}_{A/O}\right]\{\hat{\omega}_F\} - \left[\tilde{r}_{S/A}\right]\left(\{\hat{\omega}_F\} + [R_F]^T \{\hat{\omega}_S\}\right) + [R_{AB}]^T \{\dot{x}'\}\end{aligned}$$

$$\Rightarrow \boxed{\{v_S\} = -\left(\left[\tilde{r}_{A/O}\right] + \left[\tilde{r}_{S/A}\right]\right)\{\hat{\omega}_F\} - \left(\left[\tilde{r}_{S/A}\right][R_F]^T\right)\{\hat{\omega}_S\} + [R_{AB}]^T \{\dot{x}'\}}$$

Observation of this result gives the following ***partial velocity matrices*** in ***fixed frame components***.

$$\boxed{\left[{}^R v_{S,\hat{\omega}_F}\right] = -\left(\left[\tilde{r}_{A/O}\right] + \left[\tilde{r}_{S/A}\right]\right)} \quad \boxed{\left[{}^R v_{S,\hat{\omega}_S}\right] = -\left[\tilde{r}_{S/A}\right][R_F]^T} \quad \boxed{\left[{}^R v_{S,\dot{x}'}\right] = [R_{AB}]^T}$$

Recall that the ***skew symmetric matrices*** are built using ***fixed frame components*** of the indicated vectors. The following equations can be used to find these components.

$$\left[R_F\right]^T = \begin{bmatrix} C_\phi & S_\phi & 0 \\ -S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left\{r_{A/O}\right\} = \left[R_F\right]^T \begin{Bmatrix} 0 \\ a \\ 0 \end{Bmatrix} = \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ a \\ 0 \end{Bmatrix} = \begin{Bmatrix} -aS_\phi \\ aC_\phi \\ 0 \end{Bmatrix}$$

$$\left[R_{AB}\right]^T = \left(\left[{}^F R_{AB}\right]\left[R_F\right]\right)^T = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & S_\theta \\ 0 & -S_\theta & C_\theta \end{bmatrix} \begin{bmatrix} C_\phi & S_\phi & 0 \\ -S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)^T = \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & -S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix}$$

$$\Rightarrow \left[R_{AB}\right]^T = \begin{bmatrix} C_\phi & -S_\phi C_\theta & S_\phi S_\theta \\ S_\phi & C_\phi C_\theta & -C_\phi S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix}$$

$$\left\{r_{S/A}\right\} = \left[R_{AB}\right]^T \begin{Bmatrix} 0 \\ x \\ 0 \end{Bmatrix} = \begin{bmatrix} C_\phi & -S_\phi C_\theta & S_\phi S_\theta \\ S_\phi & C_\phi C_\theta & -C_\phi S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \begin{Bmatrix} 0 \\ x \\ 0 \end{Bmatrix} = x \begin{Bmatrix} -S_\phi C_\theta \\ C_\phi C_\theta \\ S_\theta \end{Bmatrix}$$

Using these results, ***explicit equations*** for the ***fixed frame components*** of the ***partial velocity matrices*** and ${}^R \underset{\sim}{v}_S$ the ***velocity*** of S can be calculated as follows.

$$\left[{}^R v_{S,\hat{\omega}_F}\right] = -\left(\left[\tilde{r}_{A/O}\right] + \left[\tilde{r}_{S/A}\right]\right) = -\left(\begin{bmatrix} 0 & 0 & aC_\phi \\ 0 & 0 & aS_\phi \\ -aC_\phi & -aS_\phi & 0 \end{bmatrix} + \begin{bmatrix} 0 & -xS_\theta & xC_\phi C_\theta \\ xS_\theta & 0 & xS_\phi C_\theta \\ -xC_\phi C_\theta & -xS_\phi C_\theta & 0 \end{bmatrix}\right)$$

$$\Rightarrow \left[{}^R v_{S,\hat{\omega}_F}\right] = \begin{bmatrix} 0 & xS_\theta & -\left(a + xC_\theta\right)C_\phi \\ -xS_\theta & 0 & -\left(a + xC_\theta\right)S_\phi \\ \left(a + xC_\theta\right)C_\phi & \left(a + xC_\theta\right)S_\phi & 0 \end{bmatrix}$$

$$\left[{}^R v_{S,\hat{\omega}_S}\right] = -\left[\tilde{r}_{S/A}\right]\left[R_F\right]^T = -\begin{bmatrix} 0 & -xS_\theta & xC_\phi C_\theta \\ xS_\theta & 0 & xS_\phi C_\theta \\ -xC_\phi C_\theta & -xS_\phi C_\theta & 0 \end{bmatrix} \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= -\begin{bmatrix} -xS_\phi S_\theta & -xC_\phi S_\theta & xC_\phi C_\theta \\ xC_\phi S_\theta & -xS_\phi S_\theta & xS_\phi C_\theta \\ -xC_\theta\left(C_\phi^2 + S_\phi^2\right) & xS_\phi C_\phi C_\theta - xS_\phi C_\phi C_\theta & 0 \end{bmatrix}$$

$$\Rightarrow \left[{}^R v_{S,\hat{\omega}_S}\right] = \begin{bmatrix} xS_\phi S_\theta & xC_\phi S_\theta & -xC_\phi C_\theta \\ -xC_\phi S_\theta & xS_\phi S_\theta & -xS_\phi C_\theta \\ xC_\theta & 0 & 0 \end{bmatrix}$$

$$\{v_S\}=\left[{}^R v_{S,\hat\omega_F}\right]\{\hat\omega_F\}+\left[{}^R v_{S,\hat\omega_S}\right]\{\hat\omega_S\}+\left[{}^R v_{S,\dot x'}\right]\{\dot x'\}$$

$$=\begin{bmatrix}0 & xS_\theta & -\left(a+xC_\theta\right)C_\phi\\ -xS_\theta & 0 & -\left(a+xC_\theta\right)S_\phi\\ \left(a+xC_\theta\right)C_\phi & \left(a+xC_\theta\right)S_\phi & 0\end{bmatrix}\begin{Bmatrix}0\\0\\\Omega\end{Bmatrix}+\begin{bmatrix}xS_\phi S_\theta & xC_\phi S_\theta & -xC_\phi C_\theta\\ -xC_\phi S_\theta & xS_\phi S_\theta & -xS_\phi C_\theta\\ xC_\theta & 0 & 0\end{bmatrix}\begin{Bmatrix}\omega\\0\\0\end{Bmatrix}$$

$$+\begin{bmatrix}C_\phi & -S_\phi C_\theta & S_\phi S_\theta\\ S_\phi & C_\phi C_\theta & -C_\phi S_\theta\\ 0 & S_\theta & C_\theta\end{bmatrix}\begin{Bmatrix}0\\\dot x\\0\end{Bmatrix}$$

$$\Rightarrow \boxed{\{v_S\}=\begin{Bmatrix}-\left(a+xC_\theta\right)\Omega C_\phi+\left(x\omega S_\theta-\dot xC_\theta\right)S_\phi\\ -\left(a+xC_\theta\right)\Omega S_\phi-\left(x\omega S_\theta-\dot xC_\theta\right)C_\phi\\ \dot xS_\theta+x\omega C_\theta\end{Bmatrix}}$$

b) The ***joints*** connecting the two bodies are ***simple revolute joints***, and the bodies share a ***common unit vector*** along the axis of the joint. Consequently, the ***body frame components*** of the relative angular velocities are the ***same*** as the ***base frame components***. That is,

$$\boxed{\{\hat\omega'_F\}=\begin{bmatrix}0 & 0 & \Omega\end{bmatrix}^T}\qquad \boxed{\{\hat\omega'_{AB}\}=\{\hat\omega'_S\}=\begin{bmatrix}\omega & 0 & 0\end{bmatrix}^T}$$

Using these results, the ***body frame components*** of the ***angular velocities*** of the bodies can be written as

$$\boxed{\{\omega'_F\}=\{\hat\omega'_F\}}\qquad \boxed{\{\omega'_S\}=\{\omega'_{AB}\}=\left[{}^F R_{AB}\right]\{\hat\omega'_F\}+\{\hat\omega'_S\}}$$

Defining the column vector of ***generalized speeds*** to be $\boxed{\{y\}\triangleq\begin{Bmatrix}\{\hat\omega'_F\}\\\{\hat\omega'_S\}\\\{\dot x'\}\end{Bmatrix}}$, the ***fixed frame components*** of the ***partial angular velocity matrices*** are observed to be

$$\boxed{\left[{}^R\omega'_{F,y}\right]_{3\times9}=\left[\left[{}^R\omega'_{F,\hat\omega'_F}\right]\ \left[{}^R\omega'_{F,\hat\omega'_S}\right]\ \left[{}^R\omega'_{F,\dot s'}\right]\right]=\left[[I]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\right]}$$

$$\boxed{\left[{}^R\omega'_{S,y}\right]_{3\times9}=\left[\left[{}^R\omega'_{S,\hat\omega'_F}\right]\ \left[{}^R\omega'_{S,\hat\omega'_S}\right]\ \left[{}^R\omega'_{S,\dot s'}\right]\right]=\left[\left[{}^F R_{AB}\right]\ \ [I]_{3\times3}\ \ [0]_{3\times3}\right]}$$

Using this result, the ***fixed frame components*** of ${}^R\underset{\sim}{v}_S$ can be written as follows.

$$\{v_S\}=-\left[R_F\right]^T\left[\tilde r'_{A/O}\right]\{\omega'_F\}-\left[R_{AB}\right]^T\left[\tilde r'_{S/A}\right]\{\omega'_S\}+\left[R_{AB}\right]^T\{\dot x'\}$$

$$=-\left[R_F\right]^T\left[\tilde r'_{A/O}\right]\left[{}^R\omega_{F,y}\right]\{y\}-\left[R_{AB}\right]^T\left[\tilde r'_{S/A}\right]\left[{}^R\omega_{S,y}\right]\{y\}+\left[R_{AB}\right]^T\{\dot x'\}$$

$$=-\left[R_F\right]^T\left[\tilde r'_{A/O}\right]\{\hat\omega'_F\}-\left[R_{AB}\right]^T\left[\tilde r'_{S/A}\right]\left(\left[{}^F R_{AB}\right]\{\hat\omega'_F\}+\{\omega'_S\}\right)+\left[R_{AB}\right]^T\{\dot x'\}$$

$$\Rightarrow\boxed{\{v_S\}=-\left(\left[R_F\right]^T\left[\tilde r'_{A/O}\right]+\left[R_{AB}\right]^T\left[\tilde r'_{S/A}\right]\left[{}^F R_{AB}\right]\right)\{\hat\omega'_F\}-\left(\left[R_{AB}\right]^T\left[\tilde r'_{S/A}\right]\right)\{\hat\omega'_S\}+\left[R_{AB}\right]^T\{\dot x'\}}$$

Observation of this result gives the following ***partial velocity matrices*** in ***fixed frame components***.

$$\boxed{\left[{}^R v_{S,\hat{\omega}'_F}\right] = -\left(\left[R_F\right]^T\left[\tilde{r}'_{A/O}\right] + \left[R_{AB}\right]^T\left[\tilde{r}'_{S/A}\right]\left[{}^F R_{AB}\right]\right)} \quad \boxed{\left[{}^R v_{S,\hat{\omega}'_S}\right] = -\left[R_{AB}\right]^T\left[\tilde{r}'_{S/A}\right]} \quad \boxed{\left[{}^R v_{S,\dot{x}'}\right] = \left[R_{AB}\right]^T}$$

The skew-symmetric matrix $\left[\tilde{r}'_{A/O}\right]$ is built with components in frame F, and the skew-symmetric matrix $\left[\tilde{r}'_{S/A}\right]$ is built with components in the AB frame. That is,

$$\boxed{\underset{\sim}{r}_{A/O} = a\,\underset{\sim}{e}_2} \quad \Rightarrow \boxed{\left[\tilde{r}'_{A/O}\right] = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ -a & 0 & 0 \end{bmatrix}} \quad \text{and} \quad \boxed{\underset{\sim}{r}_{S/A} = x\,\underset{\sim}{e}_r} \quad \Rightarrow \boxed{\left[\tilde{r}'_{S/A}\right] = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ -x & 0 & 0 \end{bmatrix}}$$

Using these results, ***explicit equations*** for the ***fixed frame components*** of the ***partial velocity matrices*** and ${}^R\underset{\sim}{v}_S$ the ***velocity*** of S can be calculated as follows.

$$\begin{aligned}
\left[{}^R v_{S,\hat{\omega}'_F}\right] &= -\left[R_F\right]^T\left[\tilde{r}'_{A/O}\right] - \left[R_{AB}\right]^T\left[\tilde{r}'_{S/A}\right]\left[{}^F R_{AB}\right] \\
&= -\begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ -a & 0 & 0 \end{bmatrix} - \begin{bmatrix} C_\phi & -S_\phi C_\theta & S_\phi S_\theta \\ S_\phi & C_\phi C_\theta & -C_\phi S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix}\begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ -x & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & S_\theta \\ 0 & -S_\theta & C_\theta \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & -aC_\phi \\ 0 & 0 & -aS_\phi \\ a & 0 & 0 \end{bmatrix} - \begin{bmatrix} -xS_\phi S_\theta & 0 & xC_\phi \\ xC_\phi S_\theta & 0 & xS_\phi \\ -xC_\theta & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & S_\theta \\ 0 & -S_\theta & C_\theta \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & -aC_\phi \\ 0 & 0 & -aS_\phi \\ a & 0 & 0 \end{bmatrix} + \begin{bmatrix} xS_\phi S_\theta & xC_\phi S_\theta & -xC_\phi C_\theta \\ -xC_\phi S_\theta & xS_\phi S_\theta & -xS_\phi C_\theta \\ xC_\theta & 0 & 0 \end{bmatrix}
\end{aligned}$$

$$\Rightarrow \boxed{\left[{}^R v_{S,\hat{\omega}'_F}\right] = \begin{bmatrix} xS_\phi S_\theta & xC_\phi S_\theta & -\left(a + xC_\theta\right)C_\phi \\ -xC_\phi S_\theta & xS_\phi S_\theta & -\left(a + xC_\theta\right)S_\phi \\ a + xC_\theta & 0 & 0 \end{bmatrix}}$$

$$\left[{}^R v_{S,\hat{\omega}'_S}\right] = -\left[R_{AB}\right]^T\left[\tilde{r}'_{S/A}\right] = -\begin{bmatrix} C_\phi & -S_\phi C_\theta & S_\phi S_\theta \\ S_\phi & C_\phi C_\theta & -C_\phi S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix}\begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ -x & 0 & 0 \end{bmatrix} = -\begin{bmatrix} -xS_\phi S_\theta & 0 & xC_\phi \\ xC_\phi S_\theta & 0 & xS_\phi \\ -xC_\theta & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \boxed{\left[{}^R v_{S,\hat{\omega}'_S}\right] = \begin{bmatrix} xS_\phi S_\theta & 0 & -xC_\phi \\ -xC_\phi S_\theta & 0 & -xS_\phi \\ xC_\theta & 0 & 0 \end{bmatrix}}$$

$$\{v_S\} = \left[{}^R v_{S,\hat{\omega}'_F}\right]\{\hat{\omega}'_F\} + \left[{}^R v_{S,\hat{\omega}'_S}\right]\{\hat{\omega}'_S\} + \left[{}^R v_{S,\dot{x}'}\right]\{\dot{x}'\}$$

$$= \begin{bmatrix} xS_\phi S_\theta & xC_\phi S_\theta & -(a+xC_\theta)C_\phi \\ -xC_\phi S_\theta & xS_\phi S_\theta & -(a+xC_\theta)S_\phi \\ a+xC_\theta & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix} + \begin{bmatrix} xS_\phi S_\theta & 0 & -xC_\phi \\ -xC_\phi S_\theta & 0 & -xS_\phi \\ xC_\theta & 0 & 0 \end{bmatrix} \begin{Bmatrix} \omega \\ 0 \\ 0 \end{Bmatrix}$$

$$+ \begin{bmatrix} C_\phi & -S_\phi C_\theta & S_\phi S_\theta \\ S_\phi & C_\phi C_\theta & -C_\phi S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \begin{Bmatrix} 0 \\ \dot{x} \\ 0 \end{Bmatrix}$$

$$\Rightarrow \boxed{\{v_S\} = \begin{Bmatrix} -(a+xC_\theta)\Omega C_\phi + (x\omega S_\theta - \dot{x}C_\theta)S_\phi \\ -(a+xC_\theta)\Omega S_\phi - (x\omega S_\theta - \dot{x}C_\theta)C_\phi \\ \dot{x}S_\theta + x\omega C_\theta \end{Bmatrix}}$$

As expected, these results are ***identical*** to those found in part (a).

c) Using Equation (58), the ***body frame components*** of ${}^R\underset{\sim}{v}_S$ can be written as follows.

$$\boxed{\{v'_S\} = -\left[{}^F R_{AB}\right][\tilde{r}'_{A/O}]\{\omega'_F\} - [\tilde{r}'_{S/A}]\{\omega'_S\} + \{\dot{x}'\}}$$

Here,

$$\boxed{\left[\tilde{r}'_{A/O}\right] = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ -a & 0 & 0 \end{bmatrix}} \quad \boxed{\left[\tilde{r}'_{S/A}\right] = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ -x & 0 & 0 \end{bmatrix}} \quad \boxed{\left[{}^F R_{AB}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & S_\theta \\ 0 & -S_\theta & C_\theta \end{bmatrix}}$$

Substituting and expanding,

$$\{v'_S\} = -\left[{}^F R_{AB}\right][\tilde{r}'_{A/O}]\{\omega'_F\} - [\tilde{r}'_{S/A}]\{\omega'_S\} + \{\dot{x}'\}$$

$$= -\left[{}^F R_{AB}\right][\tilde{r}'_{A/O}]\{\hat{\omega}'_F\} - [\tilde{r}'_{S/A}]\left(\left[{}^F R_{AB}\right]\{\hat{\omega}'_F\} + \{\hat{\omega}'_S\}\right) + \{\dot{x}'\}$$

$$= -\left(\left[{}^F R_{AB}\right][\tilde{r}'_{A/O}] + [\tilde{r}'_{S/A}]\left[{}^F R_{AB}\right]\right)\{\hat{\omega}'_F\} - [\tilde{r}'_{S/A}]\{\hat{\omega}'_S\} + \{\dot{x}'\}$$

$$= -\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & S_\theta \\ 0 & -S_\theta & C_\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ -a & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ -x & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & S_\theta \\ 0 & -S_\theta & C_\theta \end{bmatrix} \right) \begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix} - \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ -x & 0 & 0 \end{bmatrix} \begin{Bmatrix} \omega \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \dot{x} \\ 0 \end{Bmatrix}$$

$$= -\left(\begin{bmatrix} 0 & 0 & a \\ -aS_\theta & 0 & 0 \\ -aC_\theta & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -xS_\theta & xC_\theta \\ 0 & 0 & 0 \\ -x & 0 & 0 \end{bmatrix} \right) \begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ x\omega \end{Bmatrix} + \begin{Bmatrix} 0 \\ \dot{x} \\ 0 \end{Bmatrix}$$

$$= \begin{bmatrix} 0 & xS_\theta & -(a+xC_\theta) \\ aS_\theta & 0 & 0 \\ aC_\theta + x & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix} + \begin{Bmatrix} 0 \\ \dot{x} \\ x\omega \end{Bmatrix}$$

From the above equation, the following results can be observed and calculated.

$$\left[v'_{S,\hat{\omega}'_F}\right]=\begin{bmatrix}0 & xS_\theta & -\left(a+xC_\theta\right)\\ aS_\theta & 0 & 0\\ aC_\theta+x & 0 & 0\end{bmatrix}\qquad \left[v'_{S,\hat{\omega}'_S}\right]=\begin{bmatrix}0 & 0 & -x\\ 0 & 0 & 0\\ x & 0 & 0\end{bmatrix}\qquad \left[v'_{S,\dot{x}'}\right]=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\left\{v'_S\right\}=\begin{Bmatrix}-\left(a+xC_\theta\right)\Omega\\ \dot{x}\\ x\omega\end{Bmatrix}$$

Comparison with Previous Results: (Volume I, Unit 4, Exercise 4)

In Unit 4 of Volume I, ${}^R\underset{\sim}{v}_S$ the velocity of *S* in *R* is calculated in Exercise 4. The results are given to be

$${}^R\underset{\sim}{v}_S=-\Omega\left(a+xC_\theta\right)\underset{\sim}{e}_1+\left(\dot{x}C_\theta-x\omega S_\theta\right)\underset{\sim}{e}_2+\left(\dot{x}S_\theta+x\omega C_\theta\right)\underset{\sim}{k}$$

Converting this result to the ***fixed frame*** can be done as follows.

$$\left\{v_S\right\}=\left[R_F\right]^T\left\{v_S\right\}_F=\begin{bmatrix}C_\phi & -S_\phi & 0\\ S_\phi & C_\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}-\Omega\left(a+xC_\theta\right)\\ \dot{x}C_\theta-x\omega S_\theta\\ \dot{x}S_\theta+x\omega C_\theta\end{Bmatrix}$$

$$\Rightarrow\left\{v_S\right\}=\begin{Bmatrix}-\Omega C_\phi\left(a+xC_\theta\right)-S_\phi\left(\dot{x}C_\theta-x\omega S_\theta\right)\\ -\Omega S_\phi\left(a+xC_\theta\right)+\left(\dot{x}C_\theta-x\omega S_\theta\right)C_\phi\\ \dot{x}S_\theta+x\omega C_\theta\end{Bmatrix}\quad\checkmark$$

Converting this result to the *S* (or *AB*) ***frame*** can be done as follows.

$$\left\{v'_S\right\}=\left[{}^F R_{AB}\right]\left\{v_S\right\}_F=\begin{bmatrix}1 & 0 & 0\\ 0 & C_\theta & S_\theta\\ 0 & -S_\theta & C_\theta\end{bmatrix}\begin{Bmatrix}-\Omega\left(a+xC_\theta\right)\\ \dot{x}C_\theta-x\omega S_\theta\\ \dot{x}S_\theta+x\omega C_\theta\end{Bmatrix}=\begin{Bmatrix}-\Omega\left(a+xC_\theta\right)\\ \left(\dot{x}C_\theta-x\omega S_\theta\right)C_\theta+\left(\dot{x}S_\theta+x\omega C_\theta\right)S_\theta\\ -\left(\dot{x}C_\theta-x\omega S_\theta\right)S_\theta+\left(\dot{x}S_\theta+x\omega C_\theta\right)C_\theta\end{Bmatrix}$$

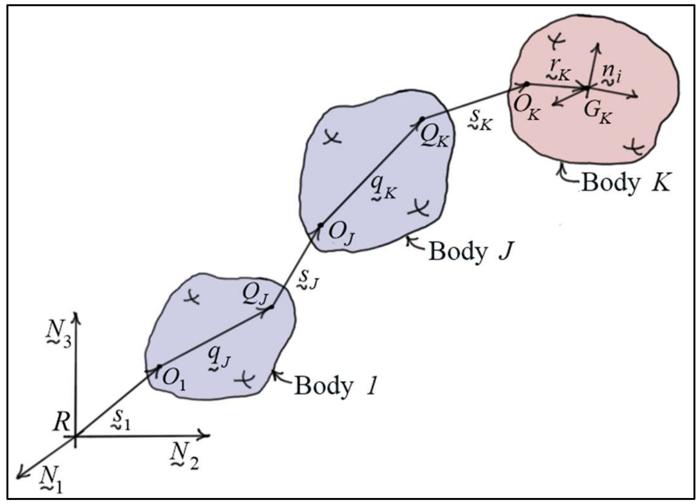
$$=\begin{Bmatrix}-\left(a+xC_\theta\right)\Omega\\ \dot{x}\left(C_\theta^2+S_\theta^2\right)\\ x\omega\left(C_\theta^2+S_\theta^2\right)\end{Bmatrix}$$

$$\Rightarrow\left\{v'_S\right\}=\begin{Bmatrix}-\left(a+xC_\theta\right)\Omega\\ \dot{x}\\ x\omega\end{Bmatrix}\quad\checkmark$$

Example 4

The diagram shows three bodies which are part of a larger multibody system. The lower numbered body array for the three bodies is as follows.

$$\mathfrak{L}\left(B_i = 1, J, K\right) = \left(\mathfrak{L}(1), \mathfrak{L}(J), \mathfrak{L}(K)\right) = \left(0, 1, J\right)$$



Calculate $\{v_K\}$ the ***fixed frame components*** of the velocity of G_K the mass center of body K using the concept of ***relative velocities***. Using that result, determine $\left[{}^R v_{K,y}\right]$ the matrix of ***fixed frame*** components of the partial velocity of G_K with respect to the system matrix of generalized speeds as defined in ***Case 1*** above which utilizes the ***lower body*** (base frame) ***components*** of the ***relative angular velocity vectors*** as generalized speeds.

Solution:

Using the concept of ***relative velocity***, $\underset{\sim}{v}_K$ the velocity of G_K can be written as follows.

$$\begin{aligned}{}^R\underset{\sim}{v}_K &= {}^R\underset{\sim}{v}_{O_1} + {}^R\underset{\sim}{v}_{Q_J/O_1} + {}^R\underset{\sim}{v}_{O_J/Q_J} + {}^R\underset{\sim}{v}_{Q_K/O_J} + {}^R\underset{\sim}{v}_{O_K/Q_K} + {}^R\underset{\sim}{v}_{G_K/O_K} \\ &= \frac{{}^R d\underset{\sim}{s}_1}{dt} + \left({}^R\underset{\sim}{\omega}_1 \times \underset{\sim}{q}_J\right) + \frac{{}^R d\underset{\sim}{s}_J}{dt} + \left({}^R\underset{\sim}{\omega}_J \times \underset{\sim}{q}_K\right) + \frac{{}^R d\underset{\sim}{s}_K}{dt} + \left({}^R\underset{\sim}{\omega}_K \times \underset{\sim}{r}_K\right)\end{aligned}$$

The ***derivatives*** of the position vectors $\underset{\sim}{s}_B \; \left(B = 1, J, K\right)$ can be expanded to give

$${}^R\underset{\sim}{v}_K = \frac{{}^R d\underset{\sim}{s}_1}{dt} + \left({}^R\underset{\sim}{\omega}_1 \times \underset{\sim}{q}_J\right) + \left(\frac{{}^1 d\underset{\sim}{s}_J}{dt} + \left({}^R\underset{\sim}{\omega}_1 \times \underset{\sim}{s}_J\right)\right) + \left({}^R\underset{\sim}{\omega}_J \times \underset{\sim}{q}_K\right) + \left(\frac{{}^J d\underset{\sim}{s}_K}{dt} + \left({}^R\underset{\sim}{\omega}_J \times \underset{\sim}{s}_K\right)\right) + \left({}^R\underset{\sim}{\omega}_K \times \underset{\sim}{r}_K\right)$$

Changing the ***order*** of all the ***cross products*** and rearranging terms gives the following.

$$\begin{aligned}{}^R\underset{\sim}{v}_K &= \frac{{}^R d\underset{\sim}{s}_1}{dt} - \left(\underset{\sim}{q}_J \times {}^R\underset{\sim}{\omega}_1\right) + \left(\frac{{}^1 d\underset{\sim}{s}_J}{dt} - \left(\underset{\sim}{s}_J \times {}^R\underset{\sim}{\omega}_1\right)\right) - \left(\underset{\sim}{q}_K \times {}^R\underset{\sim}{\omega}_J\right) + \left(\frac{{}^J d\underset{\sim}{s}_K}{dt} - \left(\underset{\sim}{s}_K \times {}^R\underset{\sim}{\omega}_J\right)\right) - \left(\underset{\sim}{r}_K \times {}^R\underset{\sim}{\omega}_K\right) \\ &= -\left(\left(\underset{\sim}{q}_J + \underset{\sim}{s}_J\right) \times {}^R\underset{\sim}{\omega}_1\right) - \left(\left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right) \times {}^R\underset{\sim}{\omega}_J\right) - \left(\underset{\sim}{r}_K \times {}^R\underset{\sim}{\omega}_K\right) + \frac{{}^R d\underset{\sim}{s}_1}{dt} + \frac{{}^1 d\underset{\sim}{s}_J}{dt} + \frac{{}^J d\underset{\sim}{s}_K}{dt}\end{aligned}$$

Finally, using the ***summation rule*** for ***angular velocities*** gives

$$\begin{aligned}{}^R\underset{\sim}{v}_K &= -\left(\left(\underset{\sim}{q}_J + \underset{\sim}{s}_J\right) \times {}^R\underset{\sim}{\omega}_1\right) - \left(\left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right) \times {}^R\underset{\sim}{\omega}_J\right) - \left(\underset{\sim}{r}_K \times {}^R\underset{\sim}{\omega}_K\right) + \frac{{}^R d\underset{\sim}{s}_1}{dt} + \frac{{}^1 d\underset{\sim}{s}_J}{dt} + \frac{{}^J d\underset{\sim}{s}_K}{dt} \\ &= -\left(\left(\underset{\sim}{q}_J + \underset{\sim}{s}_J\right) \times {}^R\underset{\sim}{\omega}_1\right) - \left(\left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right) \times \left({}^R\underset{\sim}{\omega}_1 + {}^1\underset{\sim}{\omega}_J\right)\right) - \left(\underset{\sim}{r}_K \times \left({}^R\underset{\sim}{\omega}_1 + {}^1\underset{\sim}{\omega}_J + {}^J\underset{\sim}{\omega}_K\right)\right) + \frac{{}^R d\underset{\sim}{s}_1}{dt} + \frac{{}^1 d\underset{\sim}{s}_J}{dt} + \frac{{}^J d\underset{\sim}{s}_K}{dt}\end{aligned}$$

Using this result, the ***fixed frame components*** of ${}^R\underset{\sim}{v}_K$ can be written in terms of the ***base frame components*** of the ***angular velocities*** of the bodies ***relative*** to their ***lower numbered bodies*** as follows.

$$\begin{aligned}\{v_K\} = &-\left([\tilde{q}_J]+[\tilde{s}_J]\right)\{\hat{\omega}_1\} - \left([\tilde{q}_K]+[\tilde{s}_K]\right)\left(\{\hat{\omega}_1\}+[R_1]^T\{\hat{\omega}_J\}\right)\\ &-[\tilde{r}_K]\left(\{\hat{\omega}_1\}+[R_1]^T\{\hat{\omega}_J\}+[R_J]^T\{\hat{\omega}_K\}\right)+\{\dot{s}'_1\}+[R_1]^T\{\dot{s}'_J\}+[R_J]^T\{\dot{s}'_K\}\end{aligned} \tag{59}$$

Here, ***fixed frame components*** are used to ***build*** the ***skew symmetric*** matrices $[\tilde{q}_J]$, $[\tilde{s}_J]$, $[\tilde{q}_K]$, $[\tilde{s}_K]$, and $[\tilde{r}_K]$, the column matrices $\{\hat{\omega}_B\}$ $(B=1,J,K)$ hold the ***base frame components*** of the ***angular velocities*** of the bodies ***relative*** to their ***lower numbered bodies***, and the column matrices $\{s'_B\}$ $(B=1,J,K)$ are as defined in Equation (29).

By rearranging terms, Equation (59) can be rewritten as follows.

$$\begin{aligned}\{v_K\} = &-\left([\tilde{q}_J]+[\tilde{s}_J]+[\tilde{q}_K]+[\tilde{s}_K]+[\tilde{r}_K]\right)\{\hat{\omega}_1\} - \left([\tilde{q}_K]+[\tilde{s}_K]+[\tilde{r}_K]\right)[R_1]^T\{\hat{\omega}_J\}\\ &-[\tilde{r}_K][R_J]^T\{\hat{\omega}_K\}+\{\dot{s}'_1\}+[R_1]^T\{\dot{s}'_J\}+[R_J]^T\{\dot{s}'_K\}\end{aligned} \tag{60}$$

The generalized speeds for Case 1 are defined in Equations (29) and (30). Using those generalized speeds, the following ***partial velocity matrices*** can be identified from the above result.

$$\left[{}^R v_{K,\hat{\omega}_1}\right]_{3\times3} = -\left([\tilde{q}_J]+[\tilde{s}_J]+[\tilde{q}_K]+[\tilde{s}_K]+[\tilde{r}_K]\right) \tag{61}$$

$$\left[{}^R v_{K,\hat{\omega}_J}\right]_{3\times3} = -\left([\tilde{q}_K]+[\tilde{s}_K]+[\tilde{r}_K]\right)[R_1]^T \qquad \left[{}^R v_{K,\hat{\omega}_K}\right]_{3\times3} = -[\tilde{r}_K][R_J]^T \tag{62}$$

$$\left[{}^R v_{K,\dot{s}'_1}\right]_{3\times3} = [I]_{3\times3} \qquad \left[{}^R v_{K,\dot{s}'_J}\right]_{3\times3} = [R_1]^T \qquad \left[{}^R v_{K,\dot{s}'_K}\right]_{3\times3} = [R_J]^T \tag{63}$$

These results can be put in terms of the generalized speed matrix of Equations (28)-(30) to give the following.

$$\begin{aligned}\left[{}^R v_{K,y}\right]_{3\times6N} = \Big[&\left[{}^R v_{K,\hat{\omega}_1}\right]_{3\times3} \;\cdots\; \left[{}^R v_{K,\hat{\omega}_J}\right]_{3\times3} \;\cdots\; \left[{}^R v_{K,\hat{\omega}_K}\right]_{3\times3} \;\cdots\\ &\left[{}^R v_{K,\dot{s}'_1}\right]_{3\times3} \;\cdots\; \left[{}^R v_{K,\dot{s}'_J}\right]_{3\times3} \;\cdots\; \left[{}^R v_{K,\dot{s}'_K}\right]_{3\times3} \;\cdots\Big]_{3\times6N}\end{aligned} \tag{64}$$

The matrices $\left[{}^R v_{K,\hat{\omega}_B}\right]$ $(B=1,J,K)$ fill columns $3(B-1)+j$ $(j=1,2,3)$, and the matrices $\left[{}^R v_{K,\dot{s}'_B}\right]$ fill the columns $3(N+B-1)+j$ $(j=1,2,3)$. ***All other*** entries are ***zero***.

Example 5

Using the approach described in Equations (34)-(36) and Equation (45) which utilize the ***base frame components*** of the ***relative angular velocity components***, find $\left[{}^R v_{K,y}\right]$ the matrix of ***fixed frame components*** of the ***partial velocity*** of G_K the mass center of body K of the three body system of Example 4. ***Compare*** the results.

Solution:

Partial Angular Velocities of the Bodies

Body 1: $\left[{}^R\omega_{1,y}\right]_{3\times 6N} = \left[[I]_{3\times3} \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3}\right]_{3\times 6N}$

Body J: $\left[{}^R\omega_{J,y}\right]_{3\times 6N} = \left[[I]_{3\times3} \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3} \quad [R_1]^T \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3}\right]_{3\times 6N}$

The matrix $[R_1]^T$ fills the columns $3J-2$, $3J-1$, and $3J$.

Body K:

$$\left[{}^R\omega_{K,y}\right]_{3\times 6N} = \left[[I]_{3\times3} \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3} \quad [R_1]^T \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3} \quad [R_J]^T \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3}\right]_{3\times 6N}$$

The matrix $[R_1]^T$ fills the columns $3J-2$, $3J-1$, and $3J$, and the matrix $[R_J]^T$ fills the columns $3K-2$, $3K-1$, and $3K$.

Partial Velocity of Body Origins

The process presented in Equations (34)-(36) is repeated here for convenience.

1. First, set

$$\left[{}^R v_{O_K,y}\right]_{3\times 6N} = \left[{}^R v_{O_{\mathfrak{L}(K)},y}\right]_{3\times 6N} - \left([\tilde{q}_K] + [\tilde{s}_K]\right)\left[{}^R\omega_{\mathfrak{L}(K),y}\right]_{3\times 6N}$$

2. Then, set the ***three columns*** associated with $\dot{s}'_{Ki}$ $(i=1,2,3)$ as follows.

$$\left[{}^R v_{O_K,y}\right]_{ik} = \left[R_{\mathfrak{L}(K)}\right]^T_{ij} \qquad \left(i=1,2,3;\ j=1,2,3;\ k=3N+(3K-3+j)\right)$$

For body 1, only Equation (35) applies. All entries are ***zero*** except for the three columns associated with $\{\dot{s}'_1\}$ giving the following result.

$$\left[{}^R v_{O_K,y}\right]_{ik} = [I]_{ij} \qquad \left(i=1,2,3;\ j=1,2,3;\ k=3N+j;\ K=1\right)$$

Applying these results to the three bodies gives the following.

Body 1: $\left[{}^R v_{O_1,y}\right]_{ik} = [I]_{ij}$ for $\left(i=1,2,3;\ j=1,2,3;\ k=3N+j\right)$. All other entries are ***zero***. (65)

Body J: $\left[{}^R v_{O_J,y}\right]_{3\times 6N} = \left[{}^R v_{O_1,y}\right]_{3\times 6N} - \left([\tilde{q}_J] + [\tilde{s}_J]\right)\left[{}^R\omega_{1,y}\right]_{3\times 6N}$ (66)

$\left[{}^R v_{O_J,y}\right]_{ik} = [R_1]^T_{ij} \qquad \left(i=1,2,3;\ j=1,2,3;\ k=3N+(3J-3+j)\right)$ (67)

Body K: $\left[{}^{R}v_{O_K,y}\right]_{3\times 6N} = \left[{}^{R}v_{O_J,y}\right]_{3\times 6N} - \left(\left[\tilde{q}_K\right]+\left[\tilde{s}_K\right]\right)\left[{}^{R}\omega_{J,y}\right]_{3\times 6N}$ (68)

$\left[{}^{R}v_{O_K,y}\right]_{ik} = \left[R_J\right]^T_{ij} \quad \left(i=1,2,3;\ j=1,2,3;\ k=3N+(3K-3+j)\right)$ (69)

The results in Equations (65)-(69) can be expanded to give the following.

Body 1: $\left[{}^{R}v_{O_1,\hat{\omega}_B}\right]_{3\times 3} = \left[0\right]_{3\times 3} \quad \left(B=1,\ldots,N\right)$

$\left[{}^{R}v_{O_1,\dot{s}'_1}\right]_{3\times 3} = \left[I\right]_{3\times 3}$ $\left[{}^{R}v_{O_1,\dot{s}'_B}\right]_{3\times 3} = \left[0\right]_{3\times 3} \quad \left(B=2,\ldots,N\right)$

Body J: $\left[{}^{R}v_{O_J,\hat{\omega}_1}\right]_{3\times 3} = -\left(\left[\tilde{q}_J\right]+\left[\tilde{s}_J\right]\right)$ $\left[{}^{R}v_{O_J,\hat{\omega}_B}\right]_{3\times 3} = \left[0\right]_{3\times 3} \quad \left(B\neq 1\right)$

$\left[{}^{R}v_{O_J,\dot{s}'_1}\right]_{3\times 3} = \left[I\right]_{3\times 3}$ $\left[{}^{R}v_{O_J,\dot{s}'_J}\right]_{3\times 3} = \left[R_1\right]^T_{3\times 3}$

Body K: $\left[{}^{R}v_{O_K,\hat{\omega}_1}\right]_{3\times 3} = -\left(\left[\tilde{q}_J\right]+\left[\tilde{s}_J\right]+\left[\tilde{q}_K\right]+\left[\tilde{s}_K\right]\right)$ $\left[{}^{R}v_{O_K,\hat{\omega}_J}\right]_{3\times 3} = \left[-\left(\left[\tilde{q}_K\right]+\left[\tilde{s}_K\right]\right)\left[R_1\right]^T\right]_{3\times 3}$ (70)

$\left[{}^{R}v_{O_K,\hat{\omega}_B}\right]_{3\times 3} = \left[0\right]_{3\times 3} \quad \left(B\neq 1;\ B\neq J\right)$ (71)

$\left[{}^{R}v_{O_K,\dot{s}'_1}\right]_{3\times 3} = \left[I\right]_{3\times 3}$ $\left[{}^{R}v_{O_K,\dot{s}'_J}\right]_{3\times 3} = \left[R_1\right]^T_{3\times 3}$ $\left[{}^{R}v_{O_K,\dot{s}'_K}\right]_{3\times 3} = \left[R_J\right]^T_{3\times 3}$ (72)

Partial Velocity of the Mass Center of Body K $\left(G_K\right)$

Equation (45) repeated below provides the partial velocity matrix for G_K in terms of the partial velocity matrix of O_K.

$$\left[{}^{R}v_{K,y}\right]_{3\times 6N} = \left[{}^{R}v_{O_K,y}\right]_{3\times 6N} - \left[\tilde{r}_K\right]\left[{}^{R}\omega_{K,y}\right]_{3\times 6N}$$

Using the results in Equations (70)-(72), the above equation can be expanded into the following results.

$\left[{}^{R}v_{K,\hat{\omega}_1}\right]_{3\times 3} = -\left(\left[\tilde{q}_J\right]+\left[\tilde{s}_J\right]+\left[\tilde{q}_K\right]+\left[\tilde{s}_K\right]+\left[\tilde{r}_K\right]\right)$ (73)

$\left[{}^{R}v_{K,\hat{\omega}_J}\right]_{3\times 3} = -\left(\left[\tilde{q}_K\right]+\left[\tilde{s}_K\right]+\left[\tilde{r}_K\right]\right)\left[R_1\right]^T$ $\left[{}^{R}v_{K,\hat{\omega}_K}\right]_{3\times 3} = -\left[\tilde{r}_K\right]\left[R_J\right]^T$ (74)

$\left[{}^{R}v_{O_K,\dot{s}'_1}\right]_{3\times 3} = \left[I\right]_{3\times 3}$ $\left[{}^{R}v_{O_K,\dot{s}'_J}\right]_{3\times 3} = \left[R_1\right]^T$ $\left[{}^{R}v_{O_K,\dot{s}'_K}\right]_{3\times 3} = \left[R_J\right]^T$ (75)

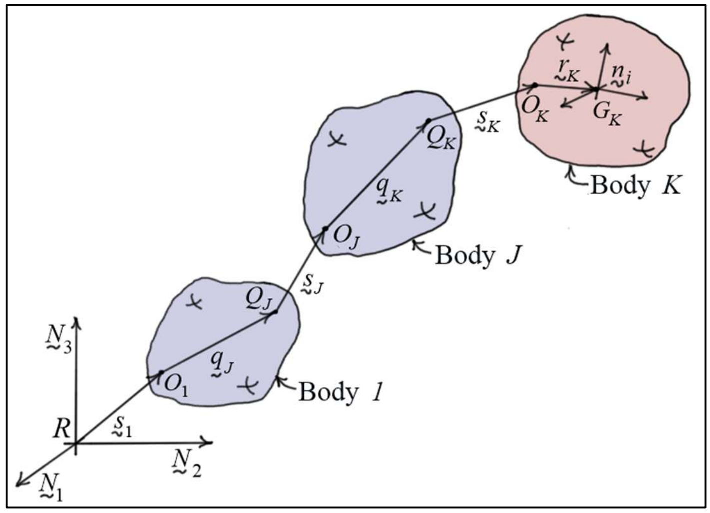
$\left[{}^{R}v_{G_K,\hat{\omega}_B}\right]_{3\times 3} = \left[0\right]_{3\times 3} \quad \left(B\neq 1;\ B\neq J;\ B\neq K\right)$ $\left[{}^{R}v_{G_K,\dot{s}'_B}\right]_{3\times 3} = \left[0\right]_{3\times 3} \quad \left(B\neq 1;\ B\neq J;\ B\neq K\right)$ (76)

These results are ***identical*** to those found in Example 4.

Example 6

The diagram again shows three bodies which are part of a larger multibody system. The lower numbered body array for the three bodies is as follows.

$$\mathfrak{L}\left(B_i = 1, J, K\right) = \left(\mathfrak{L}(1), \mathfrak{L}(J), \mathfrak{L}(K)\right) = \left(0, 1, J\right)$$



Calculate $\{v_K\}$ the ***fixed frame components*** of the velocity of G_K using the concept of ***relative velocities***. Using that result, determine $\left[{}^R v_{K,y}\right]$ the matrix of ***fixed frame*** components of the partial velocity of G_K with respect to the system matrix of generalized speeds as defined in ***Case 2*** above which utilizes the ***body frame components*** of the ***relative angular velocity vectors*** as generalized speeds.

Solution:

In example 1, ${}^R\underset{\sim}{v}_K$ the velocity of G_K in R was written as follows.

$$\begin{aligned}{}^R\underset{\sim}{v}_K &= -\left(\left(\underset{\sim}{q}_J + \underset{\sim}{s}_J\right)\times {}^R\underset{\sim}{\omega}_1\right) - \left(\left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right)\times {}^R\underset{\sim}{\omega}_J\right) - \left(\underset{\sim}{r}_K \times {}^R\underset{\sim}{\omega}_K\right) + \frac{{}^R d\underset{\sim}{s}_1}{dt} + \frac{{}^1 d\underset{\sim}{s}_J}{dt} + \frac{{}^J d\underset{\sim}{s}_K}{dt} \\ &= -\left(\left(\underset{\sim}{q}_J + \underset{\sim}{s}_J\right)\times {}^R\underset{\sim}{\omega}_1\right) - \left(\left(\underset{\sim}{q}_K + \underset{\sim}{s}_K\right)\times\left({}^R\underset{\sim}{\omega}_1 + {}^1\underset{\sim}{\omega}_J\right)\right) - \left(\underset{\sim}{r}_K \times\left({}^R\underset{\sim}{\omega}_1 + {}^1\underset{\sim}{\omega}_J + {}^J\underset{\sim}{\omega}_K\right)\right) + \frac{{}^R d\underset{\sim}{s}_1}{dt} + \frac{{}^1 d\underset{\sim}{s}_J}{dt} + \frac{{}^J d\underset{\sim}{s}_K}{dt}\end{aligned}$$

Using this result, the ***fixed frame components*** of ${}^R\underset{\sim}{v}_K$ can be written in terms of the ***body frame components*** of the ***angular velocities*** of the bodies ***relative*** to their ***lower numbered bodies*** as follows.

$$\begin{aligned}\{v_K\} &= -\left[R_1\right]^T\left(\left[\tilde{q}'_J\right]+\left[\tilde{s}'_J\right]\right)\{\hat{\omega}'_1\} - \left[R_J\right]^T\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\left(\left[{}^1R_J\right]\{\hat{\omega}'_1\}+\{\hat{\omega}'_J\}\right) \\ &\quad -\left[R_K\right]^T\left[\tilde{r}'_K\right]\left(\left[{}^1R_K\right]\{\hat{\omega}'_1\}+\left[{}^JR_K\right]\{\hat{\omega}'_J\}+\{\hat{\omega}'_K\}\right)+\{\dot{s}'_1\}+\left[R_1\right]^T\{\dot{s}'_J\}+\left[R_J\right]^T\{\dot{s}'_K\}\end{aligned}$$

Reorganizing terms gives the following result.

$$\begin{aligned}\{v_K\} &= -\left[\left[R_1\right]^T\left(\left[\tilde{q}'_J\right]+\left[\tilde{s}'_J\right]\right)+\left(\left[R_J\right]^T\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\left[{}^1R_J\right]\right)+\left(\left[R_K\right]^T\left[\tilde{r}'_K\right]\left[{}^1R_K\right]\right)\right]\{\hat{\omega}'_1\} \\ &\quad -\left[\left(\left[R_J\right]^T\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\right)+\left(\left[R_K\right]^T\left[\tilde{r}'_K\right]\left[{}^JR_K\right]\right)\right]\{\hat{\omega}'_J\} - \left[\left[R_K\right]^T\left[\tilde{r}'_K\right]\right]\{\hat{\omega}'_K\} \\ &\quad +\{\dot{s}'_1\}+\left[R_1\right]^T\{\dot{s}'_J\}+\left[R_J\right]^T\{\dot{s}'_K\}\end{aligned} \tag{77}$$

Observation of this result gives the following partial velocity matrices.

$$\left[{}^R v_{K,\hat{\omega}'_1}\right] = -\left[\left[R_1\right]^T\left(\left[\tilde{q}'_J\right]+\left[\tilde{s}'_J\right]\right)\right] - \left[\left[R_J\right]^T\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\left[{}^1R_J\right]\right] - \left[\left[R_K\right]^T\left[\tilde{r}'_K\right]\left[{}^1R_K\right]\right] \tag{78}$$

$$\left[{}^R v_{K,\hat{\omega}'_J}\right] = -\left[\left[R_J\right]^T\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\right] - \left[\left[R_K\right]^T\left[\tilde{r}'_K\right]\left[{}^J R_K\right]\right] \qquad \left[{}^R v_{K,\hat{\omega}'_K}\right] = -\left[\left[R_K\right]^T\left[\tilde{r}'_K\right]\right] \tag{79}$$

$$\left[{}^R v_{K,\dot{s}'_1}\right]_{3\times 3} = \left[I\right]_{3\times 3} \qquad \left[{}^R v_{K,\dot{s}'_J}\right]_{3\times 3} = \left[R_1\right]^T \qquad \left[{}^R v_{K,\dot{s}'_K}\right]_{3\times 3} = \left[R_J\right]^T \tag{80}$$

These results can be put in terms of the generalized speed matrix of Equations (28)-(30) to give the following.

$$\left[{}^R v_{K,y}\right]_{3\times 6N} = \left[\left[{}^R v_{K,\hat{\omega}'_1}\right] \;\cdots\; \left[{}^R v_{K,\hat{\omega}'_J}\right] \;\cdots\; \left[{}^R v_{K,\hat{\omega}'_K}\right] \;\cdots\; \left[{}^R v_{K,\dot{s}'_1}\right] \;\cdots\; \left[{}^R v_{K,\dot{s}'_J}\right] \;\cdots\; \left[{}^R v_{K,\dot{s}'_K}\right] \;\cdots\right]_{3\times 6N} \tag{81}$$

The matrices $\left[{}^R v_{K,\hat{\omega}'_B}\right]$ $\left(B=1,J,K\right)$ fill columns $3\left(B-1\right)+j$ $\left(j=1,2,3\right)$, and the matrices $\left[{}^R v_{K,\dot{s}'_B}\right]$ fill the columns $3\left(N+B-1\right)+j$ $\left(j=1,2,3\right)$. ***All other*** entries are ***zero***.

Example 7

Using the approach described in Equations (40)-(42) and Equation (48) which utilize the ***body frame components*** of the ***relative angular velocity components***, find $\left[{}^R v_{K,y}\right]$ the matrix of ***fixed frame components*** of the ***partial velocity*** of G_K the mass center of body K of the three body system of Example 6. ***Compare*** the results.

Solution:

Partial Angular Velocities of the Bodies

Body 1: $$\left[{}^R \omega'_{1,y}\right]_{3\times 6N} = \left[\left[I\right]_{3\times 3} \;\; \left[0\right]_{3\times 3} \;\cdots\; \left[0\right]_{3\times 3}\right]_{3\times 6N} \tag{82}$$

Body J: $$\left[{}^R \omega'_{J,y}\right]_{3\times 6N} = \left[\left[{}^1 R_J\right]_{3\times 3} \;\; \left[0\right]_{3\times 3} \;\cdots\; \left[0\right]_{3\times 3} \;\; \left[I\right]_{3\times 3} \;\; \left[0\right]_{3\times 3} \;\cdots\; \left[0\right]_{3\times 3}\right]_{3\times 6N} \tag{83}$$

The matrix $\left[I\right]_{3\times 3}$ fills the columns $3J-2$, $3J-1$, and $3J$.

Body K:

$$\left[{}^R \omega'_{K,y}\right]_{3\times 6N} = \left[\left[{}^1 R_K\right] \;\; \left[0\right]_{3\times 3} \;\cdots\; \left[0\right]_{3\times 3} \;\; \left[{}^J R_K\right] \;\; \left[0\right]_{3\times 3} \;\cdots\; \left[0\right]_{3\times 3} \;\; \left[I\right]_{3\times 3} \;\; \left[0\right]_{3\times 3} \;\cdots\; \left[0\right]_{3\times 3}\right]_{3\times 6N} \tag{84}$$

The matrix $\left[{}^J R_K\right]$ fills the columns $3J-2$, $3J-1$, and $3J$, and the matrix $\left[I\right]_{3\times 3}$ fills the columns $3K-2$, $3K-1$, and $3K$.

Partial Velocity of Body Origins

The process presented in Equations (40)-(42) is repeated here for convenience.

1. First, set

$$\left[{}^R v_{O_K,y}\right]_{3\times 6N} = \left[{}^R v_{O_{\mathfrak{L}(K)},y}\right]_{3\times 6N} - \left[R_{\mathfrak{L}(K)}\right]^T\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\left[{}^R \omega'_{\mathfrak{L}(K),y}\right]_{3\times 6N} \tag{85}$$

2. Then, set the ***three columns*** associated with $\dot{s}'_{Ki}$ $\left(i=1,2,3\right)$ as follows.

$$\left[{}^R v_{O_K,y}\right]_{ik} = \left[R_{\mathfrak{L}(K)}\right]^T_{ij} \qquad \left(i=1,2,3;\; j=1,2,3;\; k=3N+(3K-3+j)\right) \tag{86}$$

For body 1, only Equation (41). All entries are ***zero*** except for the three columns associated with $\{\dot{s}_1'\}$ giving the following result.

$$\boxed{\left[{}^R v_{O_K,y}\right]_{ik} = [I]_{ij}} \quad \left(i=1,2,3;\ j=1,2,3;\ k=3N+j;\ K=1\right) \tag{87}$$

Applying these results to the three bodies gives the following.

Body 1: $\boxed{\left[{}^R v_{O_1,y}\right]_{ik} = [I]_{ij}}$ for $\left(i=1,2,3;\ j=1,2,3;\ k=3N+j\right)$. All other entries are ***zero***. (88)

Body J:

$$\boxed{\left[{}^R v_{O_J,y}\right]_{3\times 6N} = \left[{}^R v_{O_1,y}\right]_{3\times 6N} - [R_1]^T\left([\tilde{q}_J'] + [\tilde{s}_J']\right)\left[{}^R\omega_{1,y}'\right]_{3\times 6N}} \tag{89}$$

$$\boxed{\left[{}^R v_{O_J,y}\right]_{ik} = [R_1]_{ij}^T} \quad \left(i=1,2,3;\ j=1,2,3;\ k=3N+(3J-3+j)\right) \tag{90}$$

Body K:

$$\boxed{\left[{}^R v_{O_K,y}\right]_{3\times 6N} = \left[{}^R v_{O_J,y}\right]_{3\times 6N} - [R_J]^T\left([\tilde{q}_K'] + [\tilde{s}_K']\right)\left[{}^R\omega_{J,y}'\right]_{3\times 6N}} \tag{91}$$

$$\boxed{\left[{}^R v_{O_K,y}\right]_{ik} = [R_J]_{ij}^T} \quad \left(i=1,2,3;\ j=1,2,3;\ k=3N+(3K-3+j)\right) \tag{92}$$

The results in Equations (88)-(92) can be expanded to give the following.

Body 1:

$$\boxed{\left[{}^R v_{O_1,\hat{\omega}_B'}\right]_{3\times 3} = [0]_{3\times 3}} \quad \left(B=1,\ldots,N\right)$$

$$\boxed{\left[{}^R v_{O_1,\dot{s}_1'}\right]_{3\times 3} = [I]_{3\times 3}} \quad \boxed{\left[{}^R v_{O_1,\dot{s}_B'}\right]_{3\times 3} = [0]_{3\times 3}} \quad \left(B=2,\ldots,N\right)$$

Body J:

$$\boxed{\left[{}^R v_{O_J,\hat{\omega}_1'}\right]_{3\times 3} = -[R_1]^T\left([\tilde{q}_J'] + [\tilde{s}_J']\right)} \quad \boxed{\left[{}^R v_{O_J,\hat{\omega}_B'}\right]_{3\times 3} = [0]_{3\times 3}} \quad \left(B\neq 1\right)$$

$$\boxed{\left[{}^R v_{O_J,\dot{s}_1'}\right]_{3\times 3} = [I]_{3\times 3}} \quad \boxed{\left[{}^R v_{O_J,\dot{s}_J'}\right]_{3\times 3} = [R_1]_{3\times 3}^T}$$

Body K:

$$\boxed{\left[{}^R v_{O_K,\hat{\omega}_1'}\right]_{3\times 3} = -[R_1]^T\left([\tilde{q}_J'] + [\tilde{s}_J']\right) - [R_J]^T\left([\tilde{q}_K'] + [\tilde{s}_K']\right)\left[{}^1 R_J\right]} \tag{93}$$

$$\boxed{\left[{}^R v_{O_K,\hat{\omega}_J'}\right]_{3\times 3} = -[R_J]^T\left([\tilde{q}_K'] + [\tilde{s}_K']\right)} \quad \boxed{\left[{}^R v_{O_K,\hat{\omega}_B'}\right]_{3\times 3} = [0]_{3\times 3}} \quad \left(B\neq 1 \text{ and } B\neq J\right) \tag{94}$$

$$\boxed{\left[{}^R v_{O_K,\dot{s}_1'}\right]_{3\times 3} = [I]_{3\times 3}} \quad \boxed{\left[{}^R v_{O_K,\dot{s}_J'}\right]_{3\times 3} = [R_1]_{3\times 3}^T} \quad \boxed{\left[{}^R v_{O_K,\dot{s}_K'}\right]_{3\times 3} = [R_J]_{3\times 3}^T} \tag{95}$$

<u>Partial Velocity of the Mass Center of Body K</u> $\left(G_K\right)$

Equation (48) repeated below provides the partial velocity matrix for G_K in terms of the partial velocity matrix of O_K.

$$\boxed{\left[{}^R v_{K,y}\right]_{3\times 6N} = \left[{}^R v_{O_K,y}\right]_{3\times 6N} - [R_K]^T[\tilde{r}_K']\left[{}^R\omega_{K,y}'\right]_{3\times 6N}}$$

Using the results in Equations (93)-(95), the above equation can be expanded into the following results.

$$\boxed{\left[{}^R v_{K,\hat{\omega}_1'}\right]_{3\times 3} = -[R_1]^T\left([\tilde{q}_J'] + [\tilde{s}_J']\right) - [R_J]^T\left([\tilde{q}_K'] + [\tilde{s}_K']\right)\left[{}^1 R_J\right] - [R_K]^T[\tilde{r}_K']\left[{}^1 R_K\right]} \tag{96}$$

$$\left[{}^R v_{K,\hat{\omega}'_J}\right]_{3\times3} = -\left[R_J\right]^T\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)-\left[R_K\right]^T\left[\tilde{r}'_K\right]\left[{}^J R_K\right] \qquad \left[{}^R v_{K,\hat{\omega}'_K}\right]_{3\times3} = -\left[R_K\right]^T\left[\tilde{r}'_K\right] \tag{97}$$

$$\left[{}^R v_{K,\dot{s}'_1}\right]_{3\times3} = \left[I\right]_{3\times3} \qquad \left[{}^R v_{K,\dot{s}'_J}\right]_{3\times3} = \left[R_1\right]^T \qquad \left[{}^R v_{K,\dot{s}'_K}\right]_{3\times3} = \left[R_J\right]^T \tag{98}$$

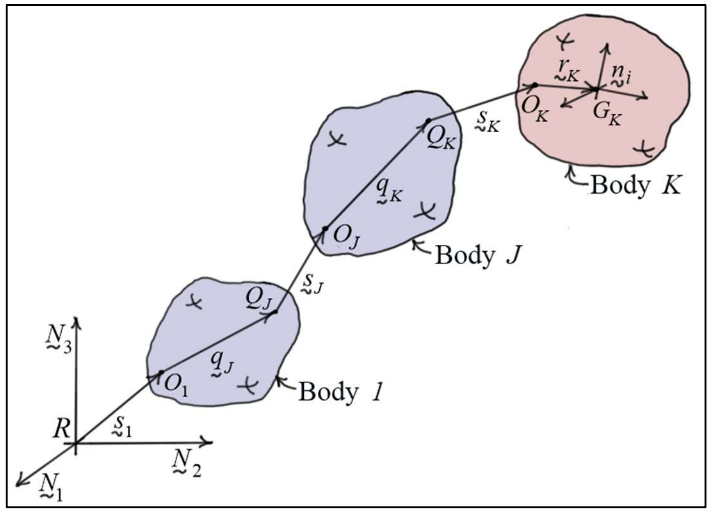
$$\left[{}^R v_{K,\hat{\omega}'_B}\right]_{3\times3} = \left[0\right]_{3\times3} \qquad \left[{}^R v_{K,\dot{s}'_B}\right]_{3\times3} = \left[0\right]_{3\times3} \qquad \left(B\neq 1 \text{ and } B\neq J \text{ and } B\neq K\right) \tag{99}$$

These results are ***identical*** to those found in Example 6.

Example 8

The diagram again shows three bodies which are part of a larger multibody system. The lower numbered body array for the three bodies is as follows.

$$\mathfrak{L}\left(B_i = 1, J, K\right) = \left(\mathfrak{L}(1), \mathfrak{L}(J), \mathfrak{L}(K)\right) = \left(0, 1, J\right)$$



Calculate $\{v'_K\}$ the ***body frame components*** of the velocity of G_K using the concept of ***relative velocities***. Using that result, determine $\left[{}^R v'_{K,y}\right]$ the matrix of ***body frame*** components of the partial velocity of G_K with respect to the system matrix of generalized speeds as defined in ***Case 2*** above which utilizes the ***body frame components*** of the ***relative angular velocity vectors*** as generalized speeds.

Solution:

In example 1, ${}^R\underset{\sim}{v}_K$ the velocity of G_K in *R* was written as follows.

$$\begin{aligned}
{}^R\underset{\sim}{v}_K &= -\left(\left(\underset{\sim}{q}_J+\underset{\sim}{s}_J\right)\times{}^R\underset{\sim}{\omega}_1\right)-\left(\left(\underset{\sim}{q}_K+\underset{\sim}{s}_K\right)\times{}^R\underset{\sim}{\omega}_J\right)-\left(\underset{\sim}{r}_K\times{}^R\underset{\sim}{\omega}_K\right)+\frac{{}^R d\underset{\sim}{s}_1}{dt}+\frac{{}^1 d\underset{\sim}{s}_J}{dt}+\frac{{}^J d\underset{\sim}{s}_K}{dt}\\
&= -\left(\left(\underset{\sim}{q}_J+\underset{\sim}{s}_J\right)\times{}^R\underset{\sim}{\omega}_1\right)-\left(\left(\underset{\sim}{q}_K+\underset{\sim}{s}_K\right)\times\left({}^R\underset{\sim}{\omega}_1+{}^1\underset{\sim}{\omega}_J\right)\right)-\left(\underset{\sim}{r}_K\times\left({}^R\underset{\sim}{\omega}_1+{}^1\underset{\sim}{\omega}_J+{}^J\underset{\sim}{\omega}_K\right)\right)+\frac{{}^R d\underset{\sim}{s}_1}{dt}+\frac{{}^1 d\underset{\sim}{s}_J}{dt}+\frac{{}^J d\underset{\sim}{s}_K}{dt}
\end{aligned}$$

Using this result, the ***body frame components*** of ${}^R\underset{\sim}{v}_K$ can be written in terms of the ***body frame components*** of the ***angular velocities*** of the bodies ***relative*** to their ***lower numbered bodies*** as follows.

$$\begin{aligned}
\{v'_K\} &= -\left[{}^1R_K\right]\left(\left[\tilde{q}'_J\right]+\left[\tilde{s}'_J\right]\right)\{\hat{\omega}'_1\}-\left[{}^J R_K\right]\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\left(\left[{}^1R_J\right]\{\hat{\omega}'_1\}+\{\hat{\omega}'_J\}\right)\\
&\quad -\left[\tilde{r}'_K\right]\left(\left[{}^1R_K\right]\{\hat{\omega}'_1\}+\left[{}^J R_K\right]\{\hat{\omega}'_J\}+\{\hat{\omega}'_K\}\right)+\left[R_K\right]\{\dot{s}'_1\}+\left[{}^1R_K\right]\{\dot{s}'_J\}+\left[{}^J R_K\right]\{\dot{s}'_K\}
\end{aligned}$$

Reorganizing terms gives the following.

$$\begin{aligned}\{v'_K\} = &-\left[\left[{}^1R_K\right]\left([\tilde{q}'_J]+[\tilde{s}'_J]\right)+\left[{}^JR_K\right]\left([\tilde{q}'_K]+[\tilde{s}'_K]\right)\left[{}^1R_J\right]+[\tilde{r}'_K]\left[{}^1R_K\right]\right]\{\hat{\omega}'_1\}\\ &-\left[\left[{}^JR_K\right]\left([\tilde{q}'_K]+[\tilde{s}'_K]\right)+[\tilde{r}'_K]\left[{}^JR_K\right]\right]\{\hat{\omega}'_J\}-[\tilde{r}'_K]\{\hat{\omega}'_K\}\\ &+[R_K]\{\dot{s}'_1\}+\left[{}^1R_K\right]\{\dot{s}'_J\}+\left[{}^JR_K\right]\{\dot{s}'_K\}\end{aligned} \tag{100}$$

Observation of this result gives the following partial velocity matrices.

$$\left[{}^Rv'_{K,\hat{\omega}'_1}\right] = -\left[\left[{}^1R_K\right]\left([\tilde{q}'_J]+[\tilde{s}'_J]\right)+\left[{}^JR_K\right]\left([\tilde{q}'_K]+[\tilde{s}'_K]\right)\left[{}^1R_J\right]+[\tilde{r}'_K]\left[{}^1R_K\right]\right] \tag{101}$$

$$\left[{}^Rv'_{K,\hat{\omega}'_J}\right] = -\left[\left[{}^JR_K\right]\left([\tilde{q}'_K]+[\tilde{s}'_K]\right)+[\tilde{r}'_K]\left[{}^JR_K\right]\right] \qquad \left[{}^Rv'_{K,\hat{\omega}'_K}\right] = -[\tilde{r}'_K] \tag{102}$$

$$\left[{}^Rv'_{K,\dot{s}'_1}\right]_{3\times3} = [R_K] \qquad \left[{}^Rv'_{K,\dot{s}'_J}\right]_{3\times3} = \left[{}^1R_K\right] \qquad \left[{}^Rv'_{K,\dot{s}'_K}\right]_{3\times3} = \left[{}^JR_K\right] \tag{103}$$

These results can be put in terms of the generalized speed matrix of Equations (28)-(30) to give the following.

$$\begin{aligned}\left[{}^Rv'_{K,y}\right]_{3\times6N} = \Big[&\left[{}^Rv'_{K,\hat{\omega}'_1}\right] \ \cdots \ \left[{}^Rv'_{K,\hat{\omega}'_J}\right] \ \cdots \ \left[{}^Rv'_{K,\hat{\omega}'_K}\right] \ \cdots\\ &\left[{}^Rv'_{K,\dot{s}'_1}\right] \ \cdots \ \left[{}^Rv'_{K,\dot{s}'_J}\right] \ \cdots \ \left[{}^Rv'_{K,\dot{s}'_K}\right] \ \cdots\Big]_{3\times6N}\end{aligned} \tag{104}$$

The matrices $\left[{}^Rv'_{K,\hat{\omega}'_B}\right]$ $(B=1,J,K)$ fill columns $3(B-1)+j \quad (j=1,2,3)$, and the matrices $\left[{}^Rv'_{K,\dot{s}'_B}\right]$ fill the columns $3(N+B-1)+j \quad (j=1,2,3)$. ***All other*** entries are ***zero***.

Example 9

Using the approach described in Equations (51)-(53) and Equation (55) which utilize the ***body frame components*** of the ***relative angular velocity components***, find $\left[{}^Rv'_{K,y}\right]$ the matrix of ***body frame components*** of the ***partial velocity*** of G_K the mass center of body *K* of the three body system of Example 8. ***Compare*** the results.

Solution:

Partial Angular Velocities of the Bodies

Body 1: $$\left[{}^R\omega'_{1,y}\right]_{3\times6N} = \left[[I]_{3\times3} \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3}\right]_{3\times6N} \tag{105}$$

Body *J*: $$\left[{}^R\omega'_{J,y}\right]_{3\times6N} = \left[\left[{}^1R_J\right]_{3\times3} \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3} \quad [I]_{3\times3} \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3}\right]_{3\times6N} \tag{106}$$

The matrix $[I]_{3\times3}$ fills the columns $3J-2$, $3J-1$, and $3J$.

Body *K*:

$$\left[{}^R\omega'_{K,y}\right]_{3\times6N} = \left[\left[{}^1R_K\right] \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3} \quad \left[{}^JR_K\right] \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3} \quad [I]_{3\times3} \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3}\right]_{3\times6N} \tag{107}$$

The matrix $\left[{}^JR_K\right]$ fills the columns $3J-2$, $3J-1$, and $3J$, and the matrix $[I]_{3\times3}$ fills the columns $3K-2$, $3K-1$, and $3K$.

Partial Velocity of Body Origins

The process presented in Equations (51)-(53) is repeated here for convenience.

1. First, set

$$\boxed{\left[{}^{R}v'_{O_K,y}\right]_{3\times 6N}=\left[{}^{\mathfrak{L}(K)}R_K\right]\left(\left[{}^{R}v'_{O_{\mathfrak{L}(K)},y}\right]_{3\times 6N}-\left([\tilde{q}'_K]+[\tilde{s}'_K]\right)\left[{}^{R}\omega'_{\mathfrak{L}(K),y}\right]_{3\times 6N}\right)} \quad (108)$$

2. Then, set the ***three columns*** associated with $\dot{s}'_{Ki}\ (i=1,2,3)$ as follows.

$$\boxed{\left[{}^{R}v'_{O_K,y}\right]_{ik}=\left[{}^{\mathfrak{L}(K)}R_K\right]_{ij}} \quad \left(i=1,2,3;\ j=1,2,3;\ k=3N+(3K-3+j)\right) \quad (109)$$

For body 1, only Equation (109) applies. All entries are ***zero*** except for the three columns associated with $\{\dot{s}'_1\}$ giving the following result.

$$\boxed{\left[{}^{R}v'_{O_K,y}\right]_{ik}=\left[R_1\right]_{ij}} \quad \left(i=1,2,3;\ j=1,2,3;\ k=3N+j;\ K=1\right) \quad (110)$$

Applying these results to the three bodies gives the following.

Body 1: $\boxed{\left[{}^{R}v'_{O_1,y}\right]_{ik}=\left[R_1\right]_{ij}}$ for $\left(i=1,2,3;\ j=1,2,3;\ k=3N+j\right)$. All other entries are ***zero***. (111)

Body J:

$$\boxed{\left[{}^{R}v'_{O_J,y}\right]_{3\times 6N}=\left[{}^{1}R_J\right]\left(\left[{}^{R}v'_{O_1,y}\right]_{3\times 6N}-\left([\tilde{q}'_J]+[\tilde{s}'_J]\right)\left[{}^{R}\omega'_{1,y}\right]_{3\times 6N}\right)} \quad (112)$$

$$\boxed{\left[{}^{R}v'_{O_J,y}\right]_{ik}=\left[{}^{1}R_J\right]_{ij}} \quad \left(i=1,2,3;\ j=1,2,3;\ k=3N+(3J-3+j)\right) \quad (113)$$

Body K:

$$\boxed{\left[{}^{R}v'_{O_K,y}\right]_{3\times 6N}=\left[{}^{J}R_K\right]\left(\left[{}^{R}v'_{O_J,y}\right]_{3\times 6N}-\left([\tilde{q}'_K]+[\tilde{s}'_K]\right)\left[{}^{R}\omega'_{J,y}\right]_{3\times 6N}\right)} \quad (114)$$

$$\boxed{\left[{}^{R}v'_{O_K,y}\right]_{ik}=\left[{}^{J}R_K\right]_{ij}} \quad \left(i=1,2,3;\ j=1,2,3;\ k=3N+(3K-3+j)\right) \quad (115)$$

The results in Equations (111)-(115) can be ***expanded*** to give the following.

Body 1:

$$\boxed{\left[{}^{R}v'_{O_1,\hat{\omega}'_B}\right]_{3\times 3}=[0]_{3\times 3}} \quad \left(B=1,\ldots,N\right)$$

$$\boxed{\left[{}^{R}v'_{O_1,\dot{s}'_1}\right]_{3\times 3}=[R_1]_{3\times 3}} \quad \boxed{\left[{}^{R}v'_{O_1,\dot{s}'_B}\right]_{3\times 3}=[0]_{3\times 3}} \quad \left(B=2,\ldots,N\right)$$

Body J:

$$\boxed{\left[{}^{R}v'_{O_J,\hat{\omega}'_1}\right]_{3\times 3}=-\left[{}^{1}R_J\right]\left([\tilde{q}'_J]+[\tilde{s}'_J]\right)} \quad \boxed{\left[{}^{R}v'_{O_J,\hat{\omega}'_B}\right]_{3\times 3}=[0]_{3\times 3}} \quad \left(B\neq 1\right)$$

$$\boxed{\left[{}^{R}v'_{O_J,\dot{s}'_1}\right]_{3\times 3}=\left[{}^{1}R_J\right][R_1]=[R_J]} \quad \boxed{\left[{}^{R}v'_{O_J,\dot{s}'_J}\right]_{3\times 3}=\left[{}^{1}R_J\right]}$$

Body K:

$$\boxed{\left[{}^{R}v'_{O_K,\hat{\omega}'_1}\right]_{3\times 3}=-\left[{}^{1}R_K\right]\left([\tilde{q}'_J]+[\tilde{s}'_J]\right)-\left[{}^{J}R_K\right]\left([\tilde{q}'_K]+[\tilde{s}'_K]\right)\left[{}^{1}R_J\right]} \quad (116)$$

$$\boxed{\left[{}^{R}v'_{O_K,\hat{\omega}'_J}\right]_{3\times 3}=-\left[{}^{J}R_K\right]\left([\tilde{q}'_K]+[\tilde{s}'_K]\right)} \quad \boxed{\left[{}^{R}v_{O_K,\hat{\omega}'_B}\right]_{3\times 3}=[0]_{3\times 3}} \quad \left(B\neq 1 \text{ and } B\neq J\right) \quad (117)$$

$$\boxed{\left[{}^{R}v'_{O_K,\dot{s}'_1}\right]_{3\times 3}=[R_K]} \quad \boxed{\left[{}^{R}v_{O_K,\dot{s}'_J}\right]_{3\times 3}=\left[{}^{1}R_K\right]} \quad \boxed{\left[{}^{R}v_{O_K,\dot{s}'_K}\right]_{3\times 3}=\left[{}^{J}R_K\right]} \quad (118)$$

Partial Velocity of the Mass Center of Body K (G_K)

Equation (55) (repeated below) provides the partial velocity matrix for G_K in terms of the partial velocity matrix of O_K.

$$\boxed{\left[{}^R v'_{K,y}\right]_{3\times 6N} = \left[{}^R v'_{O_K,y}\right]_{3\times 6N} - \left[\tilde{r}'_K\right]\left[{}^R \omega'_{K,y}\right]_{3\times 6N}}$$

Using the results in Equations (93)-(95), the above equation can be expanded into the following results.

$$\boxed{\left[{}^R v'_{K,\hat{\omega}'_1}\right]_{3\times 3} = -\left[\left[{}^1 R_K\right]\left(\left[\tilde{q}'_J\right]+\left[\tilde{s}'_J\right]\right)+\left[{}^J R_K\right]\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)\left[{}^1 R_J\right]+\left[\tilde{r}'_K\right]\left[{}^1 R_K\right]\right]} \quad (119)$$

$$\boxed{\left[{}^R v'_{K,\hat{\omega}'_J}\right]_{3\times 3} = -\left[\left[{}^J R_K\right]\left(\left[\tilde{q}'_K\right]+\left[\tilde{s}'_K\right]\right)+\left[\tilde{r}'_K\right]\left[{}^J R_K\right]\right]} \quad \boxed{\left[{}^R v'_{K,\hat{\omega}'_K}\right]_{3\times 3} = -\left[\tilde{r}'_K\right]} \quad (120)$$

$$\boxed{\left[{}^R v'_{K,\dot{s}'_1}\right]_{3\times 3} = \left[R_K\right]} \quad \boxed{\left[{}^R v_{K,\dot{s}'_J}\right]_{3\times 3} = \left[{}^1 R_K\right]} \quad \boxed{\left[{}^R v_{K,\dot{s}'_K}\right]_{3\times 3} = \left[{}^J R_K\right]} \quad (121)$$

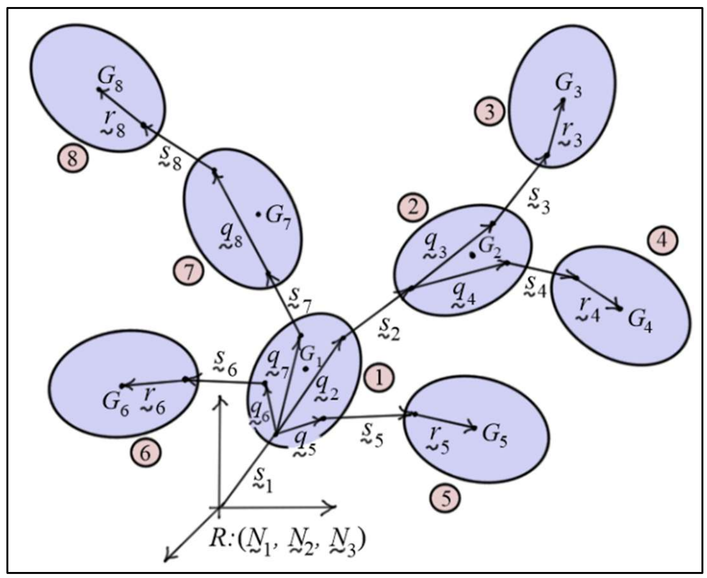
$$\boxed{\left[{}^R v_{O_K,\hat{\omega}'_B}\right]_{3\times 3} = \left[0\right]_{3\times 3}} \quad \boxed{\left[{}^R v_{O_K,\dot{s}'_B}\right]_{3\times 3} = \left[0\right]_{3\times 3}} \quad \left(B \neq 1 \text{ and } B \neq J \text{ and } B \neq K\right) \quad (122)$$

These results are ***identical*** to those found in Example 8.

Example 10

The figure shows an eight-body system numbered using the numbering scheme presented in Unit 1. Body 1 is the system ***reference body***, and the rest of the bodies are numbered in ***ascending progression*** outward along the branches. As structured, the lower numbered body array for the system is as follows.

$$\mathfrak{L}(1,\ldots,8) = (0,1,2,2,1,1,1,7)$$



The orientation of body 1 is defined ***relative*** to the fixed frame $R:\left(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3\right)$, and the orientations of all the other bodies are defined ***relative*** to their adjacent, lower numbered bodies. Using ***base frame components*** of the ***relative angular velocities*** of the bodies, the ***system generalized speed matrix*** is defined as follows.

$$\boxed{\left\{y\right\}_{48\times 1} \triangleq \begin{Bmatrix} \left\{y_1\right\}_{24\times 1} \\ \left\{y_2\right\}_{24\times 1} \end{Bmatrix}} \quad (123)$$

$$\boxed{\left\{y_1\right\}_{24\times 1} \triangleq \left[\hat{\omega}_{11} \quad \hat{\omega}_{12} \quad \hat{\omega}_{13} \quad \hat{\omega}_{21} \quad \hat{\omega}_{22} \quad \hat{\omega}_{23} \quad \cdots \quad \hat{\omega}_{71} \quad \hat{\omega}_{72} \quad \hat{\omega}_{73} \quad \hat{\omega}_{81} \quad \hat{\omega}_{82} \quad \hat{\omega}_{83}\right]^T} \quad (124)$$

$$\boxed{\left\{y_2\right\}_{24\times 1} \triangleq \left[\dot{s}'_{11} \quad \dot{s}'_{12} \quad \dot{s}'_{13} \quad \dot{s}'_{21} \quad \dot{s}'_{22} \quad \dot{s}'_{23} \quad \cdots \quad \dot{s}'_{71} \quad \dot{s}'_{72} \quad \dot{s}'_{73} \quad \dot{s}'_{81} \quad \dot{s}'_{82} \quad \dot{s}'_{83}\right]^T} \quad (125)$$

a) For each body B in the system, find the ***fixed frame components*** of the ***partial velocity matrix*** of its ***origin*** O_B and its ***mass center*** G_B. Follow the procedures outlined in Equations (34)-(36) and Equation (45). Use the results presented in Unit 2 for the ***fixed frame components*** of the ***partial angular velocity matrices*** of the bodies.
b) Use ***direct differentiation*** to find the ***fixed frame components*** of the ***velocity*** of G_8 in terms of the ***base frame components*** of the ***relative angular velocity*** vectors, ***identify*** the partial velocity matrix $\left[{}^R v_{8,y}\right]_{3\times48}$, and compare the results with those obtained in part (a).

Results from Unit 2:

$$\left[{}^R\omega_{1,y_1}\right]_{3\times24} = \left[[I]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\right]$$

$$\left[{}^R\omega_{2,y_1}\right]_{3\times24} = \left[[I]_{3\times3}\ \ [R_1]^T_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\right]$$

$$\left[{}^R\omega_{3,y_1}\right]_{3\times24} = \left[[I]_{3\times3}\ \ [R_1]^T_{3\times3}\ \ [R_2]^T_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\right]$$

$$\left[{}^R\omega_{4,y_1}\right]_{3\times24} = \left[[I]_{3\times3}\ \ [R_1]^T_{3\times3}\ \ [0]_{3\times3}\ \ [R_2]^T_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\right]$$

$$\left[{}^R\omega_{5,y_1}\right]_{3\times24} = \left[[I]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [R_1]^T_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\right]$$

$$\left[{}^R\omega_{6,y_1}\right]_{3\times24} = \left[[I]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [R_1]^T_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\right]$$

$$\left[{}^R\omega_{7,y_1}\right]_{3\times24} = \left[[I]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [R_1]^T_{3\times3}\ \ [0]_{3\times3}\right]$$

$$\left[{}^R\omega_{8,y_1}\right]_{3\times24} = \left[[I]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [0]_{3\times3}\ \ [R_1]^T_{3\times3}\ \ [R_7]^T_{3\times3}\right]$$

$$\left[{}^R\omega_{B,y_2}\right]_{3\times24} = [0]_{3\times24} \qquad (B=1,\ldots,8)$$

For each body, the system partial angular velocity matrix is partitioned as follows.

$$\left[{}^R\omega_{B,y}\right]_{3\times48} = \left[\left[{}^R\omega_{B,y_1}\right]_{3\times24}\ \ \left[{}^R\omega_{B,y_2}\right]_{3\times24}\right] = \left[\left[{}^R\omega_{B,y_1}\right]_{3\times24}\ \ [0]_{3\times24}\right] \qquad (B=1,\ldots,8)$$

Solution:

a) Body Origins:

The process presented in Equations (34)-(36) is repeated here for convenience.

1. First, set

$$\left[{}^R v_{O_K,y}\right]_{3\times6N} = \left[{}^R v_{O_{\mathfrak{L}(K)},y}\right]_{3\times6N} - \left([\tilde{q}_K]+[\tilde{s}_K]\right)\left[{}^R\omega_{\mathfrak{L}(K),y}\right]_{3\times6N}$$

2. Then, set the ***three columns*** associated with $\dot{s}'_{Ki}$ $(i=1,2,3)$ as follows.

$$\left[{}^R v_{O_K,y}\right]_{ik} = \left[R_{\mathfrak{L}(K)}\right]^T_{ij} \qquad \left(i=1,2,3;\ j=1,2,3;\ k=3N+(3K-3+j)\right)$$

For body 1, only the equation in item (2) applies. All entries are ***zero*** except for the three columns associated with $\{\dot{s}_1'\}$ giving the following result.

$$\boxed{\left[{}^R v_{O_K,y}\right]_{ik} = [I]_{ij}} \quad \left(i=1,2,3;\ j=1,2,3;\ k=3N+j;\ K=1\right)$$

Body 1:

$$\boxed{\left[{}^R v_{O_1,y_1}\right]_{3\times 24} = [0]_{3\times 24}}$$

$$\boxed{\left[{}^R v_{O_1,y_2}\right]_{3\times 24} = \left[[I]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\right]}$$

Body 2:

$$\left[{}^R v_{O_2,y}\right]_{3\times 48} = \left[{}^R v_{O_1,y}\right]_{3\times 48} - \left([\tilde{q}_2]+[\tilde{s}_2]\right)\left[{}^R \omega_{1,y}\right]_{3\times 48}$$

$$\left[{}^R v_{O_2,y}\right]_{ik} = [R_1]^T_{ij} \quad \left(i=1,2,3;\ j=1,2,3;\ k=27+j\right)$$

$$\boxed{\left[{}^R v_{O_2,y_1}\right]_{3\times 24} = \left[\left[{}^R v_{O_2,\hat{\omega}_1}\right]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\right]}$$

$$\boxed{\left[{}^R v_{O_2,\hat{\omega}_1}\right]_{3\times 3} = -\left[[\tilde{q}_2]+[\tilde{s}_2]\right]}$$

$$\boxed{\left[{}^R v_{O_2,y_2}\right]_{3\times 24} = \left[[I]_{3\times 3}\quad [R_1]^T_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\right]}$$

Body 3:

$$\left[{}^R v_{O_3,y}\right]_{3\times 48} = \left[{}^R v_{O_2,y}\right]_{3\times 48} - \left([\tilde{q}_3]+[\tilde{s}_3]\right)\left[{}^R \omega_{2,y}\right]_{3\times 48}$$

$$\left[{}^R v_{O_3,y}\right]_{ik} = [R_2]^T_{ij} \quad \left(i=1,2,3;\ j=1,2,3;\ k=30+j\right)$$

$$\boxed{\left[{}^R v_{O_3,y_1}\right]_{3\times 24} = \left[\left[{}^R v_{O_3,\hat{\omega}_1}\right]_{3\times 3}\quad \left[{}^R v_{O_3,\hat{\omega}_2}\right]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\right]}$$

$$\boxed{\left[{}^R v_{O_3,\hat{\omega}_1}\right]_{3\times 3} = -\left[[\tilde{q}_2]+[\tilde{s}_2]+[\tilde{q}_3]+[\tilde{s}_3]\right]} \qquad \boxed{\left[{}^R v_{O_3,\hat{\omega}_2}\right]_{3\times 3} = -\left[[\tilde{q}_3]+[\tilde{s}_3]\right][R_1]^T}$$

$$\boxed{\left[{}^R v_{O_3,y_2}\right]_{3\times 24} = \left[[I]_{3\times 3}\quad [R_1]^T_{3\times 3}\quad [R_2]^T_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\right]}$$

Body 4:

$$\left[{}^R v_{O_4,y}\right]_{3\times 48} = \left[{}^R v_{O_2,y}\right]_{3\times 48} - \left([\tilde{q}_4]+[\tilde{s}_4]\right)\left[{}^R \omega_{2,y}\right]_{3\times 48}$$

$$\left[{}^R v_{O_4,y}\right]_{ik} = [R_2]^T_{ij} \quad \left(i=1,2,3;\ j=1,2,3;\ k=33+j\right)$$

$$\boxed{\left[{}^R v_{O_4,y_1}\right]_{3\times 24} = \left[\left[{}^R v_{O_4,\hat{\omega}_1}\right]_{3\times 3}\quad \left[{}^R v_{O_4,\hat{\omega}_2}\right]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\right]}$$

$$\boxed{\left[{}^R v_{O_4,\hat{\omega}_1}\right]_{3\times 3} = -\left[[\tilde{q}_2]+[\tilde{s}_2]+[\tilde{q}_4]+[\tilde{s}_4]\right]} \qquad \boxed{\left[{}^R v_{O_4,\hat{\omega}_2}\right]_{3\times 3} = -\left[[\tilde{q}_4]+[\tilde{s}_4]\right][R_1]^T}$$

$$\boxed{\left[{}^R v_{O_4,y_2}\right]_{3\times 24} = \left[[I]_{3\times 3}\quad [R_1]^T_{3\times 3}\quad [0]_{3\times 3}\quad [R_2]^T_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\quad [0]_{3\times 3}\right]}$$

Body 5:

$$\left[{}^{R}v_{O_5,y}\right]_{3\times 48} = \left[{}^{R}v_{O_1,y}\right]_{3\times 48} - \left(\left[\tilde{q}_5\right]+\left[\tilde{s}_5\right]\right)\left[{}^{R}\omega_{1,y}\right]_{3\times 48}$$

$$\left[{}^{R}v_{O_5,y}\right]_{ik} = \left[R_1\right]^T_{ij} \quad \left(i=1,2,3;\ j=1,2,3;\ k=36+j\right)$$

$$\boxed{\left[{}^{R}v_{O_5,y_1}\right]_{3\times 24} = \left[\left[{}^{R}v_{O_5,\hat{\omega}_1}\right]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3}\right]}$$

$$\boxed{\left[{}^{R}v_{O_5,\hat{\omega}_1}\right]_{3\times 3} = -\left[\left[\tilde{q}_5\right]+\left[\tilde{s}_5\right]\right]}$$

$$\boxed{\left[{}^{R}v_{O_5,y_2}\right]_{3\times 24} = \left[[I]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [R_1]^T_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3}\right]}$$

Body 6:

$$\left[{}^{R}v_{O_6,y}\right]_{3\times 48} = \left[{}^{R}v_{O_1,y}\right]_{3\times 48} - \left(\left[\tilde{q}_6\right]+\left[\tilde{s}_6\right]\right)\left[{}^{R}\omega_{1,y}\right]_{3\times 48}$$

$$\left[{}^{R}v_{O_6,y}\right]_{ik} = \left[R_1\right]^T_{ij} \quad \left(i=1,2,3;\ j=1,2,3;\ k=39+j\right)$$

$$\boxed{\left[{}^{R}v_{O_6,y_1}\right]_{3\times 24} = \left[\left[{}^{R}v_{O_6,\hat{\omega}_1}\right]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3}\right]}$$

$$\boxed{\left[{}^{R}v_{O_6,\hat{\omega}_1}\right]_{3\times 3} = -\left[\left[\tilde{q}_6\right]+\left[\tilde{s}_6\right]\right]}$$

$$\boxed{\left[{}^{R}v_{O_6,y_2}\right]_{3\times 24} = \left[[I]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [R_1]^T_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3}\right]}$$

Body 7:

$$\left[{}^{R}v_{O_7,y}\right]_{3\times 48} = \left[{}^{R}v_{O_1,y}\right]_{3\times 48} - \left(\left[\tilde{q}_7\right]+\left[\tilde{s}_7\right]\right)\left[{}^{R}\omega_{1,y}\right]_{3\times 48}$$

$$\left[{}^{R}v_{O_7,y}\right]_{ik} = \left[R_1\right]^T_{ij} \quad \left(i=1,2,3;\ j=1,2,3;\ k=42+j\right)$$

$$\boxed{\left[{}^{R}v_{O_7,y_1}\right]_{3\times 24} = \left[\left[{}^{R}v_{O_7,\hat{\omega}_1}\right]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3}\right]}$$

$$\boxed{\left[{}^{R}v_{O_7,\hat{\omega}_1}\right]_{3\times 3} = -\left[\left[\tilde{q}_7\right]+\left[\tilde{s}_7\right]\right]}$$

$$\boxed{\left[{}^{R}v_{O_7,y_2}\right]_{3\times 24} = \left[[I]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [R_1]^T_{3\times 3} \quad [0]_{3\times 3}\right]}$$

Body 8:

$$\left[{}^{R}v_{O_8,y}\right]_{3\times 48} = \left[{}^{R}v_{O_7,y}\right]_{3\times 48} - \left(\left[\tilde{q}_8\right]+\left[\tilde{s}_8\right]\right)\left[{}^{R}\omega_{7,y}\right]_{3\times 48}$$

$$\left[{}^{R}v_{O_8,y}\right]_{ik} = \left[R_7\right]^T_{ij} \quad \left(i=1,2,3;\ j=1,2,3;\ k=45+j\right)$$

$$\boxed{\left[{}^{R}v_{O_8,y_1}\right]_{3\times 24} = \left[\left[{}^{R}v_{O_8,\hat{\omega}_1}\right]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad \left[{}^{R}v_{O_8,\hat{\omega}_7}\right]_{3\times 3} \quad [0]_{3\times 3}\right]}$$

$$\boxed{\left[{}^{R}v_{O_8,\hat{\omega}_1}\right]_{3\times 3} = -\left[\left[\tilde{q}_7\right]+\left[\tilde{s}_7\right]+\left[\tilde{q}_8\right]+\left[\tilde{s}_8\right]\right]} \qquad \boxed{\left[{}^{R}v_{O_8,\hat{\omega}_7}\right]_{3\times 3} = -\left(\left[\tilde{q}_8\right]+\left[\tilde{s}_8\right]\right)\left[R_1\right]^T}$$

$$\boxed{\left[{}^{R}v_{O_8,y_2}\right]_{3\times 24} = \left[[I]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [R_1]^T_{3\times 3} \quad [R_7]^T_{3\times 3}\right]}$$

Body Mass Centers:

Equation (45) repeated below provides the partial velocity matrices for the mass centers of the bodies.

$$\boxed{\left[{}^{R}v_{K,y}\right]_{3\times 6N} = \left[{}^{R}v_{O_K,y}\right]_{3\times 6N} - \left[\tilde{r}_K\right]\left[{}^{R}\omega_{K,y}\right]_{3\times 6N}}$$

Body 1:

$$\left[{}^{R}v_{1,y}\right]_{3\times 48} = \left[{}^{R}v_{O_1,y}\right]_{3\times 48} - \left[\tilde{r}_1\right]\left[{}^{R}\omega_{1,y}\right]_{3\times 48}$$

$$\boxed{\left[{}^{R}v_{1,y_1}\right]_{3\times 24} = \left[\left[{}^{R}v_{1,\hat{\omega}_1'}\right]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3}\right]}$$

$$\boxed{\left[{}^{R}v_{1,\hat{\omega}_1'}\right]_{3\times 3} = -\left[\tilde{r}_1\right]}$$

$$\boxed{\left[{}^{R}v_{1,y_2}\right]_{3\times 24} = \left[[I]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3}\right]}$$

Body 2:

$$\left[{}^{R}v_{2,y}\right]_{3\times 48} = \left[{}^{R}v_{O_2,y}\right]_{3\times 48} - \left[\tilde{r}_2\right]\left[{}^{R}\omega_{2,y}\right]_{3\times 48}$$

$$\boxed{\left[{}^{R}v_{2,y_1}\right]_{3\times 24} = \left[\left[{}^{R}v_{2,\hat{\omega}_1}\right]_{3\times 3} \quad \left[{}^{R}v_{2,\hat{\omega}_2}\right]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3}\right]}$$

$$\boxed{\left[{}^{R}v_{2,\hat{\omega}_1}\right]_{3\times 3} = -\left[\left[\tilde{q}_2\right] + \left[\tilde{s}_2\right] + \left[\tilde{r}_2\right]\right]} \quad \boxed{\left[{}^{R}v_{2,\hat{\omega}_2}\right]_{3\times 3} = -\left[\tilde{r}_2\right]\left[R_1\right]^T}$$

$$\boxed{\left[{}^{R}v_{2,y_2}\right]_{3\times 24} = \left[[I]_{3\times 3} \quad [R_1]^T_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3}\right]}$$

Body 3:

$$\left[{}^{R}v_{3,y}\right]_{3\times 48} = \left[{}^{R}v_{O_3,y}\right]_{3\times 48} - \left[\tilde{r}_3\right]\left[{}^{R}\omega_{3,y}\right]_{3\times 48}$$

$$\boxed{\left[{}^{R}v_{3,y_1}\right]_{3\times 24} = \left[\left[{}^{R}v_{3,\hat{\omega}_1}\right]_{3\times 3} \quad \left[{}^{R}v_{3,\hat{\omega}_2}\right]_{3\times 3} \quad \left[{}^{R}v_{3,\hat{\omega}_3}\right]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3}\right]}$$

$$\boxed{\left[{}^{R}v_{3,\hat{\omega}_1}\right]_{3\times 3} = -\left[\left[\tilde{q}_2\right] + \left[\tilde{s}_2\right] + \left[\tilde{q}_3\right] + \left[\tilde{s}_3\right] + \left[\tilde{r}_3\right]\right]} \quad \boxed{\left[{}^{R}v_{3,\hat{\omega}_2}\right]_{3\times 3} = -\left[\left[\tilde{q}_3\right] + \left[\tilde{s}_3\right] + \left[\tilde{r}_3\right]\right]\left[R_1\right]^T}$$

$$\boxed{\left[{}^{R}v_{3,\hat{\omega}_3}\right]_{3\times 3} = -\left[\tilde{r}_3\right]\left[R_2\right]^T}$$

$$\boxed{\left[{}^{R}v_{3,y_2}\right]_{3\times 24} = \left[[I]_{3\times 3} \quad [R_1]^T_{3\times 3} \quad [R_2]^T_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3}\right]}$$

Body 4:

$$\left[{}^{R}v_{4,y}\right]_{3\times 48} = \left[{}^{R}v_{O_4,y}\right]_{3\times 48} - \left[\tilde{r}_4\right]\left[{}^{R}\omega_{4,y}\right]_{3\times 48}$$

$$\boxed{\left[{}^{R}v_{4,y_1}\right]_{3\times 24} = \left[\left[{}^{R}v_{4,\hat{\omega}_1}\right]_{3\times 3} \quad \left[{}^{R}v_{4,\hat{\omega}_2}\right]_{3\times 3} \quad [0]_{3\times 3} \quad \left[{}^{R}v_{4,\hat{\omega}_4}\right]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3}\right]}$$

$$\boxed{\left[{}^{R}v_{4,\hat{\omega}_1}\right]_{3\times 3} = -\left[\left[\tilde{q}_2\right] + \left[\tilde{s}_2\right] + \left[\tilde{q}_4\right] + \left[\tilde{s}_4\right] + \left[\tilde{r}_4\right]\right]} \quad \boxed{\left[{}^{R}v_{4,\hat{\omega}_2}\right]_{3\times 3} = -\left[\left[\tilde{q}_4\right] + \left[\tilde{s}_4\right] + \left[\tilde{r}_4\right]\right]\left[R_1\right]^T}$$

$$\boxed{\left[{}^{R}v_{4,\hat{\omega}_3}\right]_{3\times 3} = -\left[\tilde{r}_4\right]\left[R_2\right]^T}$$

$$\boxed{\left[{}^{R}v_{4,y_2}\right]_{3\times 24} = \left[[I]_{3\times 3} \quad [R_1]^T_{3\times 3} \quad [0]_{3\times 3} \quad [R_2]^T_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3}\right]}$$

Body 5:

$$\left[{}^R v_{5,y}\right]_{3\times48} = \left[{}^R v_{O_5,y}\right]_{3\times48} - [\tilde{r}_5]\left[{}^R \omega_{5,y}\right]_{3\times48}$$

$$\boxed{\left[{}^R v_{5,y_1}\right]_{3\times24} = \left[\left[{}^R v_{5,\hat{\omega}_1}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad \left[{}^R v_{5,\hat{\omega}_5}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]}$$

$$\boxed{\left[{}^R v_{5,\hat{\omega}_1}\right]_{3\times3} = -\left[[\tilde{q}_5]+[\tilde{s}_5]+[\tilde{r}_5]\right]} \quad \boxed{\left[{}^R v_{5,\hat{\omega}_5}\right]_{3\times3} = -[\tilde{r}_5][R_1]^T}$$

$$\boxed{\left[{}^R v_{5,y_2}\right]_{3\times24} = \left[[I]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [R_1]^T_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]}$$

Body 6:

$$\left[{}^R v_{6,y}\right]_{3\times48} = \left[{}^R v_{O_6,y}\right]_{3\times48} - [\tilde{r}_6]\left[{}^R \omega_{6,y}\right]_{3\times48}$$

$$\boxed{\left[{}^R v_{6,y_1}\right]_{3\times24} = \left[\left[{}^R v_{6,\hat{\omega}_1}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad \left[{}^R v_{6,\hat{\omega}_6}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]}$$

$$\boxed{\left[{}^R v_{6,\hat{\omega}_1}\right]_{3\times3} = -\left[[\tilde{q}_6]+[\tilde{s}_6]+[\tilde{r}_6]\right]} \quad \boxed{\left[{}^R v_{6,\hat{\omega}_6}\right]_{3\times3} = -[\tilde{r}_6][R_1]^T}$$

$$\boxed{\left[{}^R v_{6,y_2}\right]_{3\times24} = \left[[I]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [R_1]^T_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]}$$

Body 7:

$$\left[{}^R v_{7,y}\right]_{3\times48} = \left[{}^R v_{O_7,y}\right]_{3\times48} - [\tilde{r}_7]\left[{}^R \omega_{7,y}\right]_{3\times48}$$

$$\boxed{\left[{}^R v_{7,y_1}\right]_{3\times24} = \left[\left[{}^R v_{7,\hat{\omega}_1}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad \left[{}^R v_{7,\hat{\omega}_7}\right]_{3\times3} \quad [0]_{3\times3}\right]}$$

$$\boxed{\left[{}^R v_{7,\hat{\omega}_1}\right]_{3\times3} = -\left[[\tilde{q}_7]+[\tilde{s}_7]+[\tilde{r}_7]\right]} \quad \boxed{\left[{}^R v_{7,\hat{\omega}_7}\right]_{3\times3} = -[\tilde{r}_7][R_1]^T}$$

$$\boxed{\left[{}^R v_{7,y_2}\right]_{3\times24} = \left[[I]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [R_1]^T_{3\times3} \quad [0]_{3\times3}\right]}$$

Body 8:

$$\left[{}^R v_{8,y}\right]_{3\times48} = \left[{}^R v_{O_8,y}\right]_{3\times48} - [\tilde{r}_8]\left[{}^R \omega_{8,y}\right]_{3\times48}$$

$$\boxed{\left[{}^R v_{8,y_1}\right]_{3\times24} = \left[\left[{}^R v_{8,\hat{\omega}_1}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad \left[{}^R v_{8,\hat{\omega}_7}\right]_{3\times3} \quad \left[{}^R v_{8,\hat{\omega}_8}\right]_{3\times3}\right]}$$

$$\boxed{\left[{}^R v_{8,\hat{\omega}_1}\right]_{3\times3} = -\left[[\tilde{q}_7]+[\tilde{s}_7]+[\tilde{q}_8]+[\tilde{s}_8]+[\tilde{r}_8]\right]} \quad \boxed{\left[{}^R v_{8,\hat{\omega}_7}\right]_{3\times3} = -\left([\tilde{q}_8]+[\tilde{s}_8]+[\tilde{r}_8]\right)[R_1]^T_{3\times3}}$$

$$\boxed{\left[{}^R v_{8,\hat{\omega}_8}\right]_{3\times3} = -[\tilde{r}_8][R_7]^T_{3\times3}}$$

$$\boxed{\left[{}^R v_{8,y_2}\right]_{3\times24} = \left[[I]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [R_1]^T_{3\times3} \quad [R_7]^T_{3\times3}\right]}$$

b) To calculate the velocity of G_8 using ***direct differentiation***, first write the position vector of G_8 as follows.

$$\underset{\sim}{p}_8 = \underset{\sim}{s}_1 + \underset{\sim}{q}_7 + \underset{\sim}{s}_7 + \underset{\sim}{q}_8 + \underset{\sim}{s}_8 + \underset{\sim}{r}_8 \qquad (126)$$

Differentiating this equation gives the following.

$$\underset{\sim}{v}_8 = \frac{{}^R d\underset{\sim}{p}_8}{dt} = \frac{{}^R d\underset{\sim}{s}_1}{dt} + \frac{{}^R d}{dt}\left(\underset{\sim}{q}_7 + \underset{\sim}{s}_7\right) + \frac{{}^R d}{dt}\left(\underset{\sim}{q}_8 + \underset{\sim}{s}_8\right) + \frac{{}^R d\underset{\sim}{r}_8}{dt}$$

$$= \frac{{}^R d\underset{\sim}{s}_1}{dt} + \left(\frac{{}^1 d}{dt}\left(\underset{\sim}{q}_7 + \underset{\sim}{s}_7\right) + {}^R\underset{\sim}{\omega}_1 \times \left(\underset{\sim}{q}_7 + \underset{\sim}{s}_7\right)\right) + \left(\frac{{}^7 d}{dt}\left(\underset{\sim}{q}_8 + \underset{\sim}{s}_8\right) + {}^R\underset{\sim}{\omega}_7 \times \left(\underset{\sim}{q}_8 + \underset{\sim}{s}_8\right)\right) + \left({}^R\underset{\sim}{\omega}_8 \times \underset{\sim}{r}_8\right)$$

$$= \frac{{}^R d\underset{\sim}{s}_1}{dt} + \frac{{}^1 d\underset{\sim}{s}_7}{dt} + \frac{{}^7 d\underset{\sim}{s}_8}{dt} + \left({}^R\underset{\sim}{\omega}_1 \times \left(\underset{\sim}{q}_7 + \underset{\sim}{s}_7\right)\right) + \left(\left({}^R\underset{\sim}{\omega}_1 + {}^1\underset{\sim}{\omega}_7\right) \times \left(\underset{\sim}{q}_8 + \underset{\sim}{s}_8\right)\right) + \left(\left({}^R\underset{\sim}{\omega}_1 + {}^1\underset{\sim}{\omega}_7 + {}^7\underset{\sim}{\omega}_8\right) \times \underset{\sim}{r}_8\right)$$

$$= \left({}^R\underset{\sim}{\omega}_1 \times \left(\underset{\sim}{q}_7 + \underset{\sim}{s}_7 + \underset{\sim}{q}_8 + \underset{\sim}{s}_8 + \underset{\sim}{r}_8\right)\right) + \left({}^1\underset{\sim}{\omega}_7 \times \left(\underset{\sim}{q}_8 + \underset{\sim}{s}_8 + \underset{\sim}{r}_8\right)\right) + \left({}^7\underset{\sim}{\omega}_8 \times \underset{\sim}{r}_8\right) + \frac{{}^R d\underset{\sim}{s}_1}{dt} + \frac{{}^1 d\underset{\sim}{s}_7}{dt} + \frac{{}^7 d\underset{\sim}{s}_8}{dt}$$

$$\boxed{\underset{\sim}{v}_8 = -\left(\left(\underset{\sim}{q}_7 + \underset{\sim}{s}_7 + \underset{\sim}{q}_8 + \underset{\sim}{s}_8 + \underset{\sim}{r}_8\right) \times {}^R\underset{\sim}{\omega}_1\right) - \left(\left(\underset{\sim}{q}_8 + \underset{\sim}{s}_8 + \underset{\sim}{r}_8\right) \times {}^1\underset{\sim}{\omega}_7\right) - \left(\underset{\sim}{r}_8 \times {}^7\underset{\sim}{\omega}_8\right) + \frac{{}^R d\underset{\sim}{s}_1}{dt} + \frac{{}^1 d\underset{\sim}{s}_7}{dt} + \frac{{}^7 d\underset{\sim}{s}_8}{dt}} \quad (127)$$

Using the ***base frame components*** of the ***relative angular velocities***, the ***fixed frame components*** of the ***velocities*** of the ***mass center*** of the body 8 can be written as follows.

$$\boxed{\begin{aligned}\{v_8\} = &-\left[[\tilde{q}_7]+[\tilde{s}_7]+[\tilde{q}_8]+[\tilde{s}_8]+[\tilde{r}_8]\right]\{\hat{\omega}_1\} - \left[[\tilde{q}_8]+[\tilde{s}_8]+[\tilde{r}_8]\right][R_1]^T\{\hat{\omega}_7\} - [\tilde{r}_8][R_7]^T\{\hat{\omega}_8\} \\ &+\{\dot{s}_1'\} + [R_1]^T\{\dot{s}_7'\} + [R_7]^T\{\dot{s}_8'\}\end{aligned}} \quad (128)$$

Using this result, the ***partial velocity matrix*** for the ***mass center*** of body 8 is observed to be as follows.

$$\boxed{\left[{}^R v_{8,y_1}\right]_{3\times 24} = \left[\left[{}^R v_{8,\hat{\omega}_1}\right]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad \left[{}^R v_{8,\hat{\omega}_7}\right]_{3\times 3} \quad \left[{}^R v_{8,\hat{\omega}_8}\right]_{3\times 3}\right]}$$

$$\boxed{\left[{}^R v_{8,\hat{\omega}_1}\right]_{3\times 3} = -\left[[\tilde{q}_7]+[\tilde{s}_7]+[\tilde{q}_8]+[\tilde{s}_8]+[\tilde{r}_8]\right]} \qquad \boxed{\left[{}^R v_{8,\hat{\omega}_7}\right]_{3\times 3} = -\left([\tilde{q}_8]+[\tilde{s}_8]+[\tilde{r}_8]\right)[R_1]^T}$$

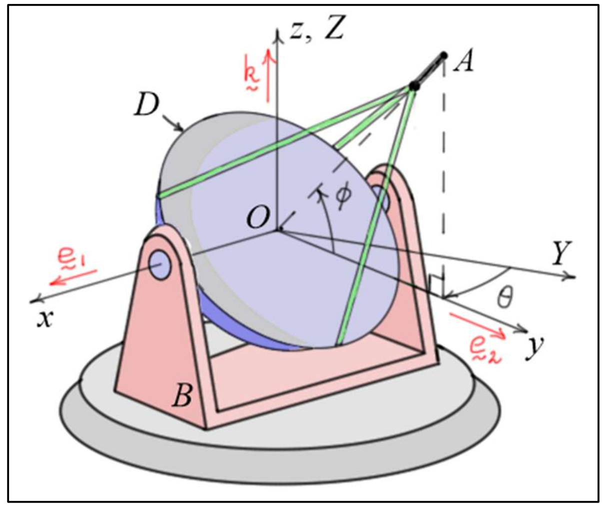
$$\boxed{\left[{}^R v_{8,\hat{\omega}_8}\right]_{3\times 3} = -[\tilde{r}_8][R_7]^T}$$

$$\boxed{\left[{}^R v_{8,y_2}\right]_{3\times 24} = \left[[I]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3} \quad [R_1]_{3\times 3}^T \quad [R_7]_{3\times 3}^T\right]}$$

These are the ***same results*** as presented above. Recall that the matrices $[\tilde{q}_B]$ $(B=1,\ldots,8)$, $[\tilde{s}_B]$ $(B=1,\ldots,8)$, and $[\tilde{r}_B]$ $(B=1,\ldots,8)$ are ***skew symmetric matrices*** built with the ***fixed frame components*** of the vectors $\underset{\sim}{q}_B$ $(B=1,\ldots,8)$, $\underset{\sim}{s}_B$ $(B=1,\ldots,8)$, and $\underset{\sim}{r}_B$ $(B=1,\ldots,8)$.

Exercises

4.1 The antenna system shown has two components, the base B and the antenna dish D. Base B rotates relative to the ground about the fixed z (or Z) axis, and dish D rotates relative to B about the rotating x-axis annotated by the unit vector $\underset{\sim}{e}_1$. At any instant, the angle between the y-axis annotated by the unit vector $\underset{\sim}{e}_2$ and the fixed Y-axis is θ, and the angle between line segment OA and the rotating y-axis is ϕ. The θ rotation is about the $-\underset{\sim}{k}$ direction, and the ϕ rotation is about the $\underset{\sim}{e}_1$ direction. Let the distance between O and A be a, and let $\omega \triangleq \dot{\theta}$ and $\Omega \triangleq \dot{\phi}$.



Reference frames: (all frames align when $\theta = \phi = 0$)

$R:(\underset{\sim}{i},\ \underset{\sim}{j},\ \underset{\sim}{k})$ (fixed frame)

$B:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{k})$ (rotating with base B)

$D:(\underset{\sim}{e}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$ (rotating with disk D)

Let $\{\hat{\omega}_B\}$ and $\{\hat{\omega}_D\}$ be the ***base frame components*** and let $\{\hat{\omega}'_B\}$ and $\{\hat{\omega}'_D\}$ be the ***body frame components*** of the ***relative angular velocities*** ${}^R\underset{\sim}{\omega}_B$ and ${}^F\underset{\sim}{\omega}_D$. Following the process of Example 1, find the ***fixed frame components*** of the ***velocity*** of point A and the ***partial velocity matrices*** associated with the relative angular velocity components using a) ***base frame*** components, and b) ***body frame*** components. c) Find the ***body frame components*** of the ***velocity*** of point A and the ***partial velocity matrix*** associated with the relative angular velocity components using ***body frame*** components.

Answers:

a) $\boxed{{}^R\underset{\sim}{\omega}_B = -\omega \underset{\sim}{k}}$ $\boxed{{}^R\underset{\sim}{\omega}_D = {}^R\underset{\sim}{\omega}_B + {}^B\underset{\sim}{\omega}_D = -\omega \underset{\sim}{k} + \Omega \underset{\sim}{e}_1}$

$\boxed{\{\hat{\omega}_B\} = [0 \quad 0 \quad -\omega]^T}$ $\boxed{\{\hat{\omega}_D\} = [\Omega \quad 0 \quad 0]^T}$

$$\boxed{\left[{}^R\omega_{B,y}\right]_{3\times 6} = \left[\left[{}^R\omega_{B,\hat{\omega}_B}\right] \quad \left[{}^R\omega_{B,\hat{\omega}_D}\right]\right] = \left[[I]_{3\times 3} \quad [0]_{3\times 3}\right]}$$

$$\boxed{\left[{}^R\omega_{D,y}\right]_{3\times 6} = \left[\left[{}^R\omega_{D,\hat{\omega}_B}\right] \quad \left[{}^R\omega_{D,\hat{\omega}_D}\right]\right] = \left[[I]_{3\times 3} \quad [R_B]^T\right]}$$

$$\boxed{\begin{aligned} \left[{}^R v_{A,y}\right] &= \left[\left[{}^R v_{A,\hat{\omega}_B}\right] \quad \left[{}^R v_{A,\hat{\omega}_D}\right]\right] = \left[\left[-\left[\tilde{r}_{A/O}\right]\right] \quad \left[-\left[\tilde{r}_{A/O}\right][R_B]^T\right]\right] \\ &= \left[\begin{bmatrix} 0 & aS_\phi & -aC_\theta C_\phi \\ -aS_\phi & 0 & aS_\theta C_\phi \\ aC_\theta C_\phi & -aS_\theta C_\phi & 0 \end{bmatrix} \begin{bmatrix} -aS_\theta S_\phi & aC_\theta S_\phi & -aC_\theta C_\phi \\ -aC_\theta S_\phi & -aS_\theta S_\phi & aS_\theta C_\phi \\ aC_\phi & 0 & 0 \end{bmatrix}\right] \end{aligned}}$$

$$\{v_A\} = \begin{Bmatrix} a\omega C_\theta C_\phi - a\Omega S_\theta S_\phi \\ -a\omega S_\theta C_\phi - a\Omega C_\theta S_\phi \\ a\Omega C_\phi \end{Bmatrix}$$

b) $\{\hat{\omega}'_B\} = [0 \quad 0 \quad -\omega]^T$ $\{\hat{\omega}'_D\} = [\Omega \quad 0 \quad 0]^T$

$$\left[{}^R v_{A,y}\right] = \left[\left[{}^R v_{A,\hat{\omega}'_B}\right] \quad \left[{}^R v_{A,\hat{\omega}'_D}\right]\right] = \left[\left[-[R_D]^T\left[\tilde{r}'_{A/O}\right]\left[{}^B R_D\right]\right] \quad \left[-[R_D]^T\left[\tilde{r}'_{A/O}\right]\right]\right]$$
$$= \left[\begin{bmatrix} -aS_\theta S_\phi & aC_\theta S_\phi & -aC_\theta C_\phi \\ -aC_\theta S_\phi & -aS_\theta S_\phi & aS_\theta C_\phi \\ aC_\phi & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} -aS_\theta S_\phi & 0 & -aC_\theta \\ -aC_\theta S_\phi & 0 & aS_\theta \\ aC_\phi & 0 & 0 \end{bmatrix}\right]$$

$$\{v_A\} = \begin{Bmatrix} a\omega C_\theta C_\phi - a\Omega S_\theta S_\phi \\ -a\omega S_\theta C_\phi - a\Omega C_\theta S_\phi \\ a\Omega C_\phi \end{Bmatrix}$$

c) $\{\hat{\omega}'_B\} = [0 \quad 0 \quad -\omega]^T$ $\{\hat{\omega}'_D\} = [\Omega \quad 0 \quad 0]^T$

$$\left[{}^R v'_{A,y}\right] = \left[\left[{}^R v'_{A,\hat{\omega}'_B}\right] \quad \left[{}^R v'_{A,\hat{\omega}'_D}\right]\right] = \left[\left[-\left[\tilde{r}'_{A/O}\right]\left[{}^B R_D\right]\right] \quad \left[-\left[\tilde{r}'_{A/O}\right]\right]\right]$$
$$= \left[\begin{bmatrix} 0 & aS_\phi & -aC_\phi \\ 0 & 0 & 0 \\ a & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & -a \\ 0 & 0 & 0 \\ a & 0 & 0 \end{bmatrix}\right]$$

$$\{v'_A\} = \begin{Bmatrix} aC_\phi \omega \\ 0 \\ a\Omega \end{Bmatrix}$$

4.2 Write a MATLAB script to ***numerically*** evaluate the ***matrix equations*** you derived in Exercise 4.1 using the data below. First calculate the ***partial angular velocity matrices*** and the ***angular velocity components***. Then calculate the ***partial velocity matrices*** and the ***velocity components*** of point *A*.

$a = 5$ (ft)
$\theta = -30$ (deg) $\quad \phi = 60$ (deg)
$\dot{\theta} = 3(\text{rad/s}) \quad \dot{\phi} = 7(\text{rad/s})$

Recall that, as shown in the diagram, the angle θ is negative.

Answers:

a) $\{\hat{\omega}_B\} = [0 \quad 0 \quad -3]^T$ (r/s) $\{\hat{\omega}_D\} = [7 \quad 0 \quad 0]^T$ (r/s)

$$\left[{}^R\omega_{B,y}\right]_{3\times 6} = \left[\left[{}^R\omega_{B,\hat{\omega}_B}\right] \quad \left[{}^R\omega_{B,\hat{\omega}_D}\right]\right] = \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right]$$

$$\left[{}^{R}\omega_{D,y}\right]_{3\times 6}=\left[\left[{}^{R}\omega_{D,\hat{\omega}_B}\right]\ \left[{}^{R}\omega_{D,\hat{\omega}_D}\right]\right]=\left[\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\ \begin{bmatrix}0.86603&0.50000&0\\0.50000&0.86603&0\\0&0&1\end{bmatrix}\right]$$

$$\left[{}^{R}v_{A,y}\right]_{3\times 6}=\left[\left[{}^{R}v_{A,\hat{\omega}_B}\right]\ \left[{}^{R}v_{A,\hat{\omega}_D}\right]\right]=\left[\left[-\left[\tilde{r}_{A/O}\right]\right]\ \left[-\left[\tilde{r}_{A/O}\right]\left[R_B\right]^T\right]\right]$$
$$=\left[\begin{bmatrix}0&4.3301&-2.1651\\-4.3301&0&-1.2500\\2.1651&1.25&0\end{bmatrix}\ \begin{bmatrix}2.1651&3.7500&-2.1651\\-3.7500&2.1651&-1.2500\\2.5000&0&0\end{bmatrix}\right]$$

$$\left\{v_A\right\}=\left[21.6506\quad -22.5000\quad 17.5000\right]^T\ \text{(ft/s)}$$

b) $\left\{\hat{\omega}'_B\right\}=\left[0\quad 0\quad -3\right]^T\ \text{(r/s)}$ $\left\{\hat{\omega}'_D\right\}=\left[7\quad 0\quad 0\right]^T\ \text{(r/s)}$

$$\left[{}^{R}\omega'_{B,y}\right]_{3\times 6}=\left[\left[{}^{R}\omega'_{B,\hat{\omega}'_B}\right]\ \left[{}^{R}\omega'_{B,\hat{\omega}'_D}\right]\right]=\left[\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\ \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\right]$$

$$\left[{}^{R}\omega'_{D,y}\right]_{3\times 6}=\left[\left[{}^{R}\omega'_{D,\hat{\omega}'_B}\right]\ \left[{}^{R}\omega'_{D,\hat{\omega}'_D}\right]\right]=\left[\begin{bmatrix}1&0&0\\0&0.5000&0.8660\\0&-0.8660&0.5000\end{bmatrix}\ \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right]$$

$$\left[{}^{R}v_{A,y}\right]_{3\times 6}=\left[\left[{}^{R}v_{A,\hat{\omega}'_B}\right]\ \left[{}^{R}v_{A,\hat{\omega}'_D}\right]\right]=\left[\left[-\left[R_D\right]^T\left[\tilde{r}'_{A/O}\right]\left[{}^{B}R_D\right]\right]\ \left[-\left[R_D\right]^T\left[\tilde{r}'_{A/O}\right]\right]\right]$$
$$=\left[\begin{bmatrix}2.1651&3.7500&-2.1651\\-3.7500&2.1651&-1.2500\\2.5000&0&0\end{bmatrix}\ \begin{bmatrix}2.1651&0&-4.3301\\-3.7500&0&-2.5000\\2.5000&0&0\end{bmatrix}\right]$$

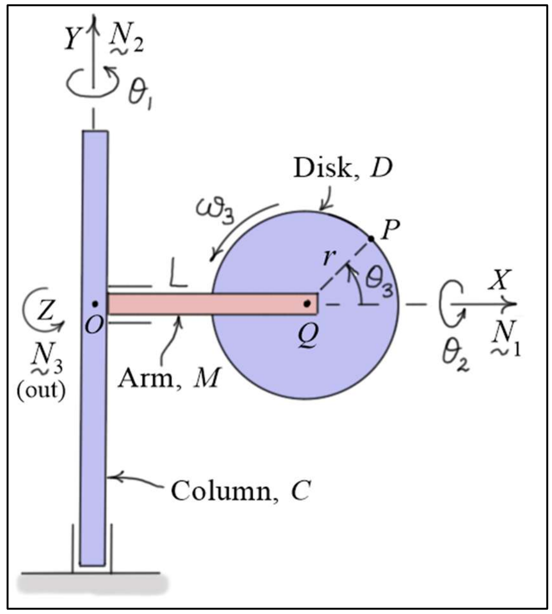
$$\left\{v_A\right\}=\left[21.6506\quad -22.5000\quad 17.5000\right]^T\ \text{(ft/s)}$$

c)

$$\left[{}^{R}v'_{A,y}\right]_{3\times 6}=\left[\left[{}^{R}v'_{A,\hat{\omega}'_B}\right]\ \left[{}^{R}v'_{A,\hat{\omega}'_D}\right]\right]=\left[\left[-\left[\tilde{r}'_{A/O}\right]\left[{}^{B}R_D\right]\right]\ \left[-\left[\tilde{r}'_{A/O}\right]\right]\right]$$
$$=\left[\begin{bmatrix}0&4.3301&-2.5000\\0&0&0\\5.0000&0&0\end{bmatrix}\ \begin{bmatrix}0&0&-5.0000\\0&0&0\\5.0000&0&0\end{bmatrix}\right]$$

$$\left\{v'_A\right\}=\left[7.5000\quad 0\quad 35.000\right]^T\ \text{(ft/s)}$$

4.3 The system shown has ***three*** bodies, the vertical ***column*** C, the horizontal ***arm*** M, and the ***disk*** D. Disk D has radius r and is oriented relative to M using angle θ_3. Arm M has length L and is oriented relative to C using angle θ_2. Column C is oriented relative to the ***fixed frame*** (X,Y,Z) using angle θ_1.



Reference Frames: (all frames align when $\theta_1 = \theta_2 = \theta_3 = 0$)

$R:(\underset{\sim}{N}_1, \underset{\sim}{N}_2, \underset{\sim}{N}_3)$ (fixed frame)
$C:(\underset{\sim}{e}_1, \underset{\sim}{N}_2, \underset{\sim}{e}_3)$ (rotating with base B)
$M:(\underset{\sim}{e}_1, \underset{\sim}{m}_2, \underset{\sim}{m}_3)$ (rotating with the arm M)
$D:(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{m}_3)$ (rotating with disk D)

Let $\{\hat{\omega}_C\}$, $\{\hat{\omega}_M\}$, and $\{\hat{\omega}_D\}$ be the ***base frame components*** and let $\{\hat{\omega}'_C\}$, $\{\hat{\omega}'_M\}$, and $\{\hat{\omega}'_D\}$ be the ***body frame components*** of the ***relative angular velocities*** ${}^R\underset{\sim}{\omega}_C$, ${}^C\underset{\sim}{\omega}_M$, and ${}^M\underset{\sim}{\omega}_D$. Following the process of Example 1, find the ***fixed frame components*** of the ***velocity*** of point P and the ***partial velocity matrices*** associated with the relative angular velocity components using a) ***base frame*** components, and b) ***body frame*** components. c) Following the process of Example 1, find the ***body frame components*** of the ***velocity*** of point P and the ***partial velocity matrices*** associated with the ***body frame components*** of the relative angular velocities.

Answers:

a) $\boxed{{}^R\underset{\sim}{\omega}_C = \omega_1 \underset{\sim}{N}_2}$ $\boxed{{}^R\underset{\sim}{\omega}_M = {}^R\underset{\sim}{\omega}_C + {}^C\underset{\sim}{\omega}_M = \omega_1 \underset{\sim}{N}_2 + \omega_2 \underset{\sim}{e}_1}$

$$\boxed{{}^R\underset{\sim}{\omega}_D = {}^R\underset{\sim}{\omega}_C + {}^C\underset{\sim}{\omega}_M + {}^M\underset{\sim}{\omega}_D = \omega_1 \underset{\sim}{N}_2 + \omega_2 \underset{\sim}{e}_1 + \omega_3 \underset{\sim}{m}_3}$$

$$\boxed{\{\hat{\omega}_C\} = [0 \quad \omega_1 \quad 0]^T} \quad \boxed{\{\hat{\omega}_M\} = [\omega_2 \quad 0 \quad 0]^T} \quad \boxed{\{\hat{\omega}_D\} = [0 \quad 0 \quad \omega_3]^T}$$

$$\boxed{\left[{}^R\omega_{C,y}\right]_{3\times 9} = \left[\left[{}^R\omega_{C,\hat{\omega}_C}\right] \quad \left[{}^R\omega_{C,\hat{\omega}_M}\right] \quad \left[{}^R\omega_{C,\hat{\omega}_D}\right]\right] = \left[[I]_{3\times 3} \quad [0]_{3\times 3} \quad [0]_{3\times 3}\right]}$$

$$\boxed{\left[{}^R\omega_{M,y}\right]_{3\times 9} = \left[\left[{}^R\omega_{M,\hat{\omega}_C}\right] \quad \left[{}^R\omega_{M,\hat{\omega}_M}\right] \quad \left[{}^R\omega_{M,\hat{\omega}_D}\right]\right] = \left[[I]_{3\times 3} \quad [R_C]^T_{3\times 3} \quad [0]_{3\times 3}\right]}$$

$$\boxed{\left[{}^R\omega_{D,y}\right]_{3\times 9} = \left[\left[{}^R\omega_{D,\hat{\omega}_C}\right] \quad \left[{}^R\omega_{D,\hat{\omega}_M}\right] \quad \left[{}^R\omega_{D,\hat{\omega}_D}\right]\right] = \left[[I]_{3\times 3} \quad [R_C]^T_{3\times 3} \quad [R_M]^T_{3\times 3}\right]}$$

$$\boxed{\left[{}^R v_{P,y}\right]_{3\times 9} = \left[\left[{}^R v_{P,\hat{\omega}_C}\right] \quad \left[{}^R v_{P,\hat{\omega}_M}\right] \quad \left[{}^R v_{P,\hat{\omega}_D}\right]\right]}$$

$$\left[{}^R v_{P,\hat{\omega}_C}\right] = \begin{bmatrix} 0 & -LS_1 - r\left(S_1C_3 - C_1S_2S_3\right) & -rC_2S_3 \\ LS_1 + r\left(S_1C_3 - C_1S_2S_3\right) & 0 & LC_1 + r\left(C_1C_3 + S_1S_2S_3\right) \\ rC_2S_3 & -LC_1 - r\left(C_1C_3 + S_1S_2S_3\right) & 0 \end{bmatrix}$$

$$\left[{}^R v_{P,\hat{\omega}_M}\right] = r\begin{bmatrix} S_1C_2S_3 & -\left(S_1C_3 - C_1S_2S_3\right) & -C_1C_2S_3 \\ -S_2S_3 & 0 & C_3 \\ C_1C_2S_3 & -\left(C_1C_3 + S_1S_2S_3\right) & S_1C_2S_3 \end{bmatrix}$$

$$\left[{}^R v_{P,\hat{\omega}_D}\right] = r\begin{bmatrix} S_1C_2S_3 & -C_2S_1C_3 & S_1S_2C_3 - C_1S_3 \\ -S_2S_3 & S_2C_3 & C_2C_3 \\ C_1C_2S_3 & -C_1C_2C_3 & S_1S_3 + C_1S_2C_3 \end{bmatrix}$$

$$\left\{v_P\right\} = -L\omega_1\begin{Bmatrix} S_1 \\ 0 \\ C_1 \end{Bmatrix} - r\omega_1\begin{Bmatrix} S_1C_3 - C_1S_2S_3 \\ 0 \\ C_1C_3 + S_1S_2S_3 \end{Bmatrix} + r\omega_2\begin{Bmatrix} S_1C_2S_3 \\ -S_2S_3 \\ C_1C_2S_3 \end{Bmatrix} + r\omega_3\begin{Bmatrix} S_1S_2C_3 - C_1S_3 \\ C_2C_3 \\ S_1S_3 + C_1S_2C_3 \end{Bmatrix}$$

b) $\quad {}^R\underset{\sim}{\omega}_C = \omega_1 \underset{\sim}{N}_2 \qquad {}^R\underset{\sim}{\omega}_M = {}^R\underset{\sim}{\omega}_C + {}^C\underset{\sim}{\omega}_M = \omega_1 \underset{\sim}{N}_2 + \omega_2 \underset{\sim}{e}_1$

$${}^R\underset{\sim}{\omega}_D = {}^R\underset{\sim}{\omega}_C + {}^C\underset{\sim}{\omega}_M + {}^M\underset{\sim}{\omega}_D = \omega_1 \underset{\sim}{N}_2 + \omega_2 \underset{\sim}{e}_1 + \omega_3 \underset{\sim}{m}_3$$

$$\left\{\hat{\omega}'_C\right\} = \begin{bmatrix} 0 & \omega_1 & 0 \end{bmatrix}^T \qquad \left\{\hat{\omega}'_M\right\} = \begin{bmatrix} \omega_2 & 0 & 0 \end{bmatrix}^T \qquad \left\{\hat{\omega}'_D\right\} = \begin{bmatrix} 0 & 0 & \omega_3 \end{bmatrix}^T$$

$$\left[{}^R\omega'_{C,y}\right]_{3\times 9} = \left[\left[{}^R\omega'_{C,\hat{\omega}'_C}\right] \quad \left[{}^R\omega'_{C,\hat{\omega}'_M}\right] \quad \left[{}^R\omega'_{C,\hat{\omega}'_D}\right]\right] = \left[\left[I\right]_{3\times 3} \quad \left[0\right]_{3\times 3} \quad \left[0\right]_{3\times 3}\right]$$

$$\left[{}^R\omega'_{M,y}\right]_{3\times 9} = \left[\left[{}^R\omega'_{M,\hat{\omega}'_C}\right] \quad \left[{}^R\omega'_{M,\hat{\omega}'_M}\right] \quad \left[{}^R\omega'_{M,\hat{\omega}'_D}\right]\right] = \left[\left[{}^C R_M\right]_{3\times 3} \quad \left[I\right]_{3\times 3} \quad \left[0\right]_{3\times 3}\right]$$

$$\left[{}^R\omega'_{D,y}\right]_{3\times 9} = \left[\left[{}^R\omega'_{D,\hat{\omega}'_C}\right] \quad \left[{}^R\omega'_{D,\hat{\omega}'_M}\right] \quad \left[{}^R\omega'_{D,\hat{\omega}'_D}\right]\right] = \left[\left[{}^C R_D\right]_{3\times 3} \quad \left[{}^M R_D\right]_{3\times 3} \quad \left[I\right]_{3\times 3}\right]$$

$$\left[{}^R v_{P,y}\right]_{3\times 9} = \left[\left[{}^R v_{P,\hat{\omega}'_C}\right] \quad \left[{}^R v_{P,\hat{\omega}'_M}\right] \quad \left[{}^R v_{P,\hat{\omega}'_D}\right]\right]$$

$$\left[{}^R v_{P,\hat{\omega}'_C}\right] = \begin{bmatrix} rS_1C_2S_3 & -LS_1 + r\left(-S_1C_3 + C_1S_2S_3\right) & -rC_1C_2S_3 \\ -rS_2S_3 & 0 & L + rC_3 \\ rC_1C_2S_3 & -LC_1 - r\left(C_1C_3 + S_1S_2S_3\right) & rS_1C_2S_3 \end{bmatrix}$$

$$\left[{}^R v_{P,\hat{\omega}'_M}\right] = \begin{bmatrix} rS_1C_2S_3 & -rS_1C_2C_3 & r\left(-C_1S_3 + S_1S_2C_3\right) \\ -rS_2S_3 & rS_2C_3 & rC_2C_3 \\ rC_1C_2S_3 & -rC_1C_2C_3 & r\left(S_1S_3 + C_1S_2C_3\right) \end{bmatrix}$$

$$\left[{}^{R}v_{P,\hat{\omega}'_D}\right]=\begin{bmatrix}0 & -rS_1C_2 & r\left(-C_1S_3+S_1S_2C_3\right)\\ 0 & rS_2 & rC_2C_3\\ 0 & -rC_1C_2 & r\left(S_1S_3+C_1S_2C_3\right)\end{bmatrix}$$

$$\left\{v_P\right\}=-L\omega_1\begin{Bmatrix}S_1\\0\\C_1\end{Bmatrix}-r\omega_1\begin{Bmatrix}S_1C_3-C_1S_2S_3\\0\\C_1C_3+S_1S_2S_3\end{Bmatrix}+r\omega_2\begin{Bmatrix}S_1C_2S_3\\-S_2S_3\\C_1C_2S_3\end{Bmatrix}+r\omega_3\begin{Bmatrix}-C_1S_3+S_1S_2C_3\\C_2C_3\\S_1S_3+C_1S_2C_3\end{Bmatrix}$$

c) ${}^{R}\underset{\sim}{\omega}_C=\omega_1\underset{\sim}{N}_2$ ${}^{R}\underset{\sim}{\omega}_M={}^{R}\underset{\sim}{\omega}_C+{}^{C}\underset{\sim}{\omega}_M=\omega_1\underset{\sim}{N}_2+\omega_2\underset{\sim}{e}_1$

$${}^{R}\underset{\sim}{\omega}_D={}^{R}\underset{\sim}{\omega}_C+{}^{C}\underset{\sim}{\omega}_M+{}^{M}\underset{\sim}{\omega}_D=\omega_1\underset{\sim}{N}_2+\omega_2\underset{\sim}{e}_1+\omega_3\underset{\sim}{m}_3$$

$\left\{\hat{\omega}'_C\right\}=\begin{bmatrix}0 & \omega_1 & 0\end{bmatrix}^T$ $\left\{\hat{\omega}'_M\right\}=\begin{bmatrix}\omega_2 & 0 & 0\end{bmatrix}^T$ $\left\{\hat{\omega}'_D\right\}=\begin{bmatrix}0 & 0 & \omega_3\end{bmatrix}^T$

$$\left[{}^{R}\omega'_{C,y}\right]_{3\times 9}=\left[\left[{}^{R}\omega'_{C,\hat{\omega}'_C}\right]\;\left[{}^{R}\omega'_{C,\hat{\omega}'_M}\right]\;\left[{}^{R}\omega'_{C,\hat{\omega}'_D}\right]\right]=\left[\left[I\right]_{3\times 3}\;\left[0\right]_{3\times 3}\;\left[0\right]_{3\times 3}\right]$$

$$\left[{}^{R}\omega'_{M,y}\right]_{3\times 9}=\left[\left[{}^{R}\omega'_{M,\hat{\omega}'_C}\right]\;\left[{}^{R}\omega'_{M,\hat{\omega}'_M}\right]\;\left[{}^{R}\omega'_{M,\hat{\omega}'_D}\right]\right]=\left[\left[{}^{C}R_M\right]_{3\times 3}\;\left[I\right]_{3\times 3}\;\left[0\right]_{3\times 3}\right]$$

$$\left[{}^{R}\omega'_{D,y}\right]_{3\times 9}=\left[\left[{}^{R}\omega'_{D,\hat{\omega}'_C}\right]\;\left[{}^{R}\omega'_{D,\hat{\omega}'_M}\right]\;\left[{}^{R}\omega'_{D,\hat{\omega}'_D}\right]\right]=\left[\left[{}^{C}R_D\right]_{3\times 3}\;\left[{}^{M}R_D\right]_{3\times 3}\;\left[I\right]_{3\times 3}\right]$$

$$\left[{}^{R}v'_{P,y}\right]_{3\times 9}=\left[\left[{}^{R}v'_{P,\hat{\omega}'_C}\right]\;\left[{}^{R}v'_{P,\hat{\omega}'_M}\right]\;\left[{}^{R}v'_{P,\hat{\omega}'_D}\right]\right]$$

$$\left[{}^{R}v'_{P,\hat{\omega}'_C}\right]=\begin{bmatrix}0 & -LS_2S_3 & LC_2S_3\\ 0 & -\left(r+LC_3\right)S_2 & \left(r+LC_3\right)C_2\\ rS_3 & -\left(L+rC_3\right)C_2 & -\left(L+rC_3\right)S_2\end{bmatrix}$$

$\left[{}^{R}v'_{P,\hat{\omega}'_M}\right]=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & r\\ rS_3 & -rC_3 & 0\end{bmatrix}$ $\left[{}^{R}v'_{P,\hat{\omega}'_D}\right]=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & r\\ 0 & -r & 0\end{bmatrix}$

$$\left\{v'_P\right\}=\begin{Bmatrix}-LS_2S_3\\-\left(r+LC_3\right)S_2\\-\left(L+rC_3\right)C_2\end{Bmatrix}\omega_1+\begin{Bmatrix}0\\0\\rS_3\end{Bmatrix}\omega_2+\begin{Bmatrix}0\\r\\0\end{Bmatrix}\omega_3$$

4.4 Write a MATLAB script to ***numerically evaluate*** the equations you derived in Exercise 4.3 using the data below. First calculate the ***partial angular velocity matrices*** and the ***angular velocity components***. Then calculate the ***partial velocity matrices*** and the ***velocity components*** of point *P*.

$L = 0.5$ (m) $\quad r = 0.25$ (m)

$\theta_1 = 20$ (deg) $\quad \theta_2 = 40$ (deg) $\quad \theta_3 = 60$ (deg)

$\omega_1 = \dot{\theta}_1 = 2$ (rad/s) $\quad \omega_2 = \dot{\theta}_2 = -3$ (rad/s) $\quad \omega_3 = \dot{\theta}_3 = 5$ (rad/s)

Answers:

a) $\{\hat{\omega}_C\} = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}^T$ (r/s) $\quad \{\hat{\omega}_M\} = \begin{bmatrix} -3 & 0 & 0 \end{bmatrix}^T$ (r/s) $\quad \{\hat{\omega}_D\} = \begin{bmatrix} 0 & 0 & 5 \end{bmatrix}^T$ (r/s)

$$\left[{}^R\omega_{C,y}\right]_{3\times 9} = \left[\left[{}^R\omega_{C,\hat{\omega}_C}\right] \; \left[{}^R\omega_{C,\hat{\omega}_M}\right] \; \left[{}^R\omega_{C,\hat{\omega}_D}\right]\right] = \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right]$$

$$\left[{}^R\omega_{M,y}\right]_{3\times 9} = \left[\left[{}^R\omega_{M,\hat{\omega}_C}\right] \; \left[{}^R\omega_{M,\hat{\omega}_M}\right] \; \left[{}^R\omega_{M,\hat{\omega}_D}\right]\right]$$

$$= \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.9397 & 0 & 0.3420 \\ 0 & 1 & 0 \\ -0.3420 & 0 & 0.9397 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right]$$

$$\left[{}^R\omega_{D,y}\right]_{3\times 9} = \left[\left[{}^R\omega_{D,\hat{\omega}_C}\right] \; \left[{}^R\omega_{D,\hat{\omega}_M}\right] \; \left[{}^R\omega_{D,\hat{\omega}_D}\right]\right]$$

$$= \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.9397 & 0 & 0.3420 \\ 0 & 1 & 0 \\ -0.3420 & 0 & 0.9397 \end{bmatrix} \begin{bmatrix} 0.9397 & 0.2198 & 0.2620 \\ 0 & 0.7660 & -0.6428 \\ -0.3420 & 0.6040 & 0.7198 \end{bmatrix}\right]$$

$$\left[{}^R v_{P,y}\right]_{3\times 9} = \left[\left[{}^R v_{P,\hat{\omega}_C}\right] \; \left[{}^R v_{P,\hat{\omega}_M}\right] \; \left[{}^R v_{P,\hat{\omega}_D}\right]\right]$$

$$\left[{}^R v_{P,\hat{\omega}_C}\right] = \begin{bmatrix} 0 & -0.0830 & -0.1659 \\ 0.0830 & 0 & 0.6349 \\ 0.1659 & -0.6349 & 0 \end{bmatrix} \quad \left[{}^R v_{P,\hat{\omega}_M}\right] = \begin{bmatrix} 0.0567 & 0.0880 & -0.1559 \\ -0.1392 & 0 & 0.1250 \\ 0.1559 & -0.1651 & 0.0567 \end{bmatrix}$$

$$\left[{}^R v_{P,\hat{\omega}_D}\right] = \begin{bmatrix} 0.0567 & -0.0328 & -0.1760 \\ -0.1392 & 0.0803 & 0.0958 \\ 0.1559 & -0.0900 & 0.1496 \end{bmatrix} \quad \{v_P\} = \begin{Bmatrix} -1.2160 \\ 0.8963 \\ -0.9896 \end{Bmatrix} \text{ (m/s)}$$

b) $\{\hat{\omega}'_C\} = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}^T$ (r/s) $\quad \{\hat{\omega}'_M\} = \begin{bmatrix} -3 & 0 & 0 \end{bmatrix}^T$ (r/s) $\quad \{\hat{\omega}'_D\} = \begin{bmatrix} 0 & 0 & 5 \end{bmatrix}^T$ (r/s)

$$\left[{}^R\omega'_{C,y}\right]_{3\times 9} = \left[\left[{}^R\omega'_{C,\hat{\omega}'_C}\right] \; \left[{}^R\omega'_{C,\hat{\omega}'_M}\right] \; \left[{}^R\omega'_{C,\hat{\omega}'_D}\right]\right] = \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right]$$

$$\left[{}^R\omega'_{M,y}\right]_{3\times 9} = \left[\left[{}^R\omega'_{M,\hat{\omega}'_C}\right]\ \left[{}^R\omega'_{M,\hat{\omega}'_M}\right]\ \left[{}^R\omega'_{M,\hat{\omega}'_D}\right]\right]$$
$$= \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7660 & 0.6428 \\ 0 & -0.6428 & 0.7660 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right]$$

$$\left[{}^R\omega'_{D,y}\right]_{3\times 9} = \left[\left[{}^R\omega'_{D,\hat{\omega}'_C}\right]\ \left[{}^R\omega'_{D,\hat{\omega}'_M}\right]\ \left[{}^R\omega'_{D,\hat{\omega}'_D}\right]\right]$$
$$= \left[\begin{bmatrix} 0.5000 & 0.6634 & 0.5567 \\ -0.8660 & 0.3830 & 0.3214 \\ 0 & -0.6428 & 0.7660 \end{bmatrix} \begin{bmatrix} 0.5000 & 0.8660 & 0 \\ -0.8660 & 0.5000 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$\left[{}^R v_{P,y}\right]_{3\times 9} = \left[\left[{}^R v_{P,\hat{\omega}'_C}\right]\ \left[{}^R v_{P,\hat{\omega}'_M}\right]\ \left[{}^R v_{P,\hat{\omega}'_D}\right]\right]$$

$$\left[{}^R v_{P,\hat{\omega}'_C}\right] = \begin{bmatrix} 0.0567 & -0.0830 & -0.1559 \\ -0.1392 & 0 & 0.6250 \\ 0.1559 & -0.6349 & 0.0567 \end{bmatrix} \qquad \left[{}^R v_{P,\hat{\omega}'_M}\right] = \begin{bmatrix} 0.0567 & -0.0328 & -0.1760 \\ -0.1392 & 0.0803 & 0.0958 \\ 0.1559 & -0.0900 & 0.1496 \end{bmatrix}$$

$$\left[{}^R v_{P,\hat{\omega}'_D}\right] = \begin{bmatrix} 0 & -0.0655 & -0.1760 \\ 0 & 0.1607 & 0.0958 \\ 0 & -0.1800 & 0.1496 \end{bmatrix} \qquad \{v_P\} = \begin{Bmatrix} -1.2160 \\ 0.8963 \\ -0.9896 \end{Bmatrix} \text{ (m/s)}$$

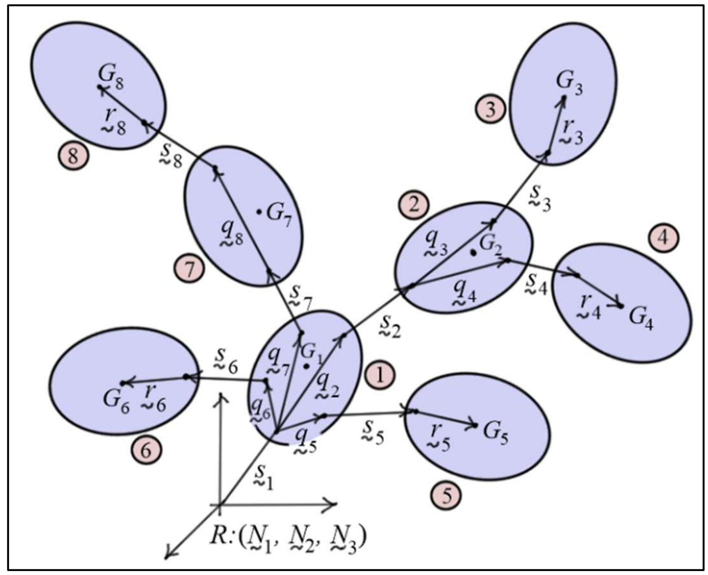
c) $$\left[{}^R v'_{P,y}\right]_{3\times 9} = \left[\left[{}^R v'_{P,\hat{\omega}'_C}\right]\ \left[{}^R v'_{P,\hat{\omega}'_M}\right]\ \left[{}^R v'_{P,\hat{\omega}'_D}\right]\right]$$

$$\left[{}^R v'_{P,\hat{\omega}'_C}\right] = \begin{bmatrix} 0 & -0.27834 & 0.33171 \\ 0 & -0.32139 & 0.38302 \\ 0.21651 & -0.47878 & -0.40174 \end{bmatrix} \qquad \left[{}^R v'_{P,\hat{\omega}'_M}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.25000 \\ 0.21651 & -0.12500 & 0 \end{bmatrix}$$

$$\left[{}^R v'_{P,\hat{\omega}'_D}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.25000 \\ 0 & -0.25000 & 0 \end{bmatrix} \qquad \{v'_P\} = \begin{Bmatrix} -0.55667 \\ 0.60721 \\ -1.6071 \end{Bmatrix} \text{ (m/s)}$$

4.5 The figure shows an eight-body system numbered using the numbering scheme presented in Unit 1. Body 1 is the system ***reference body***, and the rest of the bodies are numbered in ***ascending progression*** outward along the branches. As structured, the lower numbered body array for the system is as follows.

$$\mathcal{L}(1,\ldots,8)=(0,1,2,2,1,1,1,7)$$



The orientation of body 1 is defined ***relative*** to the fixed frame $R:\left(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3\right)$, and the orientations of all the other bodies are defined ***relative*** to their adjacent, lower numbered bodies. Using ***body frame components*** of the ***relative angular velocities*** of the bodies, the ***system generalized speed matrix*** is defined as follows.

$$\{y\}_{48\times1}\triangleq\begin{Bmatrix}\{y_1\}_{24\times1}\\\{y_2\}_{24\times1}\end{Bmatrix}$$

$$\{y_1\}_{24\times1}\triangleq\begin{bmatrix}\hat{\omega}'_{11} & \hat{\omega}'_{12} & \hat{\omega}'_{13} & \hat{\omega}'_{21} & \hat{\omega}'_{22} & \hat{\omega}'_{23} & \cdots & \hat{\omega}'_{71} & \hat{\omega}'_{72} & \hat{\omega}'_{73} & \hat{\omega}'_{81} & \hat{\omega}'_{82} & \hat{\omega}'_{83}\end{bmatrix}^T$$

$$\{y_2\}_{24\times1}\triangleq\begin{bmatrix}\dot{s}'_{11} & \dot{s}'_{12} & \dot{s}'_{13} & \dot{s}'_{21} & \dot{s}'_{22} & \dot{s}'_{23} & \cdots & \dot{s}'_{71} & \dot{s}'_{72} & \dot{s}'_{73} & \dot{s}'_{81} & \dot{s}'_{82} & \dot{s}'_{83}\end{bmatrix}^T$$

a) For each body B in the system, find the ***fixed frame components*** of the ***partial velocity matrix*** of its ***origin*** O_B and its ***mass center*** G_B. Follow the procedures outlined in Equations (40)-(42) and Equation (48). Use the results presented in Unit 2 for the ***body frame components*** of the ***partial angular velocity matrices*** of the bodies. b) Use ***direct differentiation*** to find the ***fixed frame components*** of the ***velocity*** of G_8 in terms of the ***body frame components*** of the ***relative angular velocity*** vectors, ***identify*** the partial velocity matrix $\left[{}^R v_{8,y}\right]_{3\times48}$, and compare the results with those obtained in part (a).

Answers:

Body Origins:

Body 1:

$$\left[{}^R v_{O_1,y_1}\right]_{3\times24}=[0]_{3\times24}$$

$$\left[{}^R v_{O_1,y_2}\right]_{3\times24}=\begin{bmatrix}[I]_{3\times3} & [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times3}\end{bmatrix}$$

Body 2:

$$\left[{}^R v_{O_2,y_1}\right]_{3\times24}=\begin{bmatrix}\left[{}^R v_{O_2,\hat{\omega}'_1}\right]_{3\times3} & [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times3} & [0]_{3\times3}\end{bmatrix}$$

$$\left[{}^R v_{O_2,\hat{\omega}'_1}\right]_{3\times3}=\left[-[R_1]^T\left([\tilde{q}'_2]+[\tilde{s}'_2]\right)\right]$$

$$\left[{}^{R}v_{O_2,y_2}\right]_{3\times 24} = \left[[I]_{3\times3} \quad [R_1]^T \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]$$

Body 3:

$$\left[{}^{R}v_{O_3,y_1}\right]_{3\times 24} = \left[\left[{}^{R}v_{O_3,\hat{\omega}_1'}\right]_{3\times3} \quad \left[{}^{R}v_{O_3,\hat{\omega}_2'}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]$$

$$\left[{}^{R}v_{O_3,\hat{\omega}_1'}\right]_{3\times3} = \left[-[R_1]^T\left([\tilde{q}_2'] + [\tilde{s}_2']\right) - [R_2]^T\left([\tilde{q}_3'] + [\tilde{s}_3']\right)\left[{}^{1}R_2\right]\right] \qquad \left[{}^{R}v_{O_3,\hat{\omega}_2'}\right]_{3\times3} = -[R_2]^T\left([\tilde{q}_3'] + [\tilde{s}_3']\right)$$

$$\left[{}^{R}v_{O_3,y_2}\right]_{3\times 24} = \left[[I]_{3\times3} \quad [R_1]^T \quad [R_2]^T \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]$$

Body 4:

$$\left[{}^{R}v_{O_4,y_1}\right]_{3\times 24} = \left[\left[{}^{R}v_{O_4,\hat{\omega}_1'}\right]_{3\times3} \quad \left[{}^{R}v_{O_4,\hat{\omega}_2'}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]$$

$$\left[{}^{R}v_{O_4,\hat{\omega}_1'}\right]_{3\times3} = \left[-[R_1]^T\left([\tilde{q}_2'] + [\tilde{s}_2']\right) - [R_2]^T\left([\tilde{q}_4'] + [\tilde{s}_4']\right)\left[{}^{1}R_2\right]\right] \qquad \left[{}^{R}v_{O_4,\hat{\omega}_2'}\right]_{3\times3} = -[R_2]^T\left([\tilde{q}_4'] + [\tilde{s}_4']\right)$$

$$\left[{}^{R}v_{O_4,y_2}\right]_{3\times 24} = \left[[I]_{3\times3} \quad [R_1]^T \quad [0]_{3\times3} \quad [R_2]^T \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]$$

Body 5:

$$\left[{}^{R}v_{O_5,y_1}\right]_{3\times 24} = \left[\left[{}^{R}v_{O_5,\hat{\omega}_1'}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]$$

$$\left[{}^{R}v_{O_5,\hat{\omega}_1'}\right]_{3\times3} = \left[-[R_1]^T\left([\tilde{q}_5'] + [\tilde{s}_5']\right)\right]$$

$$\left[{}^{R}v_{O_5,y_2}\right]_{3\times 24} = \left[[I]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [R_1]^T \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]$$

Body 6:

$$\left[{}^{R}v_{O_6,y_1}\right]_{3\times 24} = \left[\left[{}^{R}v_{O_6,\hat{\omega}_1'}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]$$

$$\left[{}^{R}v_{O_6,\hat{\omega}_1'}\right]_{3\times3} = \left[-[R_1]^T\left([\tilde{q}_6'] + [\tilde{s}_6']\right)\right]$$

$$\left[{}^{R}v_{O_6,y_2}\right]_{3\times 24} = \left[[I]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [R_1]^T \quad [0]_{3\times3} \quad [0]_{3\times3}\right]$$

Body 7:

$$\left[{}^{R}v_{O_7,y_1}\right]_{3\times 24} = \left[\left[{}^{R}v_{O_7,\hat{\omega}_1'}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]$$

$$\left[{}^{R}v_{O_7,\hat{\omega}_1'}\right]_{3\times3} = \left[-[R_1]^T\left([\tilde{q}_7'] + [\tilde{s}_7']\right)\right]$$

$$\left[{}^{R}v_{O_7,y_2}\right]_{3\times 24} = \left[[I]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [R_1]^T \quad [0]_{3\times3}\right]$$

Body 8:

$$\left[{}^{R}v_{O_8,y_1}\right]_{3\times 24} = \left[\left[{}^{R}v_{O_8,\hat{\omega}_1'}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad \left[{}^{R}v_{O_8,\hat{\omega}_7'}\right]_{3\times3} \quad [0]_{3\times3}\right]$$

$$\left[{}^{R}v_{O_8,\hat{\omega}_1'}\right]_{3\times3} = \left[-[R_1]^T\left([\tilde{q}_7'] + [\tilde{s}_7']\right) - [R_7]^T\left([\tilde{q}_8'] + [\tilde{s}_8']\right)\left[{}^{1}R_7\right]\right] \qquad \left[{}^{R}v_{O_8,\hat{\omega}_7'}\right]_{3\times3} = -[R_7]^T\left([\tilde{q}_8'] + [\tilde{s}_8']\right)$$

$$\left[{}^{R}v_{O_8,y_2}\right]_{3\times24}=\left[[I]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[R_1]^T\quad[R_7]^T\right]$$

Body Mass Centers:

Body 1:

$$\left[{}^{R}v_{1,y_1}\right]_{3\times24}=\left[\left[{}^{R}v_{1,\hat{\omega}'_1}\right]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\right]$$

$$\left[{}^{R}v_{1,\hat{\omega}'_1}\right]_{3\times3}=-[R_1]^T[\tilde{r}'_1]$$

$$\left[{}^{R}v_{1,y_2}\right]_{3\times24}=\left[[I]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\right]$$

Body 2:

$$\left[{}^{R}v_{2,y_1}\right]_{3\times24}=\left[\left[{}^{R}v_{2,\hat{\omega}'_1}\right]_{3\times3}\quad\left[{}^{R}v_{2,\hat{\omega}'_2}\right]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\right]$$

$$\left[{}^{R}v_{2,\hat{\omega}'_1}\right]_{3\times3}=\left[-[R_1]^T\left([\tilde{q}'_2]+[\tilde{s}'_2]\right)-[R_2]^T[\tilde{r}'_2]\left[{}^{1}R_2\right]\right]\qquad\left[{}^{R}v_{2,\hat{\omega}'_2}\right]_{3\times3}=-[R_2]^T[\tilde{r}'_2]$$

$$\left[{}^{R}v_{2,y_2}\right]_{3\times24}=\left[[I]_{3\times3}\quad[R_1]^T\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\right]$$

Body 3:

$$\left[{}^{R}v_{3,y_1}\right]_{3\times24}=\left[\left[{}^{R}v_{3,\hat{\omega}'_1}\right]_{3\times3}\quad\left[{}^{R}v_{3,\hat{\omega}'_2}\right]_{3\times3}\quad\left[{}^{R}v_{3,\hat{\omega}'_3}\right]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\right]$$

$$\left[{}^{R}v_{3,\hat{\omega}'_1}\right]_{3\times3}=\left[-[R_1]^T\left([\tilde{q}'_2]+[\tilde{s}'_2]\right)-[R_2]^T\left([\tilde{q}'_3]+[\tilde{s}'_3]\right)\left[{}^{1}R_2\right]-[R_3]^T[\tilde{r}'_3]\left[{}^{1}R_3\right]\right]$$

$$\left[{}^{R}v_{3,\hat{\omega}'_2}\right]_{3\times3}=\left[-[R_2]^T\left([\tilde{q}'_3]+[\tilde{s}'_3]\right)-[R_3]^T[\tilde{r}'_3]\left[{}^{2}R_3\right]\right]\qquad\left[{}^{R}v_{3,\hat{\omega}'_3}\right]_{3\times3}=-[R_3]^T[\tilde{r}'_3]$$

$$\left[{}^{R}v_{3,y_2}\right]_{3\times24}=\left[[I]_{3\times3}\quad[R_1]^T\quad[R_2]^T\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\right]$$

Body 4:

$$\left[{}^{R}v_{4,y_1}\right]_{3\times24}=\left[\left[{}^{R}v_{4,\hat{\omega}'_1}\right]_{3\times3}\quad\left[{}^{R}v_{4,\hat{\omega}'_2}\right]_{3\times3}\quad[0]_{3\times3}\quad\left[{}^{R}v_{4,\hat{\omega}'_4}\right]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\right]$$

$$\left[{}^{R}v_{4,\hat{\omega}'_1}\right]_{3\times3}=\left[-[R_1]^T\left([\tilde{q}'_2]+[\tilde{s}'_2]\right)-[R_2]^T\left([\tilde{q}'_4]+[\tilde{s}'_4]\right)\left[{}^{1}R_2\right]-[R_4]^T[\tilde{r}'_4]\left[{}^{1}R_4\right]\right]$$

$$\left[{}^{R}v_{4,\hat{\omega}'_2}\right]_{3\times3}=\left[-[R_2]^T\left([\tilde{q}'_4]+[\tilde{s}'_4]\right)-[R_4]^T[\tilde{r}'_4]\left[{}^{2}R_4\right]\right]\qquad\left[{}^{R}v_{4,\hat{\omega}'_4}\right]_{3\times3}=-[R_4]^T[\tilde{r}'_4]$$

$$\left[{}^{R}v_{4,y_2}\right]_{3\times24}=\left[[I]_{3\times3}\quad[R_1]^T\quad[0]_{3\times3}\quad[R_2]^T\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\right]$$

Body 5:

$$\left[{}^{R}v_{5,y_1}\right]_{3\times24}=\left[\left[{}^{R}v_{5,\hat{\omega}'_1}\right]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad\left[{}^{R}v_{5,\hat{\omega}'_5}\right]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\right]$$

$$\left[{}^{R}v_{5,\hat{\omega}'_1}\right]_{3\times3}=\left[-[R_1]^T\left([\tilde{q}'_5]+[\tilde{s}'_5]\right)-[R_5]^T[\tilde{r}'_5]\left[{}^{1}R_5\right]\right]\qquad\left[{}^{R}v_{5,\hat{\omega}'_5}\right]_{3\times3}=-[R_5]^T[\tilde{r}'_5]$$

$$\left[{}^{R}v_{5,y_2}\right]_{3\times24}=\left[[I]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[R_1]^T\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\right]$$

Body 6:

$$\left[{}^{R}v_{6,y_1}\right]_{3\times24}=\left[\left[{}^{R}v_{6,\hat{\omega}_1'}\right]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad\left[{}^{R}v_{6,\hat{\omega}_6'}\right]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\right]$$

$$\left[{}^{R}v_{6,\hat{\omega}_1'}\right]_{3\times3}=\left[-[R_1]^T\left([\tilde{q}_6']+[\tilde{s}_6']\right)-[R_6]^T[\tilde{r}_6']\left[{}^{1}R_6\right]\right] \qquad \left[{}^{R}v_{6,\hat{\omega}_6'}\right]_{3\times3}=-[R_6]^T[\tilde{r}_6']$$

$$\left[{}^{R}v_{6,y_2}\right]_{3\times24}=\left[[I]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[R_1]^T\quad[0]_{3\times3}\quad[0]_{3\times3}\right]$$

Body 7:

$$\left[{}^{R}v_{7,y_1}\right]_{3\times24}=\left[\left[{}^{R}v_{7,\hat{\omega}_1'}\right]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad\left[{}^{R}v_{7,\hat{\omega}_7'}\right]_{3\times3}\quad[0]_{3\times3}\right]$$

$$\left[{}^{R}v_{7,\hat{\omega}_1'}\right]_{3\times3}=\left[-[R_1]^T\left([\tilde{q}_7']+[\tilde{s}_7']\right)-[R_7]^T[\tilde{r}_7']\left[{}^{1}R_7\right]\right] \qquad \left[{}^{R}v_{7,\hat{\omega}_7'}\right]_{3\times3}=-[R_7]^T[\tilde{r}_7']$$

$$\left[{}^{R}v_{7,y_2}\right]_{3\times24}=\left[[I]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[R_1]^T\quad[0]_{3\times3}\right]$$

Body 8:

$$\left[{}^{R}v_{8,y_1}\right]_{3\times24}=\left[\left[{}^{R}v_{8,\hat{\omega}_1'}\right]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad\left[{}^{R}v_{8,\hat{\omega}_7'}\right]_{3\times3}\quad\left[{}^{R}v_{8,\hat{\omega}_8'}\right]_{3\times3}\right]$$

$$\left[{}^{R}v_{8,\hat{\omega}_1'}\right]_{3\times3}=\left[-[R_1]^T\left([\tilde{q}_7']+[\tilde{s}_7']\right)-[R_7]^T\left([\tilde{q}_8']+[\tilde{s}_8']\right)\left[{}^{1}R_7\right]-[R_8]^T[\tilde{r}_8']\left[{}^{1}R_8\right]\right]$$

$$\left[{}^{R}v_{8,\hat{\omega}_7'}\right]_{3\times3}=\left[-[R_7]^T\left([\tilde{q}_8']+[\tilde{s}_8']\right)-[R_8]^T[\tilde{r}_8']\left[{}^{7}R_8\right]\right] \qquad \left[{}^{R}v_{8,\hat{\omega}_8'}\right]_{3\times3}=-[R_8]^T[\tilde{r}_8']$$

$$\left[{}^{R}v_{8,y_2}\right]_{3\times24}=\left[[I]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[R_1]^T\quad[R_7]^T\right]$$

4.6 For the eight-body system described in Exercise 4.5, complete the following. a) For each body B in the system, find the ***body frame components*** of the ***partial velocity matrix*** of its ***origin*** O_B and its ***mass center*** G_B. Follow the procedures outlined in Equations (51)-(53) and Equation (55). Use the results presented in Unit 2 for the ***body frame components*** of the ***partial angular velocity matrices*** of the bodies. b) Use ***direct differentiation*** to find the ***body frame components*** of the ***velocity*** of G_8 in terms of the ***body frame components*** of the ***relative angular velocity*** vectors, ***identify*** the partial velocity matrix $\left[{}^{R}v_{8,y}'\right]_{3\times48}$, and compare the results with those obtained in part (a).

Answers:

Body Origins:

Body 1:

$$\left[{}^{R}v_{O_1,y_1}'\right]_{3\times24}=[0]_{3\times24}$$

$$\left[{}^{R}v_{O_1,y_2}'\right]_{3\times24}=\left[[R_1]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\right]$$

Body 2:

$$\left[{}^{R}v_{O_2,y_1}'\right]_{3\times24}=\left[\left[{}^{R}v_{O_2,\hat{\omega}_1'}'\right]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\quad[0]_{3\times3}\right]$$

$$\left[{}^{R}v'_{O_2,\hat{\omega}'_1}\right]_{3\times3} = -\left[{}^{1}R_2\right]\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)$$

$$\left[{}^{R}v'_{O_2,y_2}\right]_{3\times24} = \left[\left[R_2\right] \quad \left[{}^{1}R_2\right] \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]$$

Body 3:

$$\left[{}^{R}v'_{O_3,y_1}\right]_{3\times24} = \left[\left[{}^{R}v'_{O_3,\hat{\omega}'_1}\right]_{3\times3} \quad \left[{}^{R}v'_{O_3,\hat{\omega}'_2}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]$$

$$\left[{}^{R}v'_{O_3,\hat{\omega}'_1}\right]_{3\times3} = -\left[{}^{1}R_3\right]\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right) - \left[{}^{2}R_3\right]\left(\left[\tilde{q}'_3\right]+\left[\tilde{s}'_3\right]\right)\left[{}^{1}R_2\right]$$

$$\left[{}^{R}v_{O_3,\hat{\omega}'_2}\right]_{3\times3} = -\left[R_2\right]^{T}\left(\left[\tilde{q}'_3\right]+\left[\tilde{s}'_3\right]\right)$$

$$\left[{}^{R}v'_{O_3,y_2}\right]_{3\times24} = \left[\left[R_3\right] \quad \left[{}^{1}R_3\right] \quad \left[{}^{2}R_3\right] \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]$$

Body 4:

$$\left[{}^{R}v'_{O_4,y_1}\right]_{3\times24} = \left[\left[{}^{R}v'_{O_4,\hat{\omega}'_1}\right]_{3\times3} \quad \left[{}^{R}v'_{O_4,\hat{\omega}'_2}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]$$

$$\left[{}^{R}v'_{O_4,\hat{\omega}'_1}\right]_{3\times3} = -\left[{}^{1}R_4\right]\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right) - \left[{}^{2}R_4\right]\left(\left[\tilde{q}'_4\right]+\left[\tilde{s}'_4\right]\right)\left[{}^{1}R_2\right]$$

$$\left[{}^{R}v_{O_4,\hat{\omega}'_2}\right]_{3\times3} = -\left[R_2\right]^{T}\left(\left[\tilde{q}'_4\right]+\left[\tilde{s}'_4\right]\right)$$

$$\left[{}^{R}v'_{O_4,y_2}\right]_{3\times24} = \left[\left[R_4\right] \quad \left[{}^{1}R_4\right] \quad [0]_{3\times3} \quad \left[{}^{2}R_4\right] \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]$$

Body 5:

$$\left[{}^{R}v'_{O_5,y_1}\right]_{3\times24} = \left[\left[{}^{R}v'_{O_5,\hat{\omega}'_1}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]$$

$$\left[{}^{R}v'_{O_5,\hat{\omega}'_1}\right]_{3\times3} = -\left[{}^{1}R_5\right]\left(\left[\tilde{q}'_5\right]+\left[\tilde{s}'_5\right]\right)$$

$$\left[{}^{R}v'_{O_5,y_2}\right]_{3\times24} = \left[\left[R_5\right] \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad \left[{}^{1}R_5\right] \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]$$

Body 6:

$$\left[{}^{R}v'_{O_6,y_1}\right]_{3\times24} = \left[\left[{}^{R}v'_{O_6,\hat{\omega}'_1}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]$$

$$\left[{}^{R}v'_{O_6,\hat{\omega}'_1}\right]_{3\times3} = -\left[{}^{1}R_6\right]\left(\left[\tilde{q}'_6\right]+\left[\tilde{s}'_6\right]\right)$$

$$\left[{}^{R}v'_{O_6,y_2}\right]_{3\times24} = \left[\left[R_6\right] \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad \left[{}^{1}R_6\right] \quad [0]_{3\times3} \quad [0]_{3\times3}\right]$$

Body 7:

$$\left[{}^{R}v'_{O_7,y_1}\right]_{3\times24} = \left[\left[{}^{R}v'_{O_7,\hat{\omega}'_1}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3}\right]$$

$$\left[{}^{R}v'_{O_7,\hat{\omega}'_1}\right]_{3\times3} = -\left[{}^{1}R_7\right]\left(\left[\tilde{q}'_7\right]+\left[\tilde{s}'_7\right]\right)$$

$$\left[{}^{R}v'_{O_7,y_2}\right]_{3\times24} = \left[\left[R_7\right] \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad \left[{}^{1}R_7\right] \quad [0]_{3\times3}\right]$$

Body 8:

$$\left[{}^{R}v'_{O_8,y_1}\right]_{3\times24} = \left[\left[{}^{R}v'_{O_8,\hat{\omega}'_1}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad \left[{}^{R}v'_{O_8,\hat{\omega}'_7}\right]_{3\times3} \quad [0]_{3\times3}\right]$$

$$\left[{}^{R}v'_{O_8,\hat{\omega}'_1}\right]_{3\times3} = -\left[{}^{1}R_8\right]\left(\left[\tilde{q}'_7\right]+\left[\tilde{s}'_7\right]\right) - \left[{}^{7}R_8\right]\left(\left[\tilde{q}'_8\right]+\left[\tilde{s}'_8\right]\right)\left[{}^{1}R_7\right]$$

$$\left[{}^{R}v'_{O_8,\hat{\omega}'_7}\right]_{3\times3} = -\left[{}^{7}R_8\right]\left(\left[\tilde{q}'_8\right]+\left[\tilde{s}'_8\right]\right)$$

$$\left[{}^{R}v'_{O_8,y_2}\right]_{3\times 24}=\left[\left[R_8\right]\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad\left[{}^{1}R_8\right]\quad\left[{}^{7}R_8\right]\right]$$

Body Mass Centers:

Body 1:

$$\left[{}^{R}v'_{1,y_1}\right]_{3\times 24}=\left[\left[{}^{R}v'_{1,\hat{\omega}'_1}\right]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\right]_{3\times 24}$$

$$\left[{}^{R}v'_{1,\hat{\omega}'_1}\right]_{3\times 3}=-\left[\tilde{r}'_1\right]$$

$$\left[{}^{R}v'_{1,y_2}\right]_{3\times 24}=\left[\left[R_1\right]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\right]$$

Body 2:

$$\left[{}^{R}v'_{2,y_1}\right]_{3\times 24}=\left[\left[{}^{R}v'_{2,\hat{\omega}'_1}\right]_{3\times 3}\quad\left[{}^{R}v'_{2,\hat{\omega}'_2}\right]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\right]$$

$$\left[{}^{R}v'_{2,\hat{\omega}'_1}\right]_{3\times 3}=-\left[{}^{1}R_2\right]\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)-\left[\tilde{r}'_2\right]\left[{}^{1}R_2\right]\qquad\left[{}^{R}v'_{2,\hat{\omega}'_2}\right]_{3\times 3}=-\left[\tilde{r}'_2\right]$$

$$\left[{}^{R}v'_{2,y_2}\right]_{3\times 24}=\left[\left[R_2\right]\quad\left[{}^{1}R_2\right]\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\right]$$

Body 3:

$$\left[{}^{R}v'_{3,y_1}\right]_{3\times 24}=\left[\left[{}^{R}v'_{3,\hat{\omega}'_1}\right]_{3\times 3}\quad\left[{}^{R}v'_{3,\hat{\omega}'_2}\right]_{3\times 3}\quad\left[{}^{R}v'_{3,\hat{\omega}'_3}\right]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\right]$$

$$\left[{}^{R}v'_{3,\hat{\omega}'_1}\right]_{3\times 3}=-\left[{}^{1}R_3\right]\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)-\left[{}^{2}R_3\right]\left(\left[\tilde{q}'_3\right]+\left[\tilde{s}'_3\right]\right)\left[{}^{1}R_2\right]-\left[\tilde{r}'_3\right]\left[{}^{1}R_3\right]$$

$$\left[{}^{R}v'_{3,\hat{\omega}'_2}\right]_{3\times 3}=-\left[{}^{2}R_3\right]\left(\left[\tilde{q}'_3\right]+\left[\tilde{s}'_3\right]\right)-\left[\tilde{r}'_3\right]\left[{}^{2}R_3\right]\qquad\left[{}^{R}v'_{3,\hat{\omega}'_3}\right]=-\left[\tilde{r}'_3\right]$$

$$\left[{}^{R}v'_{3,y_2}\right]_{3\times 24}=\left[\left[R_3\right]\quad\left[{}^{1}R_3\right]\quad\left[{}^{2}R_3\right]\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\right]$$

Body 4:

$$\left[{}^{R}v'_{4,y_1}\right]_{3\times 24}=\left[\left[{}^{R}v'_{4,\hat{\omega}'_1}\right]_{3\times 3}\quad\left[{}^{R}v'_{4,\hat{\omega}'_2}\right]_{3\times 3}\quad[0]_{3\times 3}\quad\left[{}^{R}v'_{4,\hat{\omega}'_4}\right]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\right]$$

$$\left[{}^{R}v'_{4,\hat{\omega}'_1}\right]_{3\times 3}=-\left[{}^{1}R_4\right]\left(\left[\tilde{q}'_2\right]+\left[\tilde{s}'_2\right]\right)-\left[{}^{2}R_4\right]\left(\left[\tilde{q}'_4\right]+\left[\tilde{s}'_4\right]\right)\left[{}^{1}R_2\right]-\left[\tilde{r}'_4\right]\left[{}^{1}R_4\right]$$

$$\left[{}^{R}v'_{4,\hat{\omega}'_2}\right]_{3\times 3}=-\left[{}^{2}R_4\right]\left(\left[\tilde{q}'_4\right]+\left[\tilde{s}'_4\right]\right)-\left[\tilde{r}'_4\right]\left[{}^{2}R_4\right]\qquad\left[{}^{R}v'_{4,\hat{\omega}'_4}\right]=-\left[\tilde{r}'_4\right]$$

$$\left[{}^{R}v'_{4,y_2}\right]_{3\times 24}=\left[\left[R_4\right]\quad\left[{}^{1}R_4\right]\quad[0]_{3\times 3}\quad\left[{}^{2}R_4\right]\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\right]$$

Body 5:

$$\left[{}^{R}v'_{5,y_1}\right]_{3\times 24}=\left[\left[{}^{R}v'_{5,\hat{\omega}'_1}\right]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad\left[{}^{R}v'_{5,\hat{\omega}'_5}\right]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\right]$$

$$\left[{}^{R}v'_{5,\hat{\omega}'_1}\right]_{3\times 3}=-\left[{}^{1}R_5\right]\left(\left[\tilde{q}'_5\right]+\left[\tilde{s}'_5\right]\right)-\left[\tilde{r}'_5\right]\left[{}^{1}R_5\right]\qquad\left[{}^{R}v'_{5,\hat{\omega}'_5}\right]_{3\times 3}=-\left[\tilde{r}'_5\right]$$

$$\left[{}^{R}v'_{5,y_2}\right]_{3\times 24}=\left[\left[R_5\right]\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad\left[{}^{1}R_5\right]\quad[0]_{3\times 3}\quad[0]_{3\times 3}\quad[0]_{3\times 3}\right]$$

Body 6:

$$\left[{}^R v'_{6,y_1}\right]_{3\times24} = \left[\left[{}^R v'_{6,\hat\omega'_1}\right]_{3\times3}\quad [0]_{3\times3}\quad [0]_{3\times3}\quad [0]_{3\times3}\quad [0]_{3\times3}\quad \left[{}^R v'_{6,\hat\omega'_6}\right]_{3\times3}\quad [0]_{3\times3}\quad [0]_{3\times3}\right]$$

$$\left[{}^R v'_{6,\hat\omega'_1}\right]_{3\times3} = -\left[{}^1R_6\right]\left(\left[\tilde q'_6\right]+\left[\tilde s'_6\right]\right)-\left[\tilde r'_6\right]\left[{}^1R_6\right] \qquad \left[{}^R v'_{6,\hat\omega'_6}\right]_{3\times3} = -\left[\tilde r'_6\right]$$

$$\left[{}^R v'_{6,y_2}\right]_{3\times24} = \left[\left[R_6\right]\quad [0]_{3\times3}\quad [0]_{3\times3}\quad [0]_{3\times3}\quad [0]_{3\times3}\quad \left[{}^1R_6\right]\quad [0]_{3\times3}\quad [0]_{3\times3}\right]$$

Body 7:

$$\left[{}^R v'_{7,y_1}\right]_{3\times24} = \left[\left[{}^R v'_{7,\hat\omega'_1}\right]_{3\times3}\quad [0]_{3\times3}\quad [0]_{3\times3}\quad [0]_{3\times3}\quad [0]_{3\times3}\quad [0]_{3\times3}\quad \left[{}^R v'_{7,\hat\omega'_7}\right]_{3\times3}\quad [0]_{3\times3}\right]$$

$$\left[{}^R v'_{7,\hat\omega'_1}\right]_{3\times3} = -\left[{}^1R_7\right]\left(\left[\tilde q'_7\right]+\left[\tilde s'_7\right]\right)-\left[\tilde r'_7\right]\left[{}^1R_7\right] \qquad \left[{}^R v'_{7,\hat\omega'_7}\right]_{3\times3} = -\left[\tilde r'_7\right]$$

$$\left[{}^R v'_{7,y_2}\right]_{3\times24} = \left[\left[R_7\right]\quad [0]_{3\times3}\quad [0]_{3\times3}\quad [0]_{3\times3}\quad [0]_{3\times3}\quad [0]_{3\times3}\quad \left[{}^1R_7\right]\quad [0]_{3\times3}\right]$$

Body 8:

$$\left[{}^R v'_{8,y_1}\right]_{3\times24} = \left[\left[{}^R v'_{8,\hat\omega'_1}\right]_{3\times3}\quad [0]_{3\times3}\quad [0]_{3\times3}\quad [0]_{3\times3}\quad [0]_{3\times3}\quad [0]_{3\times3}\quad \left[{}^R v'_{8,\hat\omega'_7}\right]_{3\times3}\quad \left[{}^R v'_{8,\hat\omega'_8}\right]_{3\times3}\right]$$

$$\left[{}^R v'_{8,\hat\omega'_1}\right]_{3\times3} = -\left[{}^1R_8\right]\left(\left[\tilde q'_7\right]+\left[\tilde s'_7\right]\right)-\left[{}^7R_8\right]\left(\left[\tilde q'_8\right]+\left[\tilde s'_8\right]\right)\left[{}^1R_7\right]-\left[\tilde r'_8\right]\left[{}^1R_8\right]$$

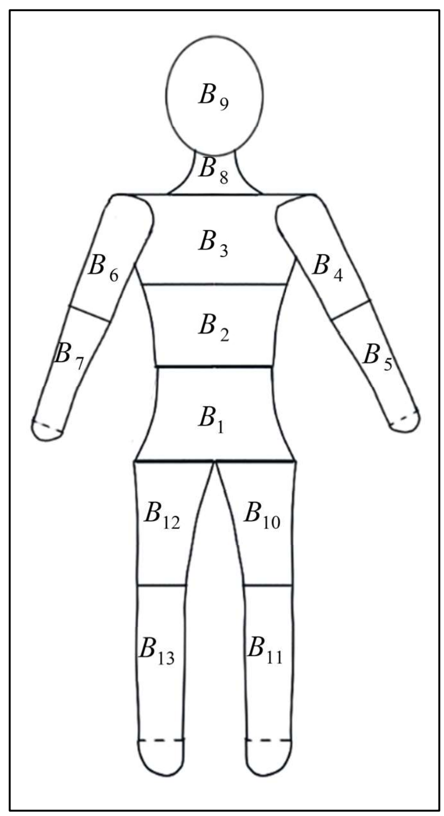
$$\left[{}^R v'_{8,\hat\omega'_7}\right]_{3\times3} = -\left[{}^7R_8\right]\left(\left[\tilde q'_8\right]+\left[\tilde s'_8\right]\right)-\left[\tilde r'_8\right]\left[{}^7R_8\right] \qquad \left[{}^R v'_{8,\hat\omega'_8}\right]_{3\times3} = -\left[\tilde r'_8\right]$$

$$\left[{}^R v'_{8,y_2}\right]_{3\times24} = \left[\left[R_8\right]\quad [0]_{3\times3}\quad [0]_{3\times3}\quad [0]_{3\times3}\quad [0]_{3\times3}\quad [0]_{3\times3}\quad \left[{}^1R_8\right]\quad \left[{}^7R_8\right]\right]$$

4.7 The figure shows a thirteen-body model of the human body numbered using the numbering scheme presented in Unit 1. Body 1 is the ***lower torso***, and it is the ***system reference body***. The rest of the bodies are numbered in ascending progression outward along the branches. As structured, the lower numbered body array for the system is as follows.

$$\mathcal{L}(1,\ldots,13) = (0,1,2,3,4,3,6,3,8,1,10,1,12)$$

The position and orientation of body 1 is defined ***relative*** to the ***fixed frame*** $R:\left(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3\right)$, and the orientations of all the other bodies are defined ***relative*** to their ***adjacent***, ***lower numbered bodies***. Body 1 can ***translate*** and ***rotate*** relative to the fixed frame, but ***all other bodies*** can ***only rotate*** relative to their lower numbered bodies, not translate. Using ***base frame components*** of the ***relative angular velocities*** of the bodies, the ***system generalized speed matrix*** is defined as follows.



$$\{y\}_{42\times1} \triangleq \begin{Bmatrix} \{y_1\}_{39\times1} \\ \{y_2\}_{3\times1} \end{Bmatrix}$$

$$\{y_1\}_{39\times1} \triangleq \begin{bmatrix} \hat{\omega}_{11} & \hat{\omega}_{12} & \hat{\omega}_{13} & \hat{\omega}_{21} & \hat{\omega}_{22} & \hat{\omega}_{23} & \cdots & \hat{\omega}_{11,1} & \hat{\omega}_{11,2} & \hat{\omega}_{11,3} & \hat{\omega}_{12,1} & \hat{\omega}_{12,2} & \hat{\omega}_{12,3} \end{bmatrix}^T$$

$$\{y_2\}_{3\times1} \triangleq \begin{bmatrix} \dot{s}'_{11} & \dot{s}'_{12} & \dot{s}'_{13} \end{bmatrix}^T$$

a) For body 7 (lower right arm), find the ***fixed frame components*** of the ***partial velocity matrix*** of its ***origin*** O_7 and its ***mass center*** G_7. Follow the procedures outlined in Equations (34)-(36) and Equation (45). Use the results presented in Unit 2 for the ***fixed frame components*** of the ***partial angular velocity matrices*** of the bodies. b) Use ***direct differentiation*** to find the ***fixed frame components*** of the ***velocity*** of G_7 in terms of the ***base frame components*** of the ***relative angular velocity*** vectors, ***identify*** the partial velocity matrix $\left[{}^R v_{7,y}\right]_{3\times42}$, and compare the results with those obtained in part (a). Assume that $s'_{Bi}\ (B=2,\ldots 13;\ i=1,2,3)$ and $\dot{s}'_{Bi}\ (B=2,\ldots 13;\ i=1,2,3)$ are all ***zero***.

Answers:

Body Origins:

$$\left[{}^R v_{O_1,y_1}\right]_{3\times39} = [0]_{3\times39} \qquad \left[{}^R v_{O_1,y_2}\right]_{3\times3} = [I]_{3\times3}$$

$$\left[{}^R v_{O_2,y_1}\right]_{3\times39} = \left[\left[{}^R v_{O_2,\hat{\omega}_1}\right]_{3\times3} \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3}\right] \qquad \left[{}^R v_{O_2,\hat{\omega}_1}\right]_{3\times3} = -[\tilde{q}_2] \qquad \left[{}^R v_{O_2,y_2}\right]_{3\times3} = [I]_{3\times3}$$

$$\left[{}^R v_{O_3,y_1}\right]_{3\times39} = \left[\left[{}^R v_{O_3,\hat{\omega}_1}\right]_{3\times3} \quad \left[{}^R v_{O_3,\hat{\omega}_2}\right]_{3\times3} \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3}\right]$$

$$\left[{}^R v_{O_3,\hat{\omega}_1}\right]_{3\times3} = -\left([\tilde{q}_2]+[\tilde{q}_3]\right) \qquad \left[{}^R v_{O_3,\hat{\omega}_2}\right]_{3\times3} = -[\tilde{q}_3][R_1]^T \qquad \left[{}^R v_{O_3,y_2}\right]_{3\times3} = [I]_{3\times3}$$

$$\left[{}^R v_{O_6,y_1}\right]_{3\times39} = \left[\left[{}^R v_{O_6,\hat{\omega}_1}\right]_{3\times3} \quad \left[{}^R v_{O_6,\hat{\omega}_2}\right]_{3\times3} \quad \left[{}^R v_{O_6,\hat{\omega}_3}\right]_{3\times3} \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3}\right]$$

$$\left[{}^R v_{O_6,\hat{\omega}_1}\right]_{3\times3} = -\left([\tilde{q}_2]+[\tilde{q}_3]+[\tilde{q}_6]\right) \qquad \left[{}^R v_{O_6,\hat{\omega}_2}\right]_{3\times3} = -\left([\tilde{q}_3]+[\tilde{q}_6]\right)[R_1]^T \qquad \left[{}^R v_{O_6,\hat{\omega}_3}\right]_{3\times3} = -[\tilde{q}_6][R_2]^T$$

$$\left[{}^R v_{O_6,y_2}\right]_{3\times3} = [I]_{3\times3}$$

$$\left[{}^R v_{O_7,y_1}\right]_{3\times39} = \left[\left[{}^R v_{O_7,\hat{\omega}_1}\right]_{3\times3} \quad \left[{}^R v_{O_7,\hat{\omega}_2}\right]_{3\times3} \quad \left[{}^R v_{O_7,\hat{\omega}_3}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad \left[{}^R v_{O_7,\hat{\omega}_6}\right]_{3\times3} \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3}\right]$$

$$\left[{}^R v_{O_7,\hat{\omega}_1}\right]_{3\times3} = -\left([\tilde{q}_2]+[\tilde{q}_3]+[\tilde{q}_6]+[\tilde{q}_7]\right) \qquad \left[{}^R v_{O_7,\hat{\omega}_2}\right]_{3\times3} = -\left([\tilde{q}_3]+[\tilde{q}_6]+[\tilde{q}_7]\right)[R_1]^T$$

$$\left[{}^R v_{O_7,\hat{\omega}_3}\right]_{3\times3} = -\left([\tilde{q}_6]+[\tilde{q}_7]\right)[R_2]^T \qquad \left[{}^R v_{O_7,\hat{\omega}_6}\right]_{3\times3} = -[\tilde{q}_7][R_3]^T \qquad \left[{}^R v_{O_7,y_2}\right]_{3\times3} = [I]_{3\times3}$$

Mass Center G_7:

$$\left[{}^R v_{7,y_1}\right]_{3\times39} = \left[\left[{}^R v_{7,\hat{\omega}_1}\right]_{3\times3} \quad \left[{}^R v_{7,\hat{\omega}_2}\right]_{3\times3} \quad \left[{}^R v_{7,\hat{\omega}_3}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad \left[{}^R v_{7,\hat{\omega}_6}\right]_{3\times3} \quad \left[{}^R v_{7,\hat{\omega}_7}\right]_{3\times3} \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3}\right]$$

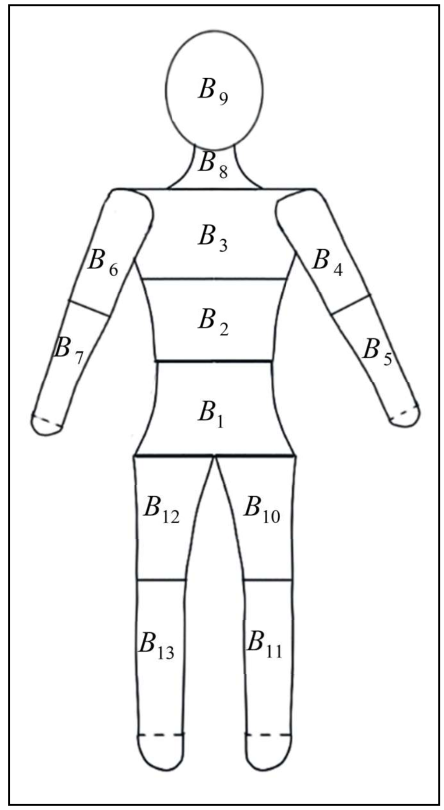
$$\left[{}^R v_{7,\hat{\omega}_1}\right]_{3\times3} = -\left([\tilde{q}_2]+[\tilde{q}_3]+[\tilde{q}_6]+[\tilde{q}_7]+[\tilde{r}_7]\right) \qquad \left[{}^R v_{7,\hat{\omega}_2}\right]_{3\times3} = -\left([\tilde{q}_3]+[\tilde{q}_6]+[\tilde{q}_7]+[\tilde{r}_7]\right)[R_1]^T$$

$$\left[{}^R v_{7,\hat{\omega}_3}\right]_{3\times3} = -\left([\tilde{q}_6]+[\tilde{q}_7]+[\tilde{r}_7]\right)[R_2]^T \qquad \left[{}^R v_{7,\hat{\omega}_6}\right]_{3\times3} = -\left([\tilde{q}_7]+[\tilde{r}_7]\right)[R_3]^T$$

$$\left[{}^R v_{7,\hat{\omega}_7}\right]_{3\times3} = -[\tilde{r}_7][R_6]^T \qquad \left[{}^R v_{7,y_2}\right]_{3\times3} = [I]_{3\times3}$$

4.8 The figure shows a thirteen-body model of the human body numbered using the numbering scheme presented in Unit 1. Body 1 is the ***lower torso***, and it is the ***system reference body***. The rest of the bodies are numbered in ascending progression outward along the branches. As structured, the lower numbered body array for the system is as follows.

$$\mathcal{L}(1,\ldots,13)=(0,1,2,3,4,3,6,3,8,1,10,1,12)$$



The position and orientation of body 1 is defined ***relative*** to the ***fixed frame*** $R:\left(\underset{\sim}{N}_1,\underset{\sim}{N}_2,\underset{\sim}{N}_3\right)$, and the orientations of all the other bodies are defined ***relative*** to their ***adjacent***, ***lower numbered bodies***. Body 1 can ***translate*** and ***rotate*** relative to the fixed frame, but ***all other bodies*** can ***only rotate*** relative to their lower numbered bodies, not translate. Using ***body frame components*** of the ***relative angular velocities*** of the bodies, the ***system generalized speed matrix*** is defined as follows.

$$\{y\}_{42\times1}\triangleq\begin{Bmatrix}\{y_1\}_{39\times1}\\\{y_2\}_{3\times1}\end{Bmatrix}$$

$$\{y_1\}_{39\times1}\triangleq\begin{bmatrix}\hat{\omega}'_{11} & \hat{\omega}'_{12} & \hat{\omega}'_{13} & \hat{\omega}'_{21} & \hat{\omega}'_{22} & \hat{\omega}'_{23} & \cdots & \hat{\omega}'_{11,1} & \hat{\omega}'_{11,2} & \hat{\omega}'_{11,3} & \hat{\omega}'_{12,1} & \hat{\omega}'_{12,2} & \hat{\omega}'_{12,3}\end{bmatrix}^T$$

$$\{y_2\}_{3\times1}\triangleq\begin{bmatrix}\dot{s}'_{11} & \dot{s}'_{12} & \dot{s}'_{13}\end{bmatrix}^T$$

a) For body 7 (lower right arm), find the ***fixed frame components*** of the ***partial velocity matrix*** of its ***origin*** O_7 and its ***mass center*** G_7. Follow the procedures outlined in Equations (40)-(42) and Equation (48). Use the results presented in Unit 2 for the ***body frame components*** of the ***partial angular velocity matrices*** of the bodies. b) Use ***direct differentiation*** to find the ***fixed frame components*** of the ***velocity*** of G_7 in terms of the ***body frame components*** of the ***relative angular velocity*** vectors, ***identify*** the partial velocity matrix $\left[{}^R v_{7,y}\right]_{3\times42}$, and compare the results with those obtained in part (a). Assume that $s'_{Bi}\ (B=2,\ldots 13;\ i=1,2,3)$ and $\dot{s}'_{Bi}\ (B=2,\ldots 13;\ i=1,2,3)$ are all ***zero***.

Answers:

Body Origins:

$$\left[{}^R v_{O_1,y_1}\right]_{3\times39}=[0]_{3\times39} \qquad \left[{}^R v_{O_1,y_2}\right]_{3\times3}=[I]_{3\times3}$$

$$\left[{}^R v_{O_2,y_1}\right]_{3\times39}=\left[\left[{}^R v_{O_2,\hat{\omega}'_1}\right]_{3\times3}\quad[0]_{3\times3}\quad\cdots\quad[0]_{3\times3}\right] \qquad \left[{}^R v_{O_2,\hat{\omega}'_1}\right]_{3\times3}=-[R_1]^T\left[\tilde{q}'_2\right] \qquad \left[{}^R v_{O_2,y_2}\right]_{3\times3}=[I]_{3\times3}$$

$$\left[{}^R v_{O_3,y_1}\right]_{3\times39}=\left[\left[{}^R v_{O_3,\hat{\omega}'_1}\right]_{3\times3}\quad\left[{}^R v_{O_3,\hat{\omega}'_2}\right]_{3\times3}\quad[0]_{3\times3}\quad\cdots\quad[0]_{3\times3}\right]$$

$$\left[{}^R v_{O_3,\hat{\omega}'_1}\right]_{3\times3}=-[R_1]^T\left[\tilde{q}'_2\right]-[R_2]^T\left[\tilde{q}'_3\right]\left[{}^1R_2\right] \qquad \left[{}^R v_{O_3,\hat{\omega}'_2}\right]_{3\times3}=-[R_2]^T\left[\tilde{q}'_3\right] \qquad \left[{}^R v_{O_3,y_2}\right]_{3\times3}=[I]_{3\times3}$$

$$\left[{}^R v_{O_6,y_1}\right]_{3\times 39} = \left[\left[{}^R v_{O_6,\hat{\omega}_1'}\right]_{3\times3} \quad \left[{}^R v_{O_6,\hat{\omega}_2'}\right]_{3\times3} \quad \left[{}^R v_{O_6,\hat{\omega}_3'}\right]_{3\times3} \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3}\right]$$

$$\left[{}^R v_{O_6,\hat{\omega}_1'}\right]_{3\times3} = -[R_1]^T[\tilde{q}_2'] - [R_2]^T[\tilde{q}_3']\left[{}^1R_2\right] - [R_3]^T[\tilde{q}_6']\left[{}^1R_3\right]$$

$$\left[{}^R v_{O_6,\hat{\omega}_2'}\right]_{3\times3} = -[R_2]^T[\tilde{q}_3'] - [R_3]^T[\tilde{q}_6']\left[{}^2R_3\right] \qquad \left[{}^R v_{O_6,\hat{\omega}_3'}\right]_{3\times3} = -[R_3]^T[\tilde{q}_6'] \qquad \left[{}^R v_{O_6,y_2}\right]_{3\times3} = [I]_{3\times3}$$

$$\left[{}^R v_{O_7,y_1}\right]_{3\times 39} = \left[\left[{}^R v_{O_7,\hat{\omega}_1'}\right]_{3\times3} \quad \left[{}^R v_{O_7,\hat{\omega}_2'}\right]_{3\times3} \quad \left[{}^R v_{O_7,\hat{\omega}_3'}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad \left[{}^R v_{O_7,\hat{\omega}_6'}\right]_{3\times3} \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3}\right]$$

$$\left[{}^R v_{O_7,\hat{\omega}_1'}\right]_{3\times3} = -[R_1]^T[\tilde{q}_2'] - [R_2]^T[\tilde{q}_3']\left[{}^1R_2\right] - [R_3]^T[\tilde{q}_6']\left[{}^1R_3\right] - [R_6]^T[\tilde{q}_7']\left[{}^1R_6\right]$$

$$\left[{}^R v_{O_7,\hat{\omega}_2'}\right]_{3\times3} = -[R_2]^T[\tilde{q}_3'] - [R_3]^T[\tilde{q}_6']\left[{}^2R_3\right] - [R_6]^T[\tilde{q}_7']\left[{}^2R_6\right]$$

$$\left[{}^R v_{O_7,\hat{\omega}_3'}\right]_{3\times3} = -[R_3]^T[\tilde{q}_6'] - [R_6]^T[\tilde{q}_7']\left[{}^3R_6\right] \qquad \left[{}^R v_{O_7,\hat{\omega}_6'}\right]_{3\times3} = -[R_6]^T[\tilde{q}_7'] \qquad \left[{}^R v_{O_7,y_2}\right]_{3\times3} = [I]_{3\times3}$$

Body 7 Mass Center G_7:

$$\left[{}^R v_{7,y_1}\right]_{3\times 39} = \left[\left[{}^R v_{7,\hat{\omega}_1'}\right]_{3\times3} \quad \left[{}^R v_{7,\hat{\omega}_2'}\right]_{3\times3} \quad \left[{}^R v_{7,\hat{\omega}_3'}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad \left[{}^R v_{7,\hat{\omega}_6'}\right]_{3\times3} \quad \left[{}^R v_{7,\hat{\omega}_7'}\right]_{3\times3} \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3}\right]$$

$$\left[{}^R v_{7,\hat{\omega}_1'}\right]_{3\times3} = -[R_1]^T[\tilde{q}_2'] - [R_2]^T[\tilde{q}_3']\left[{}^1R_2\right] - [R_3]^T[\tilde{q}_6']\left[{}^1R_3\right] - [R_6]^T[\tilde{q}_7']\left[{}^1R_6\right] - [R_7]^T[\tilde{r}_7']\left[{}^1R_7\right]$$

$$\left[{}^R v_{7,\hat{\omega}_2'}\right]_{3\times3} = -[R_2]^T[\tilde{q}_3'] - [R_3]^T[\tilde{q}_6']\left[{}^2R_3\right] - [R_6]^T[\tilde{q}_7']\left[{}^2R_6\right] - [R_7]^T[\tilde{r}_7']\left[{}^2R_7\right]$$

$$\left[{}^R v_{7,\hat{\omega}_3'}\right]_{3\times3} = -[R_3]^T[\tilde{q}_6'] - [R_6]^T[\tilde{q}_7']\left[{}^3R_6\right] - [R_7]^T[\tilde{r}_7']\left[{}^3R_7\right]$$

$$\left[{}^R v_{7,\hat{\omega}_6'}\right]_{3\times3} = -[R_6]^T[\tilde{q}_7'] - [R_7]^T[\tilde{r}_7']\left[{}^6R_7\right] \qquad \left[{}^R v_{7,\hat{\omega}_7'}\right]_{3\times3} = -[R_7]^T[\tilde{r}_7'] \qquad \left[{}^R v_{7,y_2}\right]_{3\times3} = [I]_{3\times3}$$

4.9 For the human body model described in Exercise 4.8, complete the following. a) For body 7 (lower right arm), find the ***body frame components*** of the ***partial velocity matrix*** of its ***origin*** O_7 and its ***mass center*** G_7. Follow the procedures outlined in Equations (51)-(53) and Equation (55). Use the results presented in Unit 2 for the ***body frame components*** of the ***partial angular velocity matrices*** of the bodies. b) Use ***direct differentiation*** to find the ***fixed frame components*** of the ***velocity*** of G_7 in terms of the ***body frame components*** of the ***relative angular velocity*** vectors, ***identify*** the partial velocity matrix $\left[{}^R v_{7,y}'\right]_{3\times 42}$, and compare the results with those obtained in part (a). Assume that $s_{Bi}'\ (B = 2,\ldots 13;\ i = 1,2,3)$ and $\dot{s}_{Bi}'\ (B = 2,\ldots 13;\ i = 1,2,3)$ are all ***zero***.

Answers:

Body Origins:

$$\left[{}^R v_{O_1,y}'\right]_{3\times 6N} = [0]_{3\times 39} \qquad \left[{}^R v_{O_1,y_2}'\right]_{3\times3} = [R_1]$$

$$\left[{}^R v_{O_2,y_1}'\right]_{3\times 39} = \left[\left[{}^R v_{O_2,\hat{\omega}_1'}'\right]_{3\times3} \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3}\right] \qquad \left[{}^R v_{O_2,\hat{\omega}_1'}'\right]_{3\times3} = -\left[{}^1R_2\right][\tilde{q}_2'] \qquad \left[{}^R v_{O_2,y_2}'\right]_{3\times3} = [R_2]$$

$$\left[{}^R v'_{O_3,y_1}\right]_{3\times39} = \left[\left[{}^R v'_{O_3,\hat{\omega}'_1}\right]_{3\times3} \quad \left[{}^R v'_{O_3,\hat{\omega}'_2}\right]_{3\times3} \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3}\right]$$

$$\left[{}^R v'_{O_3,\hat{\omega}'_1}\right]_{3\times3} = -\left[{}^1 R_3\right]\left[\tilde{q}'_2\right] - \left[{}^2 R_3\right]\left[\tilde{q}'_3\right]\left[{}^1 R_2\right] \qquad \left[{}^R v'_{O_3,\hat{\omega}'_2}\right]_{3\times3} = -\left[{}^2 R_3\right]\left[\tilde{q}'_3\right] \qquad \left[{}^R v'_{O_3,y_2}\right]_{3\times3} = \left[R_3\right]$$

$$\left[{}^R v'_{O_6,y_1}\right]_{3\times39} = \left[\left[{}^R v'_{O_6,\hat{\omega}'_1}\right]_{3\times3} \quad \left[{}^R v'_{O_6,\hat{\omega}'_2}\right]_{3\times3} \quad \left[{}^R v'_{O_6,\hat{\omega}'_3}\right]_{3\times3} \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3}\right]$$

$$\left[{}^R v'_{O_6,\hat{\omega}'_1}\right]_{3\times3} = -\left[{}^1 R_6\right]\left[\tilde{q}'_2\right] - \left[{}^2 R_6\right]\left[\tilde{q}'_3\right]\left[{}^1 R_2\right] - \left[{}^3 R_6\right]\left[\tilde{q}'_6\right]\left[{}^1 R_3\right]$$

$$\left[{}^R v'_{O_6,\hat{\omega}'_2}\right]_{3\times3} = -\left[{}^2 R_6\right]\left[\tilde{q}'_3\right] - \left[{}^3 R_6\right]\left[\tilde{q}'_6\right]\left[{}^2 R_3\right] \qquad \left[{}^R v'_{O_6,\hat{\omega}'_3}\right]_{3\times3} = -\left[{}^3 R_6\right]\left[\tilde{q}'_6\right] \qquad \left[{}^R v'_{O_6,y_2}\right]_{3\times3} = \left[R_6\right]$$

$$\left[{}^R v'_{O_7,y_1}\right]_{3\times39} = \left[\left[{}^R v'_{O_7,\hat{\omega}'_1}\right]_{3\times3} \quad \left[{}^R v'_{O_7,\hat{\omega}'_2}\right]_{3\times3} \quad \left[{}^R v'_{O_7,\hat{\omega}'_3}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad \left[{}^R v'_{O_7,\hat{\omega}'_6}\right]_{3\times3} \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3}\right]$$

$$\left[{}^R v'_{O_7,\hat{\omega}'_1}\right]_{3\times3} = -\left[{}^1 R_7\right]\left[\tilde{q}'_2\right] - \left[{}^2 R_7\right]\left[\tilde{q}'_3\right]\left[{}^1 R_2\right] - \left[{}^3 R_7\right]\left[\tilde{q}'_6\right]\left[{}^1 R_3\right] - \left[{}^6 R_7\right]\left[\tilde{q}'_7\right]\left[{}^1 R_6\right]$$

$$\left[{}^R v'_{O_7,\hat{\omega}'_2}\right]_{3\times3} = -\left[{}^2 R_7\right]\left[\tilde{q}'_3\right] - \left[{}^3 R_7\right]\left[\tilde{q}'_6\right]\left[{}^2 R_3\right] - \left[{}^6 R_7\right]\left[\tilde{q}'_7\right]\left[{}^2 R_6\right]$$

$$\left[{}^R v'_{O_7,\hat{\omega}'_3}\right]_{3\times3} = -\left[{}^3 R_7\right]\left[\tilde{q}'_6\right] - \left[{}^6 R_7\right]\left[\tilde{q}'_7\right]\left[{}^3 R_6\right] \qquad \left[{}^R v'_{O_7,\hat{\omega}'_6}\right]_{3\times3} = -\left[{}^6 R_7\right]\left[\tilde{q}'_7\right] \qquad \left[{}^R v'_{O_7,y_2}\right]_{3\times3} = \left[R_7\right]$$

Body 7 Mass Center G_7:

$$\left[{}^R v'_{7,y_1}\right]_{3\times39} = \left[\left[{}^R v'_{7,\hat{\omega}'_1}\right]_{3\times3} \quad \left[{}^R v'_{7,\hat{\omega}'_2}\right]_{3\times3} \quad \left[{}^R v'_{7,\hat{\omega}'_3}\right]_{3\times3} \quad [0]_{3\times3} \quad [0]_{3\times3} \quad \left[{}^R v'_{7,\hat{\omega}'_6}\right]_{3\times3} \quad \left[{}^R v'_{7,\hat{\omega}'_7}\right]_{3\times3} \quad [0]_{3\times3} \quad \cdots \quad [0]_{3\times3}\right]$$

$$\left[{}^R v'_{7,\hat{\omega}'_1}\right]_{3\times3} = -\left[{}^1 R_7\right]\left[\tilde{q}'_2\right] - \left[{}^2 R_7\right]\left[\tilde{q}'_3\right]\left[{}^1 R_2\right] - \left[{}^3 R_7\right]\left[\tilde{q}'_6\right]\left[{}^1 R_3\right] - \left[{}^6 R_7\right]\left[\tilde{q}'_7\right]\left[{}^1 R_6\right] - \left[\tilde{r}'_7\right]\left[{}^1 R_7\right]$$

$$\left[{}^R v'_{7,\hat{\omega}'_2}\right]_{3\times3} = -\left[{}^2 R_7\right]\left[\tilde{q}'_3\right] - \left[{}^3 R_7\right]\left[\tilde{q}'_6\right]\left[{}^2 R_3\right] - \left[{}^6 R_7\right]\left[\tilde{q}'_7\right]\left[{}^2 R_6\right] - \left[\tilde{r}'_7\right]\left[{}^2 R_7\right]$$

$$\left[{}^R v'_{7,\hat{\omega}'_3}\right]_{3\times3} = -\left[{}^3 R_7\right]\left[\tilde{q}'_6\right] - \left[{}^6 R_7\right]\left[\tilde{q}'_7\right]\left[{}^3 R_6\right] - \left[\tilde{r}'_7\right]\left[{}^3 R_7\right]$$

$$\left[{}^R v'_{7,\hat{\omega}'_6}\right]_{3\times3} = -\left[{}^6 R_7\right]\left[\tilde{q}'_7\right] - \left[\tilde{r}'_7\right]\left[{}^6 R_7\right] \qquad \left[{}^R v'_{7,\hat{\omega}'_7}\right]_{3\times3} = -\left[\tilde{r}'_7\right] \qquad \left[{}^R v'_{7,y_2}\right]_{3\times3} = \left[R_7\right]$$

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