

An Introduction to Three-Dimensional, Rigid Body Dynamics

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Volume III: Introduction to Multibody Kinematics

Unit 5

Acceleration

Summary

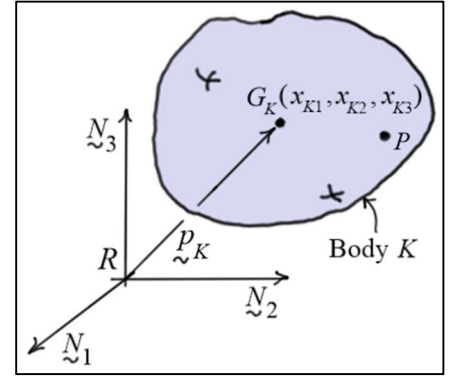
This unit focuses on *matrix-based* calculations of *acceleration*. The accelerations are expressed in terms of *partial velocity matrices* and their *time derivatives*. Both *base frame* and *body frame* components of *velocity* and *relative angular velocity vectors* are used in the calculations. Both *absolute* and *relative coordinates* are discussed. Both *orientation angle derivatives* and *angular velocity components* can be used as *generalized speeds*. Algorithms are developed for the *efficient* calculation of these quantities for multibody systems.

Explicit results are generated for some *examples* with the *purpose* of being *clear* about *how* the calculations are done. However, keep in mind that the *goal* of developing such procedures is to *implement* them into *computer algorithms*.

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Mass Center Accelerations Using Absolute Coordinates

The diagram shows a rigid body K whose orientation is given by a set of three **orientation angles** θ_{Ki} ($i=1,2,3$) and whose mass center G_K has **Cartesian coordinates** x_{Ki} ($i=1,2,3$). Both the orientation angles and the Cartesian coordinates are measured **relative** to the **fixed frame** R . The components of $\underline{p}_K$ the position vector of G_K , ${}^R\underline{v}_G$ the velocity of G_K , and ${}^R\underline{a}_G$ the acceleration of G_K in R can be written in matrix form as follows.



$$\left\{ \underline{p}_K \right\} = \begin{Bmatrix} x_{K1} \\ x_{K2} \\ x_{K3} \end{Bmatrix} \quad \left\{ \underline{v}_K \right\} = \begin{Bmatrix} \dot{x}_{K1} \\ \dot{x}_{K2} \\ \dot{x}_{K3} \end{Bmatrix} \quad \left\{ \underline{a}_K \right\} = \begin{Bmatrix} \ddot{x}_{K1} \\ \ddot{x}_{K2} \\ \ddot{x}_{K3} \end{Bmatrix} \quad (1)$$

As noted in Unit 4, if $\{\theta_K\}$ as a 3×1 matrix whose elements are the orientation angles θ_{Ki} ($i=1,2,3$), and $\{x_K\}$ as a 3×1 matrix whose elements are the mass center position coordinates x_{Ki} ($i=1,2,3$), then the velocity component matrix $\{v_K\}$ can be written as

$$\{v_K\} = \left[{}^R v_{K,\dot{\theta}_K} \right] \{\dot{\theta}_K\} + \left[{}^R v_{K,\dot{x}_K} \right] \{\dot{x}_K\} \quad (2)$$

Here, $\left[{}^R v_{K,\dot{\theta}_K} \right]$ and $\left[{}^R v_{K,\dot{x}_K} \right]$ are the **partial velocity matrices** associated with the **time derivatives** of the **orientation angles** and the **mass center position coordinates**, respectively. These matrices are easily identified by comparing Equations (1) and (2) to be

$$\left[{}^R v_{K,\dot{\theta}_K} \right] = [0]_{3 \times 3} \quad \left[{}^R v_{K,\dot{x}_K} \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3)$$

Accelerations of Other Points

As noted in Unit 4, the **fixed frame components** of the **velocities** of **other points** in body K , such as point P in the diagram above, can be written in terms of the **fixed frame components** of ${}^R\underline{v}_G$ the velocity of mass center G as follows.

$$\{v_P\} = \{v_G\} + [\tilde{\omega}_K] [R_K]^T \{p'_{P/G}\} \quad (4)$$

$$\{v_P\} = \{v_G\} + [R_K]^T [\tilde{\omega}'_K] \{p'_{P/G}\} \quad (5)$$

Equation (4) is written using the **fixed frame components** of ${}^R\omega_K$ the angular velocity of body K relative to R , and Equation (5) is written using the **body frame components** of ${}^R\omega_K$. The 3×1 matrix $\{p'_{P/G}\}$ contains the **body K components** of $r_{P/G}$ the position vector of point P relative to the mass center G .

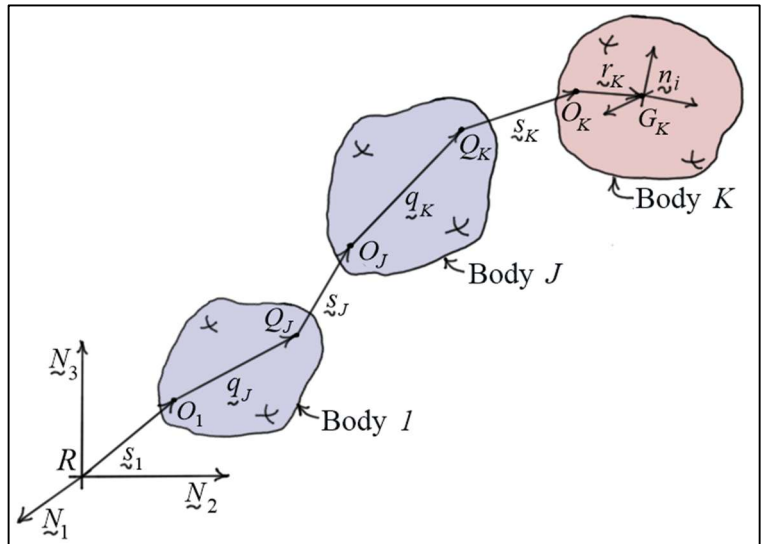
These two equations can be **differentiated** to give two expressions for the **fixed frame components** of ${}^R a_P$ the **acceleration** of P in the fixed frame.

$$\begin{aligned} \{a_P\} &= \{a_G\} + [\dot{\tilde{\omega}}_K] [R_K]^T \{p'_{P/G}\} + [\tilde{\omega}_K] [\dot{R}_K]^T \{p'_{P/G}\} \\ &= \{a_G\} + [\tilde{\alpha}_K] [R_K]^T \{p'_{P/G}\} + [\tilde{\omega}_K] [\tilde{\omega}_K] [R_K]^T \{p'_{P/G}\} \end{aligned} \quad (6)$$

$$\begin{aligned} \{a_P\} &= \{a_G\} + [R_K]^T [\dot{\tilde{\omega}}'_K] \{p'_{P/G}\} + [\dot{R}_K]^T [\tilde{\omega}'_K] \{p'_{P/G}\} \\ &= \{a_G\} + [R_K]^T [\tilde{\alpha}'_K] \{p'_{P/G}\} + [R_K]^T [\tilde{\omega}'_K] [\tilde{\omega}'_K] \{p'_{P/G}\} \end{aligned} \quad (7)$$

Accelerations of Body Origins and Mass Centers Using Relative Coordinates

To calculate the **acceleration** of the **mass center** of body K using **relative coordinates**, consider three bodies of a multibody system as shown in the diagram. As presented in Unit 4, Body 1 is the system's **reference body** and bodies J , and K form a branch off of body 1. The motion of body K is measured **relative** to body J , the motion of body J is measured **relative** to body 1, and the motion of body 1 is measured **relative** to the **fixed frame** R . Body J is the **lower numbered body** of body K , and body 1 is the **lower numbered body** of body J .



Each body has a **reference point**, an **origin**, and a **mass center**. The **reference point**, **origin**, and **mass center** for a body B are labelled Q_B , O_B , and G_B . A body's **reference point** is **fixed** in its **lower numbered body**. A body's **origin** and **mass center** are **fixed** within the **body** itself. The **reference point** for body 1 is the **origin** of the **fixed frame**.

Accelerations of the Origins of the Bodies in Fixed Frame Components

Using Fixed Frame Components of Angular Velocity

In Unit 4 the **fixed frame components** of the **velocity** of O_K the origin of body K was written in terms of the **fixed frame components** of the **velocity** of $O_{\mathcal{L}(K)}$ the origin of body $\mathcal{L}(K)$, the lower numbered body of body K , and the **fixed frame components** of the **angular velocities** of the bodies as follows.

$$\boxed{\{v_{O_K}\} = \{v_{O_{\mathcal{L}(K)}}\} - ([\tilde{q}_K] + [\tilde{s}_K])\{\omega_{\mathcal{L}(K)}\} + [R_{\mathcal{L}(K)}]^T \{s'_K\}} \quad (8)$$

The **starting condition** for this result is as follows.

$$\boxed{\{v_{O_1}\} = \{s'_1\} = [\dot{s}'_{11} \quad \dot{s}'_{12} \quad \dot{s}'_{13}]^T} \quad (9)$$

In Equations (8) and (9), $\{v_{O_K}\}$ and $\{v_{O_{\mathcal{L}(K)}}\}$ represent the **fixed frame components** of the **velocities** of the **origins** of body K and its lower numbered body $\mathcal{L}(K)$, $[\tilde{q}_K]$ and $[\tilde{s}_K]$ represent 3×3 **skew symmetric matrices** built with the **fixed frame components** of position vectors q_K and s_K , $\{\omega_{\mathcal{L}(K)}\}$ represents the **fixed frame components** of ${}^R\omega_{\mathcal{L}(K)}$ the **angular velocity** of body $\mathcal{L}(K)$ in R , $\{s'_K\}$ represents the **body $\mathcal{L}(K)$ components** of the **position vector** s_K , $\{s'_1\}$ represents the **fixed frame components** of the **position vector** s_1 , and $[R_{\mathcal{L}(K)}]^T$ represents the **transformation matrix** that **converts** body $\mathcal{L}(K)$ components into the fixed frame.

The **fixed frame components** of the **acceleration** of O_K the origin of body K can be written in terms of the **fixed frame components** of the **acceleration** of $O_{\mathcal{L}(K)}$ the origin of body $\mathcal{L}(K)$ by **differentiating** Equation (8).

$$\begin{aligned} \{a_{O_K}\} &= \{\dot{v}_{O_K}\} \\ &= \{\dot{v}_{O_{\mathcal{L}(K)}}\} - ([\dot{\tilde{q}}_K] + [\dot{\tilde{s}}_K])\{\omega_{\mathcal{L}(K)}\} - ([\ddot{q}_K] + [\ddot{s}_K])\{\omega_{\mathcal{L}(K)}\} + [\dot{R}_{\mathcal{L}(K)}]^T \{s'_K\} + [\dot{R}_{\mathcal{L}(K)}]^T \{s'_K\} \\ \Rightarrow \boxed{\{a_{O_K}\} &= \{a_{O_{\mathcal{L}(K)}}\} - ([\ddot{q}_K] + [\ddot{s}_K])\{\omega_{\mathcal{L}(K)}\} + [\dot{R}_{\mathcal{L}(K)}]^T \{s'_K\} \\ &\quad - ([\dot{\tilde{q}}_K] + [\dot{\tilde{s}}_K])\{\omega_{\mathcal{L}(K)}\} + [\tilde{\omega}_{\mathcal{L}(K)}][R_{\mathcal{L}(K)}]^T \{s'_K\}} \end{aligned} \quad (10)$$

In Equation (10), $\{a_{O_K}\}$ and $\{a_{O_{\mathcal{L}(K)}}\}$ represent the **fixed frame components** of the **accelerations** of O_K and $O_{\mathcal{L}(K)}$ the origins of bodies K and $\mathcal{L}(K)$; $\{\dot{\omega}_{\mathcal{L}(K)}\}$ represents the **fixed frame components** of the **angular acceleration** of body $\mathcal{L}(K)$; $\{\omega_{\mathcal{L}(K)}\}$ represents the **fixed frame components** of the **angular velocity** of body $\mathcal{L}(K)$; $\{s'_K\}$ and $\{\dot{s}'_K\}$ represent the **first** and **second time derivatives** of the **body $\mathcal{L}(K)$ components** of s_K ; $[\tilde{q}_K]$ and $[\dot{\tilde{q}}_K]$ are 3×3 **skew symmetric matrices** built using **fixed frame components** of the **vectors** q_K and $\frac{{}^R dq_K}{dt}$; $[\tilde{s}_K]$ and $[\dot{\tilde{s}}_K]$ are 3×3 **skew symmetric matrices** built using **fixed frame components** of the **vectors** s_K and $\frac{{}^R ds_K}{dt}$; $[\tilde{\omega}_{\mathcal{L}(K)}]$ is a 3×3 **skew symmetric matrix** built using the **fixed frame components** of ${}^R\omega_{\mathcal{L}(K)}$ the **angular velocity** of body $\mathcal{L}(K)$ in the fixed frame; and $[R_{\mathcal{L}(K)}]$ represents the **transformation matrix** that describes the **orientation** of body $\mathcal{L}(K)$ **relative** to the **fixed frame**.

The **fixed frame components** of the **position vectors** q_K and s_K are found by **transforming** the body $\mathcal{L}(K)$ components into fixed frame components. That is,

$$\boxed{\{q_K\} = [R_{\mathcal{L}(K)}]^T \{q'_K\}} \quad \text{and} \quad \boxed{\{s_K\} = [R_{\mathcal{L}(K)}]^T \{s'_K\}} \quad (11)$$

The **fixed frame components** of $\frac{{}^R d\mathbf{q}_K}{dt}$ and $\frac{{}^R d\mathbf{s}_K}{dt}$ are found by **differentiating** Equations (11) as follows.

$$\boxed{\{\dot{q}_K\} = [R_{\mathcal{L}(K)}]^T \underbrace{\{\dot{q}'_K\}}_{\text{zero}} + [\dot{R}_{\mathcal{L}(K)}]^T \{q'_K\} = [\tilde{\omega}_{\mathcal{L}(K)}] [R_{\mathcal{L}(K)}]^T \{q'_K\}} \quad (12)$$

$$\boxed{\{\dot{s}_K\} = [R_{\mathcal{L}(K)}]^T \{\dot{s}'_K\} + [\dot{R}_{\mathcal{L}(K)}]^T \{s'_K\} = [R_{\mathcal{L}(K)}]^T \{\dot{s}'_K\} + [\tilde{\omega}_{\mathcal{L}(K)}] [R_{\mathcal{L}(K)}]^T \{s'_K\}} \quad (13)$$

Equation (10) can be used to **systematically** calculate the **fixed frame components** of the **accelerations** of the **origins** of all the bodies in a multibody system starting with the acceleration of O_1 the origin of body 1 which is

$$\boxed{\{a_{O_1}\} = \{\ddot{s}'_1\} = [\ddot{s}'_{11} \quad \ddot{s}'_{12} \quad \ddot{s}'_{13}]^T} \quad (\text{fixed frame components}) \quad (14)$$

Using Body Frame Components of Angular Velocity

In Unit 4, the **fixed frame components** of the **velocity** of O_K the origin of body K were **also written** in terms of the **fixed frame components** of the **velocity** of $O_{\mathcal{L}(K)}$ the origin of body $\mathcal{L}(K)$ and the **body frame components** of the **angular velocities** of the bodies as follows.

$$\boxed{\{v_{O_K}\} = \{v_{O_{\mathcal{L}(K)}}\} - [R_{\mathcal{L}(K)}]^T ([\tilde{q}'_K] + [\tilde{s}'_K]) \{\omega'_{\mathcal{L}(K)}\} + [R_{\mathcal{L}(K)}]^T \{\dot{s}'_K\}} \quad (15)$$

As before, the **starting condition** is as follows.

$$\boxed{\{v_{O_1}\} = \{\dot{s}'_1\} = [\dot{s}'_{11} \quad \dot{s}'_{12} \quad \dot{s}'_{13}]^T} \quad (16)$$

In Equation (15), $\{v_{O_K}\}$ and $\{v_{O_{\mathcal{L}(K)}}\}$ represent the **fixed frame components** of the **velocities** of the **origins** of **body K** and its **lower numbered body $\mathcal{L}(K)$** , $[\tilde{q}'_K]$ and $[\tilde{s}'_K]$ represent 3×3 **skew symmetric matrices** built with the **body $\mathcal{L}(K)$ frame components** of position vectors $\mathbf{q}_K$ and $\mathbf{s}_K$, and $\{\omega'_{\mathcal{L}(K)}\}$ represents the **body frame components** of ${}^R \omega_{\mathcal{L}(K)}$ the **angular velocity** of body $\mathcal{L}(K)$ in R .

The **fixed frame components** of the **acceleration** of O_K the origin of body K can be written in terms of the **fixed frame components** of the acceleration of $O_{\mathcal{L}(K)}$ the origin of body $\mathcal{L}(K)$ by **differentiating** Equation (15).

$$\begin{aligned} \{a_{O_K}\} &= \{\dot{v}_{O_K}\} = \{\dot{v}_{O_{\mathcal{L}(K)}}\} - [R_{\mathcal{L}(K)}]^T ([\tilde{q}'_K] + [\tilde{s}'_K]) \{\dot{\omega}'_{\mathcal{L}(K)}\} - [R_{\mathcal{L}(K)}]^T \left(\underbrace{[\dot{\tilde{q}}'_K]}_{\text{zero}} + [\dot{\tilde{s}}'_K] \right) \{\omega'_{\mathcal{L}(K)}\} \\ &\quad - [\dot{R}_{\mathcal{L}(K)}]^T ([\tilde{q}'_K] + [\tilde{s}'_K]) \{\omega'_{\mathcal{L}(K)}\} + [R_{\mathcal{L}(K)}]^T \{\dot{s}'_K\} + [\dot{R}_{\mathcal{L}(K)}]^T \{s'_K\} \\ &= \{a_{O_{\mathcal{L}(K)}}\} - [R_{\mathcal{L}(K)}]^T ([\tilde{q}'_K] + [\tilde{s}'_K]) \{\dot{\omega}'_{\mathcal{L}(K)}\} - [R_{\mathcal{L}(K)}]^T [\dot{\tilde{s}}'_K] \{\omega'_{\mathcal{L}(K)}\} \\ &\quad - [R_{\mathcal{L}(K)}]^T [\tilde{\omega}'_{\mathcal{L}(K)}] ([\tilde{q}'_K] + [\tilde{s}'_K]) \{\omega'_{\mathcal{L}(K)}\} + [R_{\mathcal{L}(K)}]^T \{\dot{s}'_K\} + [R_{\mathcal{L}(K)}]^T [\tilde{\omega}'_{\mathcal{L}(K)}] \{s'_K\} \end{aligned}$$

$$\begin{aligned} \{a_{O_K}\} = & \{a_{O_{\mathcal{L}(K)}}\} - [R_{\mathcal{L}(K)}]^T ([\tilde{q}'_K] + [\tilde{s}'_K]) \{\dot{\omega}'_{\mathcal{L}(K)}\} + [R_{\mathcal{L}(K)}]^T \{\ddot{s}'_K\} \\ & - [R_{\mathcal{L}(K)}]^T ([[\tilde{\omega}'_{\mathcal{L}(K)}]([\tilde{q}'_K] + [\tilde{s}'_K]) + [\tilde{s}'_K]) \{\omega'_{\mathcal{L}(K)}\} + [R_{\mathcal{L}(K)}]^T [\tilde{\omega}'_{\mathcal{L}(K)}] \{\dot{s}'_K\} \end{aligned} \quad (17)$$

The terms in this equation can be **reordered** and **combined** to give the following **alternate form**.

$$\begin{aligned} \{a_{O_K}\} = & \{a_{O_{\mathcal{L}(K)}}\} + [R_{\mathcal{L}(K)}]^T [\alpha'_{\mathcal{L}(K)}] (\{q'_K\} + \{s'_K\}) + [R_{\mathcal{L}(K)}]^T \{\ddot{s}'_K\} \\ & + [R_{\mathcal{L}(K)}]^T [\tilde{\omega}'_{\mathcal{L}(K)}] [\tilde{\omega}'_{\mathcal{L}(K)}] (\{q'_K\} + \{s'_K\}) + 2[R_{\mathcal{L}(K)}]^T [\tilde{\omega}'_{\mathcal{L}(K)}] \{\dot{s}'_K\} \end{aligned} \quad (18)$$

In Equations (17) and (18), $\{a_{O_K}\}$ and $\{a_{O_{\mathcal{L}(K)}}\}$ represent the **fixed frame components** of the **accelerations** of O_K and $O_{\mathcal{L}(K)}$ the **origins** of bodies K and $\mathcal{L}(K)$; $\{\dot{\omega}'_{\mathcal{L}(K)}\}$ represents the **body frame components** of the **angular acceleration** of body $\mathcal{L}(K)$ in R ; $\{\omega'_{\mathcal{L}(K)}\}$ represents the **body frame components** of the **angular velocity** of body $\mathcal{L}(K)$ in R ; $\{\dot{s}'_K\}$ and $\{\ddot{s}'_K\}$ represent the **first** and **second time derivatives** of $\{s'_K\}$; $[\tilde{s}'_K]$ is a 3×3 **skew symmetric matrices** built using $\{\dot{s}'_K\} = [\dot{s}'_{K1} \quad \dot{s}'_{K2} \quad \dot{s}'_{K3}]^T$; $[\tilde{\omega}'_{\mathcal{L}(K)}]$ is a 3×3 **skew symmetric matrix** built using the **body frame components** of ${}^R\omega_{\mathcal{L}(K)}$ the **angular velocity** of body $\mathcal{L}(K)$ in R ; and $[R_{\mathcal{L}(K)}]$ represents the **transformation matrix** that describes the **orientation** of body $\mathcal{L}(K)$ **relative** to the **fixed frame**.

Equation (17) (or (18)) can be used to **systematically** calculate the **accelerations** of the **origins** of all the bodies in a multibody system starting with the acceleration of O_1 the origin of body 1 which is

$$\{a_{O_1}\} = \{\dot{s}'_1\} = [\dot{s}'_{11} \quad \dot{s}'_{12} \quad \dot{s}'_{13}]^T \quad (19)$$

Accelerations of the Mass Centers of the Bodies in Fixed Frame Components

Using Fixed Frame Components of Angular Velocity

In Unit 4, the **fixed frame components** of the **velocity** of G_K the **mass center** of body K was written in terms of the **fixed frame components** of the **velocity** of O_K the **origin** of body K and the **fixed frame components** of the **angular velocity** of body K as follows.

$$\{v_K\} = \{v_{O_K}\} - [\tilde{r}_K] \{\omega_K\} \quad (20)$$

Here, $\{v_K\}$ represents the **fixed frame components** of ${}^Rv_{G_K}$ the **velocity** of G_K in R , $\{v_{O_K}\}$ represents the **fixed frame components** of ${}^Rv_{O_K}$ the **velocity** of O_K in R , $\{\omega_K\}$ represents the **fixed frame components** of ${}^R\omega_K$ the **angular velocity** of body K in R , and $[\tilde{r}_K]$ represents a 3×3 **skew symmetric matrix** built using the **fixed frame components** of position vector r_K . The **fixed frame components** of r_K can be calculated by transforming $\{r'_K\}$ the body K components into the fixed frame as follows.

$$\{r_K\} = [R_K]^T \{r'_K\} \quad (21)$$

The **fixed frame components** of the **acceleration** of G_K the **mass center** of body K can be found by **differentiating** Equation (20) as follows.

$$\begin{aligned} \{a_K\} &= \{\dot{v}_K\} = \{\dot{v}_{O_K}\} - [\tilde{r}_K] \{\dot{\omega}_K\} - [\dot{\tilde{r}}_K] \{\omega_K\} \\ \Rightarrow \{a_K\} &= \{a_{O_K}\} - [\tilde{r}_K] \{\dot{\omega}_K\} - [\dot{\tilde{r}}_K] \{\omega_K\} \end{aligned} \quad (22)$$

Here, $\{a_K\}$ represents the **fixed frame components** of ${}^R a_{G_K}$ the **acceleration** of G_K in R , $\{a_{O_K}\}$ represents the **fixed frame components** of ${}^R a_{O_K}$ the **acceleration** of O_K in R , $\{\dot{\omega}_K\}$ represents the **fixed frame components** of ${}^R \alpha_K$ the **angular acceleration** of body K in R , and $[\dot{\tilde{r}}_K]$ represents a 3×3 **skew symmetric matrix** built using the **fixed frame components** of $\frac{{}^R d r_K}{dt}$. The **fixed frame components** of $\frac{{}^R d r_K}{dt}$ can be calculated by differentiating Equation (21) as follows.

$$\{\dot{r}_K\} = [\dot{R}_K]^T \{r'_K\} + [R_K]^T \underbrace{\{\dot{r}'_K\}}_{\text{zero}} = [\tilde{\omega}_K] [R_K]^T \{r'_K\} \quad (23)$$

Using Body Frame Components of Angular Velocity

In Unit 4, the **fixed frame components** of the **velocity** of G_K the **mass center** of body K was also written in terms of the **fixed frame components** of the **velocity** of O_K the **origin** of body K and the **body frame components** of the **angular velocity** of body K as follows.

$$\{v_K\} = \{v_{O_K}\} - [R_K]^T [\tilde{r}'_K] \{\omega'_K\} \quad (24)$$

In this result, the terms $\{v_K\}$ and $\{v_{O_K}\}$ represent **fixed frame components** of the velocities of G_K and O_K in R , the term $\{\omega'_K\}$ represents the **body frame components** of ${}^R \omega_K$, and the 3×3 skew symmetric matrix $[\tilde{r}'_K]$ is built using **body K frame components** of the position vector r_K .

The **fixed frame components** of the **acceleration** of G_K the mass center of body K can be found by **differentiating** Equation (24) as follows.

$$\begin{aligned} \{a_K\} &= \{\dot{v}_K\} = \{\dot{v}_{O_K}\} - [R_K]^T [\tilde{r}'_K] \{\dot{\omega}'_K\} - [R_K]^T \underbrace{[\dot{\tilde{r}}'_K]}_{\text{zero}} \{\omega'_K\} - [\dot{R}_K]^T [\tilde{r}'_K] \{\omega'_K\} \\ \{a_K\} &= \{a_{O_K}\} - [R_K]^T [\tilde{r}'_K] \{\dot{\omega}'_K\} - [R_K]^T [\tilde{\omega}'_K] [\tilde{r}'_K] \{\omega'_K\} \end{aligned} \quad (25)$$

The terms in this equation can be **reordered** and to give the following **alternate form**.

$$\{a_K\} = \{a_{O_K}\} + [R_K]^T [\alpha'_K] \{r'_K\} + [R_K]^T [\tilde{\omega}'_K] [\tilde{\omega}'_K] \{r'_K\} \quad (26)$$

Here, $\{a_K\}$ represents the **fixed frame components** of ${}^R\mathcal{a}_{G_K}$ the **acceleration** of G_K in R , $\{a_{O_K}\}$ represents the **fixed frame components** of ${}^R\mathcal{a}_{O_K}$ the **acceleration** of O_K in R , $\{\dot{\omega}'_K\}$ represents the **body frame components** of ${}^R\mathcal{a}_{O_K}$ the **angular acceleration** of body K in R , and $[\tilde{\omega}'_K]$ is a 3×3 **skew symmetric matrix** built with the **body frame components** of ${}^R\omega_K$.

Accelerations of the Origins of Bodies in Body Frame Components

In Unit 4, the **body frame components** of ${}^R\mathcal{v}_{O_K}$ the velocity of O_K the origin of body K were written in terms of the **body frame components** of ${}^R\mathcal{v}_{O_{\mathcal{L}(K)}}$ the velocity of $O_{\mathcal{L}(K)}$ the origin of body $\mathcal{L}(K)$ as follows.

$$\{\mathcal{v}'_{O_K}\} = \left[{}^{\mathcal{L}(K)}R_K \right] \left(\{\mathcal{v}'_{O_{\mathcal{L}(K)}}\} + \{\dot{s}'_K\} + [\tilde{\omega}'_{\mathcal{L}(K)}] (\{q'_K\} + \{s'_K\}) \right) = \left[{}^{\mathcal{L}(K)}R_K \right] \{\mathcal{v}_{O_K}\}_{\mathcal{L}(K)} \quad (27)$$

Here,

$$\begin{aligned} \{\mathcal{v}_{O_K}\}_{\mathcal{L}(K)} &= \{\mathcal{v}'_{O_{\mathcal{L}(K)}}\} + \{\dot{s}'_K\} + [\tilde{\omega}'_{\mathcal{L}(K)}] (\{q'_K\} + \{s'_K\}) \\ &= \left[{}^{\mathcal{L}^2(K)}R_{\mathcal{L}(K)} \right] \{\mathcal{v}_{O_{\mathcal{L}(K)}}\}_{\mathcal{L}^2(K)} + \{\dot{s}'_K\} + [\tilde{\omega}'_{\mathcal{L}(K)}] (\{q'_K\} + \{s'_K\}) \end{aligned} \quad (28)$$

Note that, the **components** of the **velocity** of the **origin** of body K are **first expressed** in its **lower numbered body** $\mathcal{L}(K)$ and then **converted** into components in body K using the transformation matrix $\left[{}^{\mathcal{L}(K)}R_K \right]$. The **components** of the **velocity** of the **origin** of **body** $\mathcal{L}(K)$ are **first expressed** in its **lower numbered body** $\mathcal{L}^2(K)$ and then **converted** into components in body $\mathcal{L}(K)$ using the transformation matrix $\left[{}^{\mathcal{L}^2(K)}R_{\mathcal{L}(K)} \right]$.

Using Equation (28), the **fixed frame components** of ${}^R\mathcal{v}_{O_K}$ can be found by pre-multiplying the body $\mathcal{L}(K)$ components by $\left[R_{\mathcal{L}(K)} \right]^T$ as follows.

$$\begin{aligned} \{\mathcal{v}_{O_K}\} &= \left[R_{\mathcal{L}(K)} \right]^T \{\mathcal{v}_{O_K}\}_{\mathcal{L}(K)} = \left[R_{\mathcal{L}(K)} \right]^T \left(\left[{}^{\mathcal{L}^2(K)}R_{\mathcal{L}(K)} \right] \{\mathcal{v}_{O_{\mathcal{L}(K)}}\}_{\mathcal{L}^2(K)} + \{\dot{s}'_K\} + [\tilde{\omega}'_{\mathcal{L}(K)}] (\{q'_K\} + \{s'_K\}) \right) \\ &= \left[R_{\mathcal{L}^2(K)} \right]^T \{\mathcal{v}_{O_{\mathcal{L}(K)}}\}_{\mathcal{L}^2(K)} + \left[R_{\mathcal{L}(K)} \right]^T \{\dot{s}'_K\} + \left[R_{\mathcal{L}(K)} \right]^T [\tilde{\omega}'_{\mathcal{L}(K)}] (\{q'_K\} + \{s'_K\}) \end{aligned} \quad (29)$$

The **fixed frame components** of ${}^R\mathcal{a}_{O_K}$ the **acceleration** of O_K in R can be found by **differentiating** Equation (29).

$$\begin{aligned} \{a_{O_K}\} &= \{\dot{\mathcal{v}}_{O_K}\} \\ &= \left[R_{\mathcal{L}^2(K)} \right]^T \{\dot{\mathcal{v}}_{O_{\mathcal{L}(K)}}\}_{\mathcal{L}^2(K)} + \left[\dot{R}_{\mathcal{L}^2(K)} \right]^T \{\mathcal{v}_{O_{\mathcal{L}(K)}}\}_{\mathcal{L}^2(K)} + \left[R_{\mathcal{L}(K)} \right]^T \{\dot{s}'_K\} + \left[\dot{R}_{\mathcal{L}(K)} \right]^T \{s'_K\} \\ &\quad + \left[R_{\mathcal{L}(K)} \right]^T [\dot{\tilde{\omega}}'_{\mathcal{L}(K)}] (\{q'_K\} + \{s'_K\}) + \left[R_{\mathcal{L}(K)} \right]^T [\tilde{\omega}'_{\mathcal{L}(K)}] \{\dot{s}'_K\} + \left[\dot{R}_{\mathcal{L}(K)} \right]^T [\tilde{\omega}'_{\mathcal{L}(K)}] (\{q'_K\} + \{s'_K\}) \\ &\quad \vdots \end{aligned}$$

$$\begin{aligned}
& \vdots \\
& = \left[R_{\mathcal{L}^2(K)} \right]^T \left\{ \dot{v}_{O_{\mathcal{L}(K)}} \right\}_{\mathcal{L}^2(K)} + \left[R_{\mathcal{L}^2(K)} \right]^T \left[\tilde{\omega}'_{\mathcal{L}^2(K)} \right] \left\{ v_{O_{\mathcal{L}(K)}} \right\}_{\mathcal{L}^2(K)} + \left[R_{\mathcal{L}(K)} \right]^T \{ \ddot{s}'_K \} + \left[R_{\mathcal{L}(K)} \right]^T \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \{ \dot{s}'_K \} \\
& + \left[R_{\mathcal{L}(K)} \right]^T \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] (\{ q'_K \} + \{ s'_K \}) + \left[R_{\mathcal{L}(K)} \right]^T \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \{ \dot{s}'_K \} + \left[R_{\mathcal{L}(K)} \right]^T \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] (\{ q'_K \} + \{ s'_K \}) \\
& = \left[R_{\mathcal{L}^2(K)} \right]^T \left(\left\{ \dot{v}_{O_{\mathcal{L}(K)}} \right\}_{\mathcal{L}^2(K)} + \left[\tilde{\omega}'_{\mathcal{L}^2(K)} \right] \left\{ v_{O_{\mathcal{L}(K)}} \right\}_{\mathcal{L}^2(K)} \right) + \left[R_{\mathcal{L}(K)} \right]^T \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] (\{ q'_K \} + \{ s'_K \}) \\
& + \left[R_{\mathcal{L}(K)} \right]^T \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] (\{ q'_K \} + \{ s'_K \}) + \left[R_{\mathcal{L}(K)} \right]^T \{ \ddot{s}'_K \} + 2 \left[R_{\mathcal{L}(K)} \right]^T \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \{ \dot{s}'_K \}
\end{aligned}$$

Or,

$$\begin{aligned}
\{ a_{O_K} \} &= \left[R_{\mathcal{L}^2(K)} \right]^T \left\{ a_{O_{\mathcal{L}(K)}} \right\}_{\mathcal{L}^2(K)} + \left[R_{\mathcal{L}(K)} \right]^T \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] (\{ q'_K \} + \{ s'_K \}) + \left[R_{\mathcal{L}(K)} \right]^T \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] (\{ q'_K \} + \{ s'_K \}) \\
&+ \left[R_{\mathcal{L}(K)} \right]^T \{ \ddot{s}'_K \} + 2 \left[R_{\mathcal{L}(K)} \right]^T \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \{ \dot{s}'_K \}
\end{aligned}$$

These components can be **converted** into the body K system by **pre-multiplying** by $[R_K]$.

$$\begin{aligned}
\{ a'_{O_K} \} &= [R_K] \{ a_{O_K} \} \\
&= [R_K] \left[R_{\mathcal{L}^2(K)} \right]^T \left\{ a_{O_{\mathcal{L}(K)}} \right\}_{\mathcal{L}^2(K)} + [R_K] \left[R_{\mathcal{L}(K)} \right]^T \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] (\{ q'_K \} + \{ s'_K \}) \\
&+ [R_K] \left[R_{\mathcal{L}(K)} \right]^T \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] (\{ q'_K \} + \{ s'_K \}) + [R_K] \left[R_{\mathcal{L}(K)} \right]^T \{ \ddot{s}'_K \} + 2 [R_K] \left[R_{\mathcal{L}(K)} \right]^T \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \{ \dot{s}'_K \}
\end{aligned}$$

Or,

$$\begin{aligned}
\{ a'_{O_K} \} &= \left[{}^{\mathcal{L}^2(K)} R_K \right] \left\{ a_{O_{\mathcal{L}(K)}} \right\}_{\mathcal{L}^2(K)} \\
&+ \left[{}^{\mathcal{L}(K)} R_K \right] \left(\left[\tilde{\omega}'_{\mathcal{L}(K)} \right] (\{ q'_K \} + \{ s'_K \}) + \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] (\{ q'_K \} + \{ s'_K \}) + \{ \ddot{s}'_K \} + 2 \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \{ \dot{s}'_K \} \right) \quad (30) \\
&= \left[{}^{\mathcal{L}^2(K)} R_K \right] \left\{ a_{O_{\mathcal{L}(K)}} \right\}_{\mathcal{L}^2(K)} + \left[{}^{\mathcal{L}(K)} R_K \right] \left\{ a_{O_K/O_{\mathcal{L}(K)}} \right\}_{\mathcal{L}(K)}
\end{aligned}$$

Here, $\{ a_{O_{\mathcal{L}(K)}} \}_{\mathcal{L}^2(K)}$ represents the **components** of ${}^R a_{O_{\mathcal{L}(K)}}$ in the **body** $\mathcal{L}^2(K)$ **frame**, and $\{ a_{O_K/O_{\mathcal{L}(K)}} \}_{\mathcal{L}(K)}$ represents the **components** of ${}^R a_{O_K/O_{\mathcal{L}(K)}}$ in the **body** $\mathcal{L}(K)$ **frame**. Specifically,

$$\{ a_{O_K/O_{\mathcal{L}(K)}} \}_{\mathcal{L}(K)} = \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] (\{ q'_K \} + \{ s'_K \}) + \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] (\{ q'_K \} + \{ s'_K \}) + \{ \ddot{s}'_K \} + 2 \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \{ \dot{s}'_K \} \quad (31)$$

This result can **also** be found by **reverting** to **vector equations** and applying the **derivative rule**. In Unit 4, ${}^R v_{O_K}$ the **velocity** of O_K the origin of body K was written in terms of ${}^R v_{O_{\mathcal{L}(K)}}$ the **velocity** of $O_{\mathcal{L}(K)}$ the origin of body $\mathcal{L}(K)$ as follows.

$${}^R v_{O_K} = {}^R v_{O_{\mathcal{L}(K)}} + \frac{{}^{\mathcal{L}(K)} d}{dt} (s_K) + {}^R \omega_{\mathcal{L}(K)} \times (q_K + s_K)$$

The **acceleration** of O_K can be found by **differentiating** this result using the **derivative rule** as follows.

$$\begin{aligned}
{}^R \underline{a}_{O_K} &= \frac{{}^R d {}^R \underline{v}_{O_K}}{dt} = \frac{{}^R d {}^R \underline{v}_{O_{\mathcal{L}(K)}}}{dt} + \frac{{}^R d}{dt} \left(\frac{{}^{\mathcal{L}(K)} d}{dt} (\underline{s}_K) \right) + \frac{{}^R d}{dt} \left({}^R \underline{\omega}_{\mathcal{L}(K)} \times (\underline{q}_K + \underline{s}_K) \right) \\
&= {}^R \underline{a}_{O_{\mathcal{L}(K)}} + \frac{{}^{\mathcal{L}(K)} d}{dt} \left(\frac{{}^{\mathcal{L}(K)} d}{dt} (\underline{s}_K) \right) + \left({}^R \underline{\omega}_{\mathcal{L}(K)} \times \frac{{}^{\mathcal{L}(K)} d}{dt} (\underline{s}_K) \right) + \left({}^R \underline{\alpha}_{\mathcal{L}(K)} \times (\underline{q}_K + \underline{s}_K) \right) + {}^R \underline{\omega}_{\mathcal{L}(K)} \times \left(\frac{{}^R d}{dt} (\underline{q}_K + \underline{s}_K) \right) \\
&= {}^R \underline{a}_{O_{\mathcal{L}(K)}} + \frac{{}^{\mathcal{L}(K)} d^2}{dt^2} (\underline{s}_K) + \left({}^R \underline{\omega}_{\mathcal{L}(K)} \times \frac{{}^{\mathcal{L}(K)} d}{dt} (\underline{s}_K) \right) + \left({}^R \underline{\alpha}_{\mathcal{L}(K)} \times (\underline{q}_K + \underline{s}_K) \right) \\
&\quad + {}^R \underline{\omega}_{\mathcal{L}(K)} \times \left(\frac{{}^{\mathcal{L}(K)} d}{dt} (\underline{q}_K + \underline{s}_K) + {}^R \underline{\omega}_{\mathcal{L}(K)} \times (\underline{q}_K + \underline{s}_K) \right)
\end{aligned}$$

Rearranging terms gives the following result.

$$\begin{aligned}
{}^R \underline{a}_{O_K} &= {}^R \underline{a}_{O_{\mathcal{L}(K)}} + \left({}^R \underline{\alpha}_{\mathcal{L}(K)} \times (\underline{q}_K + \underline{s}_K) \right) + \left({}^R \underline{\omega}_{\mathcal{L}(K)} \times \left({}^R \underline{\omega}_{\mathcal{L}(K)} \times (\underline{q}_K + \underline{s}_K) \right) \right) \\
&\quad + \frac{{}^{\mathcal{L}(K)} d^2}{dt^2} (\underline{s}_K) + 2 \left({}^R \underline{\omega}_{\mathcal{L}(K)} \times \frac{{}^{\mathcal{L}(K)} d}{dt} (\underline{s}_K) \right) \\
&= {}^R \underline{a}_{O_{\mathcal{L}(K)}} + {}^R \underline{a}_{O_K/O_{\mathcal{L}(K)}}
\end{aligned} \tag{32}$$

Using **body frame components** of the **angular velocities** of the bodies in Equation (32), the **body frame components** of the **acceleration** of O_K the origin of body K can be written as follows.

$$\begin{aligned}
{}^R \underline{a}_{O_{\mathcal{L}(K)}} &\rightarrow \left[{}^{\mathcal{L}(K)} R_K \right] \left\{ \underline{a}_{O_{\mathcal{L}(K)}} \right\}_{\mathcal{L}^2(K)} \\
{}^R \underline{a}_{O_K/O_{\mathcal{L}(K)}} &\rightarrow \left[{}^{\mathcal{L}(K)} R_K \right] \left\{ \underline{a}_{O_K/O_{\mathcal{L}(K)}} \right\}_{\mathcal{L}(K)} = \left[{}^{\mathcal{L}(K)} R_K \right] \left(\left[\dot{\tilde{\omega}}'_{\mathcal{L}(K)} \right] (\{q'_K\} + \{s'_K\}) + \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] (\{q'_K\} + \{s'_K\}) \right) \\
&\quad + \left[{}^{\mathcal{L}(K)} R_K \right] \left(\{\dot{s}'_K\} + 2 \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \{s'_K\} \right)
\end{aligned}$$

Or,

$$\begin{aligned}
\{a'_{O_K}\} &= \left[{}^{\mathcal{L}(K)} R_K \right] \left\{ \underline{a}_{O_{\mathcal{L}(K)}} \right\}_{\mathcal{L}^2(K)} + \left[{}^{\mathcal{L}(K)} R_K \right] \left(\left[\dot{\tilde{\omega}}'_{\mathcal{L}(K)} \right] (\{q'_K\} + \{s'_K\}) + \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] (\{q'_K\} + \{s'_K\}) \right) \\
&\quad + \left[{}^{\mathcal{L}(K)} R_K \right] \left(\{\dot{s}'_K\} + 2 \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \{s'_K\} \right)
\end{aligned} \tag{33}$$

This result is **identical** to that in Equation (30). It is a **recursive relationship** for finding the **accelerations** of the **origins** of the bodies in terms of the **accelerations** of the **origins** of their **lower numbered bodies**. The starting condition is

$$\{a'_{O_1}\} = [R_1] \{\dot{s}'_1\}$$

Here, $\{a'_{O_K}\}$ represents the **body K components** of ${}^R \underline{a}_{O_K}$, $\left\{ \underline{a}_{O_{\mathcal{L}(K)}} \right\}_{\mathcal{L}^2(K)}$ represents the **body $\mathcal{L}^2(K)$ components**

of ${}^R \underline{a}_{O_{\mathcal{L}(K)}}$, the **skew symmetric matrix** $\left[\dot{\tilde{\omega}}'_{\mathcal{L}(K)} \right] = \left[\tilde{\alpha}'_{\mathcal{L}(K)} \right]$ is built using the **body frame components** of ${}^R \underline{\alpha}_{\mathcal{L}(K)}$

the **angular acceleration** of body $\mathcal{L}(K)$ in R , and the **skew symmetric matrix** $\left[\tilde{\omega}'_{\mathcal{L}(K)} \right]$ is built using the **body**

frame components of ${}^R\omega_{\mathcal{L}(K)}$. The **transformation matrix** $\left[{}^{\mathcal{L}^2(K)}R_K\right]$ **converts** body $\mathcal{L}^2(K)$ components into body K components, and the **transformation matrix** $\left[{}^{\mathcal{L}(K)}R_K\right]$ **converts** body $\mathcal{L}(K)$ components into body K components.

Accelerations of the Mass Centers of Bodies in Body Frame Components

Given the **accelerations** of the **body origins**, the **accelerations** of the **mass centers** of the bodies can be calculated by using the concept of **relative acceleration**.

$${}^R\mathbf{a}_K \triangleq {}^R\mathbf{a}_{G_K} = {}^R\mathbf{a}_{O_K} + {}^R\mathbf{a}_{G_K/O_K} = {}^R\mathbf{a}_{O_K} + \left({}^R\boldsymbol{\alpha}_K \times \mathbf{r}_{G_K/O_K}\right) + {}^R\boldsymbol{\omega}_K \times \left({}^R\boldsymbol{\omega}_K \times \mathbf{r}_{G_K/O_K}\right) \quad (34)$$

Using the **body frame components** of the **angular velocities** of the bodies, the **body frame components** of ${}^R\mathbf{a}_K$ the **acceleration** of G_K the mass center of body K can be written as follows.

$$\begin{aligned} \{a'_K\} &= \{a'_{O_K}\} + [\tilde{\alpha}'_K]\{r'_K\} + [\tilde{\omega}'_K][\tilde{\omega}'_K]\{r'_K\} \\ &= \left[{}^{\mathcal{L}^2(K)}R_K\right]\{a_{O_{\mathcal{L}^2(K)}}\}_{\mathcal{L}^2(K)} + \left[{}^{\mathcal{L}(K)}R_K\right]\left([\tilde{\omega}'_{\mathcal{L}(K)}](\{q'_K\} + \{s'_K\}) + [\tilde{\omega}'_{\mathcal{L}(K)}][\tilde{\omega}'_{\mathcal{L}(K)}](\{q'_K\} + \{s'_K\})\right) \\ &\quad + \left[{}^{\mathcal{L}(K)}R_K\right](\{\ddot{s}'_K\} + 2[\tilde{\omega}'_{\mathcal{L}(K)}]\{\dot{s}'_K\}) + [\tilde{\alpha}'_K]\{r'_K\} + [\tilde{\omega}'_K][\tilde{\omega}'_K]\{r'_K\} \\ &= \{a_{O_{\mathcal{L}(K)}}\}_K + \left[{}^{\mathcal{L}(K)}R_K\right]\left([\tilde{\omega}'_{\mathcal{L}(K)}](\{q'_K\} + \{s'_K\}) + [\tilde{\omega}'_{\mathcal{L}(K)}][\tilde{\omega}'_{\mathcal{L}(K)}](\{q'_K\} + \{s'_K\})\right) \\ &\quad + \left[{}^{\mathcal{L}(K)}R_K\right](\{\ddot{s}'_K\} + 2[\tilde{\omega}'_{\mathcal{L}(K)}]\{\dot{s}'_K\}) + [\tilde{\alpha}'_K]\{r'_K\} + [\tilde{\omega}'_K][\tilde{\omega}'_K]\{r'_K\} \end{aligned} \quad (35)$$

Accelerations of Body Origins in Terms of Partial Velocities and Their Time Derivatives

Generalized Speeds

Unit 4 presented procedures for calculating **partial velocities** for **body origins** and **mass centers** of the N bodies of a multibody system. The **partial velocities** were defined for **three choices** of **generalized speeds**. In each case the column vector of generalized speeds was **partitioned** as follows.

$$\{\mathbf{y}\}_{6N \times 1} \triangleq \begin{Bmatrix} \{\mathbf{y}_1\}_{3N \times 1} \\ \{\mathbf{y}_2\}_{3N \times 1} \end{Bmatrix} \quad (36)$$

In each case, the column matrix $\{\mathbf{y}_2\}_{3N \times 1}$ is defined to be

$$\{\mathbf{y}_2\}_{3N \times 1} \triangleq [\dot{s}'_{11} \quad \dot{s}'_{12} \quad \dot{s}'_{13} \quad \cdots \quad \dot{s}'_{j1} \quad \dot{s}'_{j2} \quad \dot{s}'_{j3} \quad \cdots \quad \dot{s}'_{K1} \quad \dot{s}'_{K2} \quad \dot{s}'_{K3} \quad \cdots \quad \dot{s}'_{N1} \quad \dot{s}'_{N2} \quad \dot{s}'_{N3}]^T \quad (37)$$

Recall that $\{s'_K\} = [s'_{K1} \quad s'_{K2} \quad s'_{K3}]^T$ represents the body $\mathcal{L}(K)$ components of the position vector $\mathbf{s}_K$. The column matrix $\{\mathbf{y}_1\}_{3N \times 1}$ contains **either** the **base frame** components of the **relative angular velocity vectors**, the **body frame** components of the **relative angular velocity vectors**, or the **time derivatives** of the **orientation angles**

of the bodies. When using *time derivatives* of *orientation angles* as generalized speeds, *either base frame* or *body frame components* of the relative angular velocity vectors can be used.

Partial Velocities

Fixed Frame Components Using Base Frame Components of the Relative Angular Velocity Vectors

In Unit 4, $\{v_{O_K}\}$ the *fixed frame components* of the *velocity* of O_K the origin of body K were written using the *base frame components* of the *relative angular velocity vectors* as follows.

$$\begin{aligned} \{v_{O_K}\} &= \begin{bmatrix} R v_{O_{\mathcal{L}(K)},y} \end{bmatrix} \{y\} - ([\tilde{q}_K] + [\tilde{s}_K]) \begin{bmatrix} R \omega_{\mathcal{L}(K),y} \end{bmatrix} \{y\} + [R_{\mathcal{L}(K)}]^T \{\dot{s}'_K\} \\ &= \left(\begin{bmatrix} R v_{O_{\mathcal{L}(K)},y} \end{bmatrix} - ([\tilde{q}_K] + [\tilde{s}_K]) \begin{bmatrix} R \omega_{\mathcal{L}(K),y} \end{bmatrix} + [R_{\mathcal{L}(K)}]^T [S_{s'_K}] \right) \{y\} \\ &= \begin{bmatrix} R v_{O_K,y} \end{bmatrix} \{y\} \end{aligned} \quad (38)$$

Matrix $\begin{bmatrix} R v_{O_K,y} \end{bmatrix}$ represents the *fixed frame components* of the *partial velocity matrix* of O_K , matrix $\begin{bmatrix} R v_{O_{\mathcal{L}(K)},y} \end{bmatrix}$ represents the *fixed frame components* of the *partial velocity matrix* of $O_{\mathcal{L}(K)}$, matrix $\begin{bmatrix} R \omega_{\mathcal{L}(K),y} \end{bmatrix}$ represents the *fixed frame components* of the *partial angular velocity matrix* for body $\mathcal{L}(K)$, and $[S_{s'_K}]$ is a matrix that selects $\{\dot{s}'_K\}$ from the state vector $\{y\}$.

$$[S_{s'_K}]_{3 \times 6N} = \begin{bmatrix} [0]_{3 \times 3N} & \underbrace{[0]_{3 \times 3}}_1 & \cdots & \underbrace{[0]_{3 \times 3}}_{K-1} & \underbrace{[I]_{3 \times 3}}_K & \underbrace{[0]_{3 \times 3}}_{K+1} & \cdots & \underbrace{[0]_{3 \times 3}}_N \end{bmatrix}_{3 \times 6N} \quad (39)$$

The matrix $[I]_{3 \times 3}$ represents a 3×3 identity matrix.

The partial velocity matrix $\begin{bmatrix} R v_{O_K,y} \end{bmatrix}$ is *constructed* using the following procedure relating the *partial velocity* of O_K to the *partial velocity* of $O_{\mathcal{L}(K)}$ the origin of body $\mathcal{L}(K)$. See the details in Unit 4.

1. First, set

$$\begin{bmatrix} R v_{O_K,y} \end{bmatrix}_{3 \times 6N} = \begin{bmatrix} R v_{O_{\mathcal{L}(K)},y} \end{bmatrix}_{3 \times 6N} - ([\tilde{q}_K] + [\tilde{s}_K]) \begin{bmatrix} R \omega_{\mathcal{L}(K),y} \end{bmatrix}_{3 \times 6N} \quad (40)$$

2. Then, set the *three columns* associated with $\dot{s}'_{Ki}$ ($i = 1, 2, 3$) as follows.

$$\begin{bmatrix} R v_{O_K,y} \end{bmatrix}_{ik} = \begin{bmatrix} R_{\mathcal{L}(K)} \end{bmatrix}_{ij}^T \quad (i = 1, 2, 3; j = 1, 2, 3; k = 3N + (3K - 3 + j)) \quad (41)$$

For body 1, only Equation (41) applies. All entries are *zero* except for the three columns associated with $\{\dot{s}'_1\}$ giving the following result.

$$\begin{bmatrix} R v_{O_K,y} \end{bmatrix}_{ik} = [I]_{ij} \quad (i = 1, 2, 3; j = 1, 2, 3; k = 3N + j; K = 1)$$

The matrices $[\tilde{q}_K]$ and $[\tilde{s}_K]$ are 3×3 **skew symmetric matrices** built using the **fixed frame components** of the **position vectors** q_K and s_K . See Equation (11).

Fixed Frame Components Using Body Frame Components of the Relative Angular Velocity Vectors

In Unit 4, $\{v_{O_K}\}$ the **fixed frame components** of the **velocity** of O_K the origin of body K were also written using the **body frame components** of the **relative angular velocity vectors** as follows.

$$\begin{aligned} \{v_{O_K}\} &= \begin{bmatrix} R_{v_{O_{\mathcal{L}(K)},y}} \end{bmatrix} \{y\} - \begin{bmatrix} R_{\mathcal{L}(K)} \end{bmatrix}^T ([\tilde{q}_K] + [\tilde{s}'_K]) \begin{bmatrix} R_{\omega'_{\mathcal{L}(K),y}} \end{bmatrix} \{y\} + \begin{bmatrix} R_{\mathcal{L}(K)} \end{bmatrix}^T \{\dot{s}'_K\} \\ &= \left(\begin{bmatrix} R_{v_{O_{\mathcal{L}(K)},y}} \end{bmatrix} - \begin{bmatrix} R_{\mathcal{L}(K)} \end{bmatrix}^T ([\tilde{q}_K] + [\tilde{s}'_K]) \begin{bmatrix} R_{\omega'_{\mathcal{L}(K),y}} \end{bmatrix} + \begin{bmatrix} R_{\mathcal{L}(K)} \end{bmatrix}^T [S_{s'_K}] \right) \{y\} \\ &= \begin{bmatrix} R_{v_{O_K,y}} \end{bmatrix} \{y\} \end{aligned} \quad (42)$$

Matrix $\begin{bmatrix} R_{v_{O_K,y}} \end{bmatrix}$ represents the **fixed frame components** of the **partial velocity matrix** of O_K , matrix $\begin{bmatrix} R_{v_{O_{\mathcal{L}(K)},y}} \end{bmatrix}$ represents the **fixed frame components** of the **partial velocity matrix** of $O_{\mathcal{L}(K)}$, matrix $\begin{bmatrix} R_{\omega'_{\mathcal{L}(K),y}} \end{bmatrix}$ represents the **body frame components** of the **partial angular velocity matrix** for body $\mathcal{L}(K)$, and $[S_{s'_K}]$ is a matrix that selects $\{\dot{s}'_K\}$ from the state vector $\{y\}$ as defined in Equation (39).

The partial velocity matrix $\begin{bmatrix} R_{v_{O_K,y}} \end{bmatrix}$ is **constructed** using the following procedure that relates the **partial velocity** of O_K to the **partial velocity** of $O_{\mathcal{L}(K)}$ the origin of body $\mathcal{L}(K)$ the lower numbered body of body K . See the details in Unit 4.

1. First, set

$$\begin{bmatrix} R_{v_{O_K,y}} \end{bmatrix}_{3 \times 6N} = \begin{bmatrix} R_{v_{O_{\mathcal{L}(K)},y}} \end{bmatrix}_{3 \times 6N} - \begin{bmatrix} R_{\mathcal{L}(K)} \end{bmatrix}^T ([\tilde{q}'_K] + [\tilde{s}'_K]) \begin{bmatrix} R_{\omega'_{\mathcal{L}(K),y}} \end{bmatrix}_{3 \times 6N} \quad (43)$$

2. Then, set the **three columns** associated with $\dot{s}'_{K_i}$ ($i = 1, 2, 3$) as follows.

$$\begin{bmatrix} R_{v_{O_K,y}} \end{bmatrix}_{ik} = \begin{bmatrix} R_{\mathcal{L}(K)} \end{bmatrix}_{ij}^T \quad (i = 1, 2, 3; j = 1, 2, 3; k = 3N + (3K - 3 + j)) \quad (44)$$

For body 1, only Equation (44) applies. All entries are **zero** except for the three columns associated with $\{\dot{s}'_1\}$ giving the following result.

$$\begin{bmatrix} R_{v_{O_K,y}} \end{bmatrix}_{ik} = [I]_{ij} \quad (i = 1, 2, 3; j = 1, 2, 3; k = 3N + j; K = 1)$$

The matrices $[\tilde{q}'_K]$ and $[\tilde{s}'_K]$ are 3×3 **skew symmetric matrices** built using the **body $\mathcal{L}(K)$ frame components** of the **position vectors** q_K and s_K .

Body Frame Components Using Body Frame Components of the Relative Angular Velocity Vectors

Finally, using Equations (27) and (28), $\{v'_{O_K}\}$ the **body frame components** of the **velocity** of O_K the origin of body K can be written using the **body frame components** of the **relative angular velocity vectors** as follows.

$$\{\dot{v}'_{O_K}\} = \left[\begin{smallmatrix} \mathcal{L}(K) \\ R_K \end{smallmatrix} \right] \left(\{\dot{v}'_{O_{\mathcal{L}(K)}}\} + \{\dot{s}'_K\} + [\tilde{\omega}'_{\mathcal{L}(K)}] (\{q'_K\} + \{s'_K\}) \right) = \left[\begin{smallmatrix} \mathcal{L}(K) \\ R_K \end{smallmatrix} \right] \{\dot{v}_{O_K}\}_{\mathcal{L}(K)} \quad (45)$$

$$\begin{aligned} \{\dot{v}_{O_K}\}_{\mathcal{L}(K)} &= \{\dot{v}'_{O_{\mathcal{L}(K)}}\} + \{\dot{s}'_K\} + [\tilde{\omega}'_{\mathcal{L}(K)}] (\{q'_K\} + \{s'_K\}) \\ &= \left[\begin{smallmatrix} \mathcal{L}^2(K) \\ R_{\mathcal{L}(K)} \end{smallmatrix} \right] \{\dot{v}_{O_{\mathcal{L}(K)}}\}_{\mathcal{L}^2(K)} + \{\dot{s}'_K\} + [\tilde{\omega}'_{\mathcal{L}(K)}] (\{q'_K\} + \{s'_K\}) \end{aligned} \quad (46)$$

The $\mathcal{L}(K)$ components of ${}^R\dot{v}_{O_K}$ can also be written in terms of **partial velocity** and **partial angular velocity matrices** as follows.

$$\begin{aligned} \{\dot{v}_{O_K}\}_{\mathcal{L}(K)} &= \left[\begin{smallmatrix} \mathcal{L}^2(K) \\ R_{\mathcal{L}(K)} \end{smallmatrix} \right] \{\dot{v}_{O_{\mathcal{L}(K)}}\}_{\mathcal{L}^2(K)} + \{\dot{s}'_K\} + [\tilde{\omega}'_{\mathcal{L}(K)}] (\{q'_K\} + \{s'_K\}) \\ &= \left[\begin{smallmatrix} \mathcal{L}^2(K) \\ R_{\mathcal{L}(K)} \end{smallmatrix} \right] \left[{}^R\dot{v}_{O_{\mathcal{L}(K)},y} \right]_{\mathcal{L}^2(K)} \{y\} - ([\tilde{q}'_K] + [\tilde{s}'_K]) \{\omega'_{\mathcal{L}(K)}\} + \{\dot{s}'_K\} \\ &= \left(\left[\begin{smallmatrix} \mathcal{L}^2(K) \\ R_{\mathcal{L}(K)} \end{smallmatrix} \right] \left[{}^R\dot{v}_{O_{\mathcal{L}(K)},y} \right]_{\mathcal{L}^2(K)} - ([\tilde{q}'_K] + [\tilde{s}'_K]) \left[{}^R\omega'_{\mathcal{L}(K),y} \right] + [S_{s'_K}] \right) \{y\} \\ &= \left(\left[\begin{smallmatrix} \mathcal{L}^2(K) \\ R_{\mathcal{L}(K)} \end{smallmatrix} \right] \left[{}^R\dot{v}_{O_{\mathcal{L}(K)},y} \right]_{\mathcal{L}^2(K)} + \left[{}^R\hat{v}_{O_K,y} \right]_{\mathcal{L}(K)} \right) \{y\} \\ &= \left[{}^R\dot{v}_{O_K,y} \right]_{\mathcal{L}(K)} \{y\} \end{aligned} \quad (47)$$

Matrix $\left[{}^R\dot{v}_{O_{\mathcal{L}(K)},y} \right]_{\mathcal{L}^2(K)}$ is the **partial velocity matrix** of $O_{\mathcal{L}(K)}$ the origin of body $\mathcal{L}(K)$ **expressed** in **body** $\mathcal{L}^2(K)$ **components**. Matrix $\left[{}^R\hat{v}_{O_K,y} \right]_{\mathcal{L}(K)}$ is the **portion** of the **partial velocity matrix** of O_K associated with ${}^R\dot{v}_{O_K/O_{\mathcal{L}(K)}}$ the **velocity** of O_K **relative** to $O_{\mathcal{L}(K)}$ and is defined as follows.

$$\left[{}^R\hat{v}_{O_K,y} \right]_{\mathcal{L}(K)} \triangleq \left[S_{s'_K} \right] - ([\tilde{q}'_K] + [\tilde{s}'_K]) \left[{}^R\omega'_{\mathcal{L}(K),y} \right] \quad (48)$$

As before, $[S_{s'_K}]$ is a matrix that selects $\{\dot{s}'_K\}$ from the state vector $\{y\}$.

The matrix $\left[{}^R\dot{v}_{O_K,y} \right]_{\mathcal{L}(K)}$ **represents** the **body** $\mathcal{L}(K)$ **components** of the **partial velocity matrix** of O_K and is **constructed** using the following procedure.

1. First, set

$$\left[{}^R\dot{v}_{O_K,y} \right]_{\mathcal{L}(K)} = \left[\begin{smallmatrix} \mathcal{L}^2(K) \\ R_{\mathcal{L}(K)} \end{smallmatrix} \right] \left[{}^R\dot{v}_{O_{\mathcal{L}(K)},y} \right]_{\mathcal{L}^2(K)} - ([\tilde{q}'_K] + [\tilde{s}'_K]) \left[{}^R\omega'_{\mathcal{L}(K),y} \right] \quad (49)$$

2. Then, set the **three columns** associated with $\dot{s}'_{Ki}$ ($i=1,2,3$) as follows.

$$\left(\left[{}^R\dot{v}_{O_K,y} \right]_{\mathcal{L}(K)} \right)_{ik} = [I]_{ij} \quad (i=1,2,3; j=1,2,3; k=3N+(3K-3+j)) \quad (50)$$

For body 1, only Equation (50) applies. All entries are **zero** except for the three columns associated with $\{\dot{s}'_1\}$ giving the following result.

$$\left(\left[{}^R\dot{v}_{O_1,y} \right]_R \right)_{ik} = [I]_{ij} \quad (i=1,2,3; j=1,2,3; k=3N+j) \quad (51)$$

Accelerations

Fixed Frame Components

The **fixed frame components** of the **acceleration** of O_K the origin of body K can be found by **differentiating** the **fixed frame velocity components** in **terms** of the **partial velocity** of O_K as follows.

$$\{a_{O_K}\} = \{\dot{v}_{O_K}\} = [{}^R v_{O_K,y}] \{\dot{y}\} + [{}^R \dot{v}_{O_K,y}] \{y\} \quad (52)$$

Using **base frame components** of the **relative angular velocity vectors**, the **partial velocity matrix** can be calculated using the process described in Equations (40) and (41). The matrix $[{}^R \dot{v}_{O_K,y}]$ can be calculated by differentiating $[{}^R v_{O_K,y}]$ as defined in Equation (38). That is,

$$[{}^R \dot{v}_{O_K,y}] = [{}^R \dot{v}_{O_{\mathcal{E}(K)},y}] - ([\tilde{q}_K] + [\tilde{s}_K]) [{}^R \dot{\omega}_{\mathcal{E}(K),y}] - ([\dot{\tilde{q}}_K] + [\dot{\tilde{s}}_K]) [{}^R \omega_{\mathcal{E}(K),y}] + [\tilde{\omega}_{\mathcal{E}(K)}] [R_{\mathcal{E}(K)}]^T [S_{s'_K}] \quad (53)$$

This result can be **constructed** using the following procedure.

1. First, set

$$[{}^R \dot{v}_{O_K,y}]_{3 \times 6N} = [{}^R \dot{v}_{O_{\mathcal{E}(K)},y}]_{3 \times 6N} - ([\tilde{q}_K] + [\tilde{s}_K]) [{}^R \dot{\omega}_{\mathcal{E}(K),y}]_{3 \times 6N} - ([\dot{\tilde{q}}_K] + [\dot{\tilde{s}}_K]) [{}^R \omega_{\mathcal{E}(K),y}]_{3 \times 6N} \quad (54)$$

2. Then, set the **three columns** associated with s'_{Ki} ($i = 1, 2, 3$) as follows.

$$[{}^R \dot{v}_{O_K,y}]_{ik} = [\dot{R}_{\mathcal{E}(K)}]_{ij}^T = [\tilde{\omega}_{\mathcal{E}(K)}] [R_{\mathcal{E}(K)}]^T_{ij} \quad (i = 1, 2, 3; j = 1, 2, 3; k = 3N + (3K - 3 + j)) \quad (55)$$

For body 1, all entries are **zero**.

$$[{}^R \dot{v}_{O_1,y}]_{3 \times 6N} = [0]_{3 \times 6N} \quad (56)$$

The matrices $[\dot{\tilde{q}}_K]$ and $[\dot{\tilde{s}}_K]$ are 3×3 **skew symmetric matrices** built using the **fixed frame components** of $\frac{{}^R dq_K}{dt}$ and $\frac{{}^R ds_K}{dt}$. Calculation of these components is shown in Equations (12) and (13) above.

Using the **body frame components** of the **relative angular velocity vectors**, the **partial velocity matrix** can be calculated using the process described in Equations (43) and (44). The matrix $[{}^R \dot{v}_{O_K,y}]$ can be calculated by differentiating $[{}^R v_{O_K,y}]$ as defined in Equation (42). That is,

$$\begin{aligned} [{}^R \dot{v}_{O_K,y}] &= [{}^R \dot{v}_{O_{\mathcal{E}(K)},y}] - [\dot{R}_{\mathcal{E}(K)}]^T ([\tilde{q}'_K] + [\tilde{s}'_K]) [{}^R \omega'_{\mathcal{E}(K),y}] - [R_{\mathcal{E}(K)}]^T ([\tilde{q}'_K] + [\tilde{s}'_K]) [{}^R \dot{\omega}'_{\mathcal{E}(K),y}] \\ &\quad - [R_{\mathcal{E}(K)}]^T [\tilde{s}'_K] [{}^R \omega'_{\mathcal{E}(K),y}] + [\dot{R}_{\mathcal{E}(K)}]^T [S_{s'_K}] \\ &= [{}^R \dot{v}_{O_{\mathcal{E}(K)},y}] - [R_{\mathcal{E}(K)}]^T [\tilde{\omega}'_{\mathcal{E}(K)}] ([\tilde{q}'_K] + [\tilde{s}'_K]) [{}^R \omega'_{\mathcal{E}(K),y}] - [R_{\mathcal{E}(K)}]^T ([\tilde{q}'_K] + [\tilde{s}'_K]) [{}^R \dot{\omega}'_{\mathcal{E}(K),y}] \\ &\quad - [R_{\mathcal{E}(K)}]^T [\tilde{s}'_K] [{}^R \omega'_{\mathcal{E}(K),y}] + [R_{\mathcal{E}(K)}]^T [\tilde{\omega}'_{\mathcal{E}(K)}] [S_{s'_K}] \end{aligned}$$

$$\begin{aligned} \left[{}^R \dot{v}_{O_K, y} \right] = & \left[{}^R \dot{v}_{O_{\mathcal{L}(K)}, y} \right] - \left[R_{\mathcal{L}(K)} \right]^T \left(\left([\tilde{q}'_K] + [\tilde{s}'_K] \right) \left[{}^R \dot{\omega}'_{\mathcal{L}(K), y} \right] + \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \left([\tilde{q}'_K] + [\tilde{s}'_K] \right) \left[{}^R \omega'_{\mathcal{L}(K), y} \right] \right. \\ & \left. + [\tilde{s}'_K] \left[{}^R \omega'_{\mathcal{L}(K), y} \right] \right) + \left[R_{\mathcal{L}(K)} \right]^T \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \left[S_{s'_K} \right] \end{aligned} \quad (57)$$

This matrix can be **constructed** using the following procedure.

1. First, set

$$\begin{aligned} \left[{}^R \dot{v}_{O_K, y} \right]_{3 \times 6N} = & \left[{}^R \dot{v}_{O_{\mathcal{L}(K)}, y} \right]_{3 \times 6N} - \left[R_{\mathcal{L}(K)} \right]^T \left(\left([\tilde{q}'_K] + [\tilde{s}'_K] \right) \left[{}^R \dot{\omega}'_{\mathcal{L}(K), y} \right]_{3 \times 6N} \right. \\ & \left. - \left[R_{\mathcal{L}(K)} \right]^T \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \left([\tilde{q}'_K] + [\tilde{s}'_K] \right) \left[{}^R \omega'_{\mathcal{L}(K), y} \right]_{3 \times 6N} - \left[R_{\mathcal{L}(K)} \right]^T \left[\tilde{s}'_K \right] \left[{}^R \omega'_{\mathcal{L}(K), y} \right]_{3 \times 6N} \right) \end{aligned} \quad (58)$$

2. Then, set the **three columns** associated with $\dot{s}'_{K_i}$ ($i = 1, 2, 3$) as follows.

$$\left[{}^R \dot{v}_{O_K, y} \right]_{ik} = \left[\dot{R}_{\mathcal{L}(K)} \right]_{ij}^T = \left[\left[R_{\mathcal{L}(K)} \right]^T \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \right]_{ij} \quad (i = 1, 2, 3; j = 1, 2, 3; k = 3N + (3K - 3 + j)) \quad (59)$$

For body 1, all entries are **zero**.

$$\left[{}^R \dot{v}_{O_1, y} \right]_{3 \times 6N} = [0]_{3 \times 6N} \quad (60)$$

The **time derivatives** of the **partial angular velocity matrices** that appear in Equations (54) and (58) are presented in Unit 3.

Body Frame Components

As noted in Equation (47), the **body** $\mathcal{L}(K)$ **frame components** of the **velocity** of O_K the **origin** of body K can be written as follows.

$$\begin{aligned} \{v_{O_K}\}_{\mathcal{L}(K)} = & \left(\left[{}^{\mathcal{L}^2(K)} R_{\mathcal{L}(K)} \right] \left[{}^R v_{O_{\mathcal{L}(K)}, y} \right]_{\mathcal{L}^2(K)} - \left([\tilde{q}'_K] + [\tilde{s}'_K] \right) \left[{}^R \dot{\omega}'_{\mathcal{L}(K), y} \right] + \left[S_{s'_K} \right] \right) \{y\} \\ = & \left(\left[{}^{\mathcal{L}^2(K)} R_{\mathcal{L}(K)} \right] \left[{}^R v_{O_{\mathcal{L}(K)}, y} \right]_{\mathcal{L}^2(K)} + \left[{}^R \hat{v}_{O_K, y} \right]_{\mathcal{L}(K)} \right) \{y\} \\ = & \left[{}^R v_{O_K, y} \right]_{\mathcal{L}(K)} \{y\} \end{aligned} \quad (61)$$

Note that, the **components** of the **velocity** of the **origin** of body $\mathcal{L}(K)$ are **first expressed** in its **lower numbered body** $\mathcal{L}^2(K)$ and then **converted** into components in body $\mathcal{L}(K)$ using the **transformation matrix** $\left[{}^{\mathcal{L}^2(K)} R_{\mathcal{L}(K)} \right]$. The **components** of the **velocity** of the **origin** of body K are **first expressed** in its **lower numbered body** $\mathcal{L}(K)$ and then **converted** into components in body K using the transformation matrix $\left[{}^{\mathcal{L}(K)} R_K \right]$. The partial velocity matrix $\left[{}^R \hat{v}_{O_K, y} \right]_{\mathcal{L}(K)}$ is defined in Equation (48).

To assist in the **differentiation** of Equation (61) to find the **body frame components** of ${}^R a_{O_K}$, a more **detailed construction** of the partial velocity matrix $\left[{}^R v_{O_K, y} \right]_{\mathcal{L}(K)}$ is **warranted**. First, consider calculating ${}^R v_{O_K}$ using the concept of relative velocity as follows.

$$\begin{aligned}
{}^R\mathbf{v}_{O_K} &= {}^R\mathbf{v}_{O_K/O_{\mathcal{L}(K)}} + {}^R\mathbf{v}_{O_{\mathcal{L}(K)}/O_{\mathcal{L}^2(K)}} + \cdots + {}^R\mathbf{v}_{O_{\mathcal{L}^{u_K-1}(K)}/O_1} + {}^R\mathbf{v}_{O_1} \\
&= \sum_{r=0}^{u_K} {}^R\mathbf{v}_{O_{\mathcal{L}^r(K)}/O_{\mathcal{L}^{r+1}(K)}} \\
&= {}^R\mathbf{v}_{O_K/O_{\mathcal{L}(K)}} + \sum_{r=1}^{u_K-1} {}^R\mathbf{v}_{O_{\mathcal{L}^r(K)}/O_{\mathcal{L}^{r+1}(K)}} + {}^R\mathbf{v}_{O_1}
\end{aligned} \tag{62}$$

Recall that the **integer** u_K is the integer **value** such that $\mathcal{L}^{u_K}(K) = 1$. $\mathcal{L}^{u_K}(K)$ refers to body 1, the **system's reference body**. Hence, Equation (62) represents the **sum** of the **relative velocities** of **all** the **body origins** from body K to body 1 using the **body connection array**.

The terms in Equation (62) can be written in component forms as follows.

Body 1:

$${}^R\mathbf{v}_{O_1} \rightarrow \left\{ \mathbf{v}_{O_1} \right\}_R = \left\{ \dot{s}'_1 \right\} \quad \text{fixed frame components}$$

All other bodies:

$$\begin{aligned}
{}^R\mathbf{v}_{O_B/O_{\mathcal{L}(B)}} &\rightarrow \left\{ \mathbf{v}_{O_B/O_{\mathcal{L}(B)}} \right\}_{\mathcal{L}(B)} = \left[{}^R\hat{\mathbf{v}}_{O_B,y} \right]_{\mathcal{L}(B)} \{y\} \\
&= \left(\left[S'_{s'_B} \right] - \left([\tilde{q}'_B] + [\tilde{s}'_B] \right) \left[{}^R\omega'_{\mathcal{L}(B),y} \right] \right) \{y\}
\end{aligned} \quad \text{body } \mathcal{L}(B) \text{ components}$$

Using this concept, Equation (61) can be rewritten as follows.

$$\begin{aligned}
\left\{ \mathbf{v}_{O_K} \right\}_{\mathcal{L}(K)} &= \sum_{r=0}^{u_K} \left[{}^{\mathcal{L}^{r+1}(K)}R_{\mathcal{L}(K)} \right] \left[{}^R\hat{\mathbf{v}}_{O_{\mathcal{L}^r(K)},y} \right]_{\mathcal{L}^{r+1}(K)} \{y\} \\
&= \left[R_{\mathcal{L}(K)} \right] \left[S'_{s'_1} \right] \{y\} + \sum_{r=0}^{u_K-1} \left[{}^{\mathcal{L}^{r+1}(K)}R_{\mathcal{L}(K)} \right] \left[{}^R\hat{\mathbf{v}}_{O_{\mathcal{L}^r(K)},y} \right]_{\mathcal{L}^{r+1}(K)} \{y\} \\
&= \left(\left[R_{\mathcal{L}(K)} \right] \left[S'_{s'_1} \right] + \sum_{r=0}^{u_K-1} \left[{}^{\mathcal{L}^{r+1}(K)}R_{\mathcal{L}(K)} \right] \left[{}^R\hat{\mathbf{v}}_{O_{\mathcal{L}^r(K)},y} \right]_{\mathcal{L}^{r+1}(K)} \right) \{y\} \\
&= \left[{}^R\mathbf{v}_{O_K,y} \right]_{\mathcal{L}(K)} \{y\}
\end{aligned} \tag{63}$$

The **accelerations** of the **origins** of the bodies can now be found by **differentiating** the **fixed frame components** of the **velocities** of these points. The **body** $\mathcal{L}(K)$ **components** of ${}^R\mathbf{v}_{O_K}$ given in Equation (63) can be **converted** into **fixed frame components** by pre-multiplying both sides of the equation by $\left[R_{\mathcal{L}(K)} \right]^T$ and noting that $\left[R_{\mathcal{L}(K)} \right]^T \left[{}^{\mathcal{L}^{r+1}(K)}R_{\mathcal{L}(K)} \right] = \left[R_{\mathcal{L}^{r+1}(K)} \right]^T$. The result can then be written as follows.

$$\begin{aligned}
\{v_{O_K}\} &= \sum_{r=0}^{u_K} \left[R_{\mathcal{L}^{r+1}(K)} \right]^T \left[{}^R \hat{v}_{O_{\mathcal{L}^r(K)}, y} \right]_{\mathcal{L}^{r+1}(K)} \{y\} \\
&= \left(\left[S_{s'_1} \right] + \sum_{r=0}^{u_K-1} \left[R_{\mathcal{L}^{r+1}(K)} \right]^T \left[{}^R \hat{v}_{O_{\mathcal{L}^r(K)}, y} \right]_{\mathcal{L}^{r+1}(K)} \right) \{y\} \\
&= \left[{}^R v_{O_K, y} \right] \{y\}
\end{aligned} \tag{64}$$

Differentiating this result gives

$$\begin{aligned}
\{a_{O_K}\} &= \left[{}^R v_{O_K, y} \right] \{\dot{y}\} + \left[{}^R \dot{v}_{O_K, y} \right] \{y\} \\
&= \left(\left[S_{s'_1} \right] + \sum_{r=0}^{u_K-1} \left[R_{\mathcal{L}^{r+1}(K)} \right]^T \left[{}^R \hat{v}_{O_{\mathcal{L}^r(K)}, y} \right]_{\mathcal{L}^{r+1}(K)} \right) \{\dot{y}\} \\
&\quad + \left(\underbrace{\left[\dot{S}_{s'_1} \right]}_{\text{zero}} + \sum_{r=0}^{u_K-1} \left[R_{\mathcal{L}^{r+1}(K)} \right]^T \left[{}^R \dot{\hat{v}}_{O_{\mathcal{L}^r(K)}, y} \right]_{\mathcal{L}^{r+1}(K)} + \sum_{r=0}^{u_K-1} \left[\dot{R}_{\mathcal{L}^{r+1}(K)} \right]^T \left[{}^R \hat{v}_{O_{\mathcal{L}^r(K)}, y} \right]_{\mathcal{L}^{r+1}(K)} \right) \{y\}
\end{aligned}$$

Or,

$$\begin{aligned}
\{a_{O_K}\} &= \left(\left[S_{s'_1} \right] + \sum_{r=0}^{u_K-1} \left[R_{\mathcal{L}^{r+1}(K)} \right]^T \left[{}^R \hat{v}_{O_{\mathcal{L}^r(K)}, y} \right]_{\mathcal{L}^{r+1}(K)} \right) \{\dot{y}\} \\
&\quad + \left(\sum_{r=0}^{u_K-1} \left[R_{\mathcal{L}^{r+1}(K)} \right]^T \left[{}^R \dot{\hat{v}}_{O_{\mathcal{L}^r(K)}, y} \right]_{\mathcal{L}^{r+1}(K)} + \sum_{r=0}^{u_K-1} \left[R_{\mathcal{L}^{r+1}(K)} \right]^T \left[\tilde{\omega}'_{\mathcal{L}^{r+1}(K)} \right] \left[{}^R \hat{v}_{O_{\mathcal{L}^r(K)}, y} \right]_{\mathcal{L}^{r+1}(K)} \right) \{y\}
\end{aligned}$$

The **body frame components** of ${}^R a_{O_K}$ can be found by **pre-multiplying** by $[R_K]$.

$$\begin{aligned}
\{a'_{O_K}\} &= \left([R_K] \left[S_{s'_1} \right] + \sum_{r=0}^{u_K-1} [R_K] \left[R_{\mathcal{L}^{r+1}(K)} \right]^T \left[{}^R \hat{v}_{O_{\mathcal{L}^r(K)}, y} \right]_{\mathcal{L}^{r+1}(K)} \right) \{\dot{y}\} \\
&\quad + \left(\sum_{r=0}^{u_K-1} [R_K] \left[R_{\mathcal{L}^{r+1}(K)} \right]^T \left[{}^R \dot{\hat{v}}_{O_{\mathcal{L}^r(K)}, y} \right]_{\mathcal{L}^{r+1}(K)} + \sum_{r=0}^{u_K-1} [R_K] \left[R_{\mathcal{L}^{r+1}(K)} \right]^T \left[\tilde{\omega}'_{\mathcal{L}^{r+1}(K)} \right] \left[{}^R \hat{v}_{O_{\mathcal{L}^r(K)}, y} \right]_{\mathcal{L}^{r+1}(K)} \right) \{y\} \\
&= \left([R_K] \left[S_{s'_1} \right] + \sum_{r=0}^{u_K-1} \left[\mathcal{L}^{r+1}(K) R_K \right] \left[{}^R \hat{v}_{O_{\mathcal{L}^r(K)}, y} \right]_{\mathcal{L}^{r+1}(K)} \right) \{\dot{y}\} \\
&\quad + \left(\sum_{r=0}^{u_K-1} \left[\mathcal{L}^{r+1}(K) R_K \right] \left[{}^R \dot{\hat{v}}_{O_{\mathcal{L}^r(K)}, y} \right]_{\mathcal{L}^{r+1}(K)} + \sum_{r=0}^{u_K-1} \left[\mathcal{L}^{r+1}(K) R_K \right] \left[\tilde{\omega}'_{\mathcal{L}^{r+1}(K)} \right] \left[{}^R \hat{v}_{O_{\mathcal{L}^r(K)}, y} \right]_{\mathcal{L}^{r+1}(K)} \right) \{y\}
\end{aligned}$$

Rearranging terms gives the following.

$$\begin{aligned}
\{a'_{O_K}\} &= \left([R_K] \left[S_{s'_1} \right] + \sum_{r=0}^{u_K-1} \left[\mathcal{L}^{r+1}(K) R_K \right] \left[{}^R \hat{v}_{O_{\mathcal{L}^r(K)}, y} \right]_{\mathcal{L}^{r+1}(K)} \right) \{\dot{y}\} \\
&\quad + \left(\sum_{r=0}^{u_K-1} \left[\mathcal{L}^{r+1}(K) R_K \right] \left(\left[{}^R \dot{\hat{v}}_{O_{\mathcal{L}^r(K)}, y} \right]_{\mathcal{L}^{r+1}(K)} + \left[\tilde{\omega}'_{\mathcal{L}^{r+1}(K)} \right] \left[{}^R \hat{v}_{O_{\mathcal{L}^r(K)}, y} \right]_{\mathcal{L}^{r+1}(K)} \right) \right) \{y\}
\end{aligned} \tag{65}$$

Separating out the terms associated with O_K , Equation (65) can be rewritten as follows.

$$\begin{aligned} \{a'_{O_K}\} = & \left([R_K] [S_{s'_1}] + \sum_{r=1}^{u_K-1} \left[{}^{\mathcal{L}^{r+1}(K)} R_K \right] \left[{}^R \hat{v}_{O_{\mathcal{L}^r(K)}, y} \right]_{\mathcal{L}^{r+1}(K)} \right) \{\dot{y}\} \\ & + \left(\sum_{r=1}^{u_K-1} \left[{}^{\mathcal{L}^{r+1}(K)} R_K \right] \left(\left[{}^R \dot{\hat{v}}_{O_{\mathcal{L}^r(K)}, y} \right]_{\mathcal{L}^{r+1}(K)} + \left[\tilde{\omega}'_{\mathcal{L}^{r+1}(K)} \right] \left[{}^R \hat{v}_{O_{\mathcal{L}^r(K)}, y} \right]_{\mathcal{L}^{r+1}(K)} \right) \right) \{y\} \\ & + \left[{}^{\mathcal{L}(K)} R_K \right] \left[{}^R \hat{v}_{O_K, y} \right]_{\mathcal{L}(K)} \{\dot{y}\} + \left[{}^{\mathcal{L}(K)} R_K \right] \left(\left[{}^R \dot{\hat{v}}_{O_K, y} \right]_{\mathcal{L}(K)} + \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \left[{}^R \hat{v}_{O_K, y} \right]_{\mathcal{L}(K)} \right) \{y\} \end{aligned}$$

Or,

$$\begin{aligned} \{a'_{O_K}\} = & \{a_{O_{\mathcal{L}(K)}}\}_K + \{a'_{O_K/O_{\mathcal{L}(K)}}\} \\ = & \{a_{O_{\mathcal{L}(K)}}\}_K \\ & + \left[{}^{\mathcal{L}(K)} R_K \right] \left[{}^R \hat{v}_{O_K, y} \right]_{\mathcal{L}(K)} \{\dot{y}\} + \left[{}^{\mathcal{L}(K)} R_K \right] \left(\left[{}^R \dot{\hat{v}}_{O_K, y} \right]_{\mathcal{L}(K)} + \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \left[{}^R \hat{v}_{O_K, y} \right]_{\mathcal{L}(K)} \right) \{y\} \end{aligned} \quad (66)$$

Here, letting $J = \mathcal{L}(K)$, the terms in Equation (66) can be written as follows.

$$\begin{aligned} \{a_{O_J}\}_K = & \left[{}^J R_K \right] \{a'_{O_J}\} \\ = & \left[{}^J R_K \right] \left(\left[R_J \right] [S_{s'_1}] \{\dot{y}\} + \left(\sum_{r=0}^{u_J-1} \left[{}^{\mathcal{L}^{r+1}(J)} R_J \right] \left[{}^R \hat{v}_{O_{\mathcal{L}^r(J)}, y} \right]_{\mathcal{L}^{r+1}(J)} \right) \{\dot{y}\} \right) \\ & + \left[{}^J R_K \right] \left(\left(\sum_{r=0}^{u_J-1} \left[{}^{\mathcal{L}^{r+1}(J)} R_J \right] \left(\left[{}^R \dot{\hat{v}}_{O_{\mathcal{L}^r(J)}, y} \right]_{\mathcal{L}^{r+1}(J)} + \left[\tilde{\omega}'_{\mathcal{L}^{r+1}(J)} \right] \left[{}^R \hat{v}_{O_{\mathcal{L}^r(J)}, y} \right]_{\mathcal{L}^{r+1}(J)} \right) \right) \{y\} \right) \end{aligned} \quad (67)$$

$$\{a'_{O_K/O_J}\} = \left[{}^J R_K \right] \left(\left[{}^R \hat{v}_{O_K, y} \right]_J \{\dot{y}\} + \left(\left[{}^R \dot{\hat{v}}_{O_K, y} \right]_J + \left[\tilde{\omega}'_J \right] \left[{}^R \hat{v}_{O_K, y} \right]_J \right) \{y\} \right) \quad (68)$$

Together, Equations (66)-(68) form a **procedure** for **calculating** the **accelerations** of the **origins** of the bodies of the system **recursively**.

To **provide further clarification** to the terms in Equation (65)-(68), the **fixed frame components** of ${}^R a_{O_B/O_{\mathcal{L}(B)}}$ the **acceleration** of the **origin** of a body B **relative** to the **origin** of its **lower numbered body** $\mathcal{L}(B)$ can also be calculated from the $\mathcal{L}(B)$ components of ${}^R v_{O_B/O_{\mathcal{L}(B)}}$ by **differentiating** as follows.

$$\{a_{O_B/O_{\mathcal{L}(B)}}\} = \frac{d}{dt} \left(\left[R_{\mathcal{L}(B)} \right]^T \left\{ v_{O_B/O_{\mathcal{L}(B)}} \right\}_{\mathcal{L}(B)} \right) = \frac{d}{dt} \left(\left[R_{\mathcal{L}(B)} \right]^T \left[{}^R \hat{v}_{O_B, y} \right]_{\mathcal{L}(B)} \{y\} \right) \quad (69)$$

Expanding this result gives the following.

$$\begin{aligned} \{a_{O_B/O_{\mathcal{L}(B)}}\} = & \left[R_{\mathcal{L}(B)} \right]^T \left[{}^R \hat{v}_{O_B, y} \right]_{\mathcal{L}(B)} \{\dot{y}\} + \left[R_{\mathcal{L}(B)} \right]^T \left[{}^R \dot{\hat{v}}_{O_B, y} \right]_{\mathcal{L}(B)} \{y\} \\ & + \left[R_{\mathcal{L}(B)} \right]^T \left[\tilde{\omega}'_{\mathcal{L}(B)} \right] \left[{}^R \hat{v}_{O_B, y} \right]_{\mathcal{L}(B)} \{y\} \end{aligned} \quad (70)$$

The **body frame components** of ${}^R \underline{a}_{O_B/O_{\mathcal{E}(B)}}$ can now be found by **pre-multiplying** this result by $[R_B]$.

$$\begin{aligned} \{a'_{O_B/O_{\mathcal{E}(B)}}\} &= [R_B] \{a_{O_B/O_{\mathcal{E}(B)}}\} \\ &= [{}^{\mathcal{E}(B)}R_B] \left([{}^R \hat{v}_{O_B,y}]_{\mathcal{E}(B)} \{\dot{y}\} + \left([{}^R \dot{\hat{v}}_{O_B,y}]_{\mathcal{E}(B)} + [\tilde{\omega}'_{\mathcal{E}(B)}] [{}^R \hat{v}_{O_B,y}]_{\mathcal{E}(B)} \right) \{y\} \right) \end{aligned} \quad (71)$$

Here,

$$[{}^R \hat{v}_{O_B,y}]_{\mathcal{E}(B)} = [S'_{s'_B}] - ([\tilde{q}'_B] + [\tilde{s}'_B]) [{}^R \omega'_{\mathcal{E}(B),y}] \quad (72)$$

$$[{}^R \dot{\hat{v}}_{O_B,y}]_{\mathcal{E}(B)} = \underbrace{[\dot{S}'_{s'_B}]}_{\text{zero}} - ([\tilde{q}'_B] + [\tilde{s}'_B]) [{}^R \dot{\omega}'_{\mathcal{E}(B),y}] - [\dot{\tilde{s}}'_B] [{}^R \omega'_{\mathcal{E}(B),y}] \quad (73)$$

Using these observations, Equation (65) can also be rewritten as follows.

$$\{a'_{O_K}\} = [R_K] \{a_{O_1}\}_R + \sum_{r=0}^{u_K-1} [{}^{\mathcal{E}^{r+1}(K)}R_K] \{a_{O_{\mathcal{E}^r(K)}/O_{\mathcal{E}^{r+1}(K)}}\}_{\mathcal{E}^{r+1}(K)} \quad (74)$$

Accelerations of Body Mass Centers in Terms of Partial Velocities and Their Time Derivatives

The **accelerations** of the **mass centers** of the **bodies** can be calculated in terms of their **partial velocity matrices** and their **time derivatives** as detailed below.

Partial Velocities - Fixed Frame Components

In Unit 4, $\{v_K\}$ the **fixed frame components** of the **velocity** of G_K the mass center of body K were written in terms of the **fixed frame components** of the **velocity** of O_K the origin of body K and the **base frame components** of the **relative angular velocity vectors** as follows.

$$\begin{aligned} \{v_K\} &= \{v_{O_K}\} - [\tilde{r}_K] \{\omega_K\} \\ &= [{}^R v_{O_K,y}] \{y\} - [\tilde{r}_K] [{}^R \omega_{K,y}] \{y\} \\ &= [{}^R v_{K,y}] \{y\} \end{aligned} \quad (75)$$

Here,

$$[{}^R v_{K,y}]_{3 \times 6N} = [{}^R v_{O_K,y}]_{3 \times 6N} - [\tilde{r}_K] [{}^R \omega_{K,y}]_{3 \times 6N} \quad (76)$$

Matrices $[{}^R v_{K,y}]$ and $[{}^R v_{O_K,y}]$ represent the **fixed frame components** of the **partial velocity matrices** for G_K and O_K , matrix $[{}^R \omega_{K,y}]$ represents the **fixed frame components** of the **partial angular velocity matrix** for body K , and matrix $[\tilde{r}_K]$ represents a 3×3 **skew symmetric matrix** built using the **fixed frame components** of the position vector $\underline{r}_K$. The partial velocity matrix $[{}^R v_{O_K,y}]$ is defined in Equation (38).

In Unit 4, $\{v_K\}$ the **fixed frame components** of the **velocity** of G_K were also written in terms of the **fixed frame components** of the **velocity** of O_K and the **body frame components** of the **relative angular velocity vectors** as follows.

$$\begin{aligned}\{v_K\} &= \{v_{O_K}\} - [R_K]^T [\tilde{r}'_K] \{\omega'_K\} \\ &= [{}^R v_{O_K,y}] \{y\} - [R_K]^T [\tilde{r}'_K] [{}^R \omega'_{K,y}] \{y\} \\ &= [{}^R v_{K,y}] \{y\}\end{aligned}\quad (77)$$

Here,

$$[{}^R v_{K,y}]_{3 \times 6N} = [{}^R v_{O_K,y}]_{3 \times 6N} - [R_K]^T [\tilde{r}'_K] [{}^R \omega'_{K,y}]_{3 \times 6N} \quad (78)$$

Matrices $[{}^R v_{K,y}]$ and $[{}^R v_{O_K,y}]$ represent the **fixed frame components** of the **partial velocity matrices** for G_K and O_K , matrix $[{}^R \omega'_{K,y}]$ represents the **body frame components** of the **partial angular velocity matrix** for body K , and matrix $[\tilde{r}'_K]$ represents a 3×3 **skew symmetric matrix** built using the **body frame components** of the position vector r_K . The partial velocity matrix $[{}^R v_{O_K,y}]$ is defined in Equation (42).

Partial Velocities - Body Frame Components

In Unit 4, $\{v'_K\}$ the **body frame components** of the **velocity** of G_K were also written in terms of the **body frame components** of the **velocity** of O_K and the **body frame components** of the **relative angular velocity vectors** as follows.

$$\begin{aligned}\{v'_K\} &= \{v'_{O_K}\} - [\tilde{r}'_K] \{\omega'_K\} \\ &= [{}^{\mathcal{L}(K)} R_K] \{v_{O_K}\}_{\mathcal{L}(K)} - [\tilde{r}'_K] \{\omega'_K\} \\ &= [{}^{\mathcal{L}(K)} R_K] [{}^R v_{O_K,y}]_{\mathcal{L}(K)} \{y\} - [\tilde{r}'_K] [{}^R \omega'_{K,y}] \{y\} \\ &= [{}^R v'_{K,y}] \{y\}\end{aligned}\quad (79)$$

Here,

$$[{}^R v'_{K,y}]_{3 \times 6N} = [{}^{\mathcal{L}(K)} R_K] [{}^R v_{O_K,y}]_{\mathcal{L}(K)} - [\tilde{r}'_K] [{}^R \omega'_{K,y}]_{3 \times 6N} \quad (80)$$

Matrix $[{}^R v'_{K,y}]$ represents the **body frame components** of the **partial velocity matrix** for G_K , matrix $[{}^R v_{O_K,y}]_{\mathcal{L}(K)}$ represents the **body $\mathcal{L}(K)$ components** of the **partial velocity matrix** of O_K , matrix $[{}^R \omega'_{K,y}]$ represents the **body frame components** of the **partial angular velocity matrix** for body K , and matrix $[\tilde{r}'_K]$ represents a 3×3 **skew symmetric matrix** built using the **body frame components** of the position vector r_K . The **partial velocity matrix** $[{}^R v_{O_K,y}]_{\mathcal{L}(K)}$ is defined in Equation (47).

Accelerations – Fixed Frame Components

The **fixed frame components** of the **acceleration** of G_K the mass center of body K can be found by **differentiating** the **fixed frame velocity components** in terms of the **partial velocity** of G_K as follows.

$$\boxed{\{a_K\} = \{\dot{v}_K\} = \begin{bmatrix} {}^R v_{K,y} \end{bmatrix} \{\dot{y}\} + \begin{bmatrix} {}^R \dot{v}_{K,y} \end{bmatrix} \{y\}} \quad (81)$$

Using **base frame components** of the **relative angular velocity vectors**, the **partial velocity matrix** can be calculated using Equation (76). That equation can be **differentiated** to define a process for calculating $\begin{bmatrix} {}^R \dot{v}_{K,y} \end{bmatrix}$ as follows.

$$\boxed{\begin{bmatrix} {}^R \dot{v}_{K,y} \end{bmatrix}_{3 \times 6N} = \begin{bmatrix} {}^R \dot{v}_{O_K,y} \end{bmatrix}_{3 \times 6N} - [\tilde{r}_K] \begin{bmatrix} {}^R \dot{\omega}_{K,y} \end{bmatrix}_{3 \times 6N} - \begin{bmatrix} \dot{\tilde{r}}_K \end{bmatrix} \begin{bmatrix} {}^R \omega_{K,y} \end{bmatrix}_{3 \times 6N}} \quad (82)$$

The matrix $\begin{bmatrix} {}^R \dot{v}_{O_K,y} \end{bmatrix}$ is the **time derivative** of $\begin{bmatrix} {}^R v_{O_K,y} \end{bmatrix}$ the **partial velocity matrix** of O_K and is defined in Equation (53), and the matrix $\begin{bmatrix} \dot{\tilde{r}}_K \end{bmatrix}$ is a 3×3 **skew symmetric matrix** built using the **fixed frame components** of $\frac{{}^R dr_K}{dt}$. Calculation of these components is shown in Equation (23).

Using **body frame components** of the **relative angular velocity vectors**, the partial velocity matrix can be calculated using Equation (78). That equation can be **differentiated** to define a process for calculating $\begin{bmatrix} {}^R \dot{v}_{K,y} \end{bmatrix}$ as follows.

$$\begin{aligned} \begin{bmatrix} {}^R \dot{v}_{K,y} \end{bmatrix}_{3 \times 6N} &= \begin{bmatrix} {}^R \dot{v}_{O_K,y} \end{bmatrix}_{3 \times 6N} - [R_K]^T [\tilde{r}'_K] \begin{bmatrix} {}^R \dot{\omega}'_{K,y} \end{bmatrix}_{3 \times 6N} - [\dot{R}_K]^T [\tilde{r}'_K] \begin{bmatrix} {}^R \omega'_{K,y} \end{bmatrix}_{3 \times 6N} - [R_K]^T \underbrace{\begin{bmatrix} \dot{\tilde{r}}'_K \end{bmatrix}}_{\text{zero}} \begin{bmatrix} {}^R \omega'_{K,y} \end{bmatrix}_{3 \times 6N} \\ \boxed{\begin{bmatrix} {}^R \dot{v}_{K,y} \end{bmatrix}_{3 \times 6N} &= \begin{bmatrix} {}^R \dot{v}_{O_K,y} \end{bmatrix}_{3 \times 6N} - [R_K]^T [\tilde{r}'_K] \begin{bmatrix} {}^R \dot{\omega}'_{K,y} \end{bmatrix}_{3 \times 6N} - [R_K]^T [\tilde{\omega}'_K] [\tilde{r}'_K] \begin{bmatrix} {}^R \omega'_{K,y} \end{bmatrix}_{3 \times 6N}} \end{aligned} \quad (83)$$

The matrix $\begin{bmatrix} {}^R \dot{v}_{O_K,y} \end{bmatrix}$ is the **time derivative** of $\begin{bmatrix} {}^R v_{O_K,y} \end{bmatrix}$ the partial velocity matrix of O_K and is defined in Equation (57). As noted above, the **time derivatives** of the **partial angular velocity matrices** that appear in Equations (82) and (83) are presented in Unit 3.

Accelerations – Body Frame Components

In Equation (79), the **body frame components** of the **velocity** of G_K the **mass center** of body K are written as follows.

$$\boxed{\begin{aligned} \{v'_K\} &= \left(\begin{bmatrix} {}^{\mathcal{E}(K)} R_K \end{bmatrix} \begin{bmatrix} {}^R v_{O_K,y} \end{bmatrix}_{\mathcal{E}(K)} - [\tilde{r}'_K] \begin{bmatrix} {}^R \omega'_{K,y} \end{bmatrix} \right) \{y\} \\ &\triangleq \begin{bmatrix} {}^R v'_{K,y} \end{bmatrix} \{y\} \end{aligned}} \quad (84)$$

The **partial velocity matrix** $\begin{bmatrix} R v'_{K,y} \end{bmatrix}$ is defined by using **components resolved** in the **bodies** between body 1 and body K , so taking the **derivative** of Equation (84) to find the **body frame components** of ${}^R \underline{a}_K$ is **not a straightforward process**. The concept of **relative acceleration** is used here instead.

$$\begin{aligned} {}^R \underline{a}_K &= {}^R \underline{a}_{O_K} + {}^R \underline{a}_{G_K/O_K} \\ &= {}^R \underline{a}_{O_K} + \left({}^R \underline{\omega}_K \times \underline{r}_K \right) + {}^R \underline{\omega}_K \times \left({}^R \underline{\omega}_K \times \underline{r}_K \right) \end{aligned} \quad (85)$$

Using this result, the **body frame components** of ${}^R \underline{a}_K$ the acceleration of G_K can be written as follows.

$$\{a'_K\} = \{a'_{O_K}\} + [\tilde{\alpha}'_K] \{r'_K\} + [\tilde{\omega}'_K] [\tilde{\omega}'_K] \{r'_K\} \quad (86)$$

The **body frame components** of ${}^R \underline{a}_{O_K}$ can be found using Equations (66)-(68). Using those results gives

$$\begin{aligned} \{a'_K\} &= \{a'_{O_K}\} + [\tilde{\alpha}'_K] \{r'_K\} + [\tilde{\omega}'_K] [\tilde{\omega}'_K] \{r'_K\} \\ &= \{a_{O_{\mathcal{E}(K)}}\}_K + \{a'_{O_K/O_{\mathcal{E}(K)}}\} + [\tilde{\alpha}'_K] \{r'_K\} + [\tilde{\omega}'_K] [\tilde{\omega}'_K] \{r'_K\} \\ \{a'_{O_B/O_{\mathcal{E}(B)}}\} &= [R_B] \{a_{O_B/O_{\mathcal{E}(B)}}\} \\ &= [\mathcal{E}^{(B)} R_B] \left(\left[{}^R \hat{v}_{O_B,y} \right]_{\mathcal{E}(B)} \{y\} + \left(\left[{}^R \hat{v}_{O_B,y} \right]_{\mathcal{E}(B)} + [\tilde{\omega}'_{\mathcal{E}(B)}] \left[{}^R \hat{v}_{O_B,y} \right]_{\mathcal{E}(B)} \right) \{y\} \right) \end{aligned}$$

Equations (69)-(74) **show** that $\{a'_{O_K}\}$ can be calculated as the **sum** of the **relative accelerations** of **all** the **reference points** from body K to body 1 using the **body connection array**. Using this result, the above equation can be rewritten as follows.

$$\begin{aligned} \{a'_K\} &= \{a'_{O_K}\} + [\tilde{\alpha}'_K] \{r'_K\} + [\tilde{\omega}'_K] [\tilde{\omega}'_K] \{r'_K\} \\ &= [R_K] \{a_{O_1}\}_R + \sum_{r=0}^{u_K-1} \left[\mathcal{E}^{r+1(K)} R_K \right] \left\{ a_{O_{\mathcal{E}^r(K)}/O_{\mathcal{E}^{r+1}(K)}} \right\}_{\mathcal{E}^{r+1}(K)} + [\tilde{\alpha}'_K] \{r'_K\} + [\tilde{\omega}'_K] [\tilde{\omega}'_K] \{r'_K\} \end{aligned} \quad (87)$$

Comparison of Acceleration Results

Body Origins – Fixed Frame Components

To **compare** the results of the methods outlined above, the equations written in terms of the partial velocity and partial angular velocity matrices can be expanded. The expanded results can then be compared with the earlier equations. Equation (52) is **expanded** below using both, first using **base frame components** for the **relative angular velocities**, and then using **body frame components** for the **relative angular velocities**.

Using **base frame components** for the **relative angular velocities**, the **fixed frame components** of the **acceleration** of O_K can be expanded as follows.

$$\begin{aligned}
\{a_{O_K}\} &= \begin{bmatrix} {}^R v_{O_K,y} \end{bmatrix} \{\dot{y}\} + \begin{bmatrix} {}^R \dot{v}_{O_K,y} \end{bmatrix} \{y\} \\
&= \left(\begin{bmatrix} {}^R v_{O_{\mathcal{E}(K)},y} \end{bmatrix} - ([\tilde{q}_K] + [\tilde{s}_K]) \begin{bmatrix} {}^R \omega_{\mathcal{E}(K),y} \end{bmatrix} + [R_{\mathcal{E}(K)}]^T [S_{s'_K}] \right) \{\dot{y}\} \\
&\quad + \left(\begin{bmatrix} {}^R \dot{v}_{O_{\mathcal{E}(K)},y} \end{bmatrix} - ([\tilde{q}_K] + [\tilde{s}_K]) \begin{bmatrix} {}^R \dot{\omega}_{\mathcal{E}(K),y} \end{bmatrix} - ([\dot{\tilde{q}}_K] + [\dot{\tilde{s}}_K]) \begin{bmatrix} {}^R \omega_{\mathcal{E}(K),y} \end{bmatrix} \right) \{y\} \\
&\quad + [\tilde{\omega}_{\mathcal{E}(K)}] [R_{\mathcal{E}(K)}]^T [S_{s'_K}] \{y\} \\
&= \left(\begin{bmatrix} {}^R v_{O_{\mathcal{E}(K)},y} \end{bmatrix} \{\dot{y}\} + \begin{bmatrix} {}^R \dot{v}_{O_{\mathcal{E}(K)},y} \end{bmatrix} \{y\} \right) - ([\tilde{q}_K] + [\tilde{s}_K]) \left(\begin{bmatrix} {}^R \omega_{\mathcal{E}(K),y} \end{bmatrix} \{\dot{y}\} + \begin{bmatrix} {}^R \dot{\omega}_{\mathcal{E}(K),y} \end{bmatrix} \{y\} \right) \\
&\quad + [R_{\mathcal{E}(K)}]^T \{\ddot{s}'_K\} - ([\dot{\tilde{q}}_K] + [\dot{\tilde{s}}_K]) \begin{bmatrix} {}^R \omega_{\mathcal{E}(K),y} \end{bmatrix} \{y\} + [\tilde{\omega}_{\mathcal{E}(K)}] [R_{\mathcal{E}(K)}]^T \{\dot{s}'_K\}
\end{aligned} \tag{88}$$

To simplify this result, note the following.

$$\begin{aligned}
\{\omega_{\mathcal{E}(K)}\} &= \begin{bmatrix} {}^R \omega_{\mathcal{E}(K),y} \end{bmatrix} \{y\} \quad (\text{fixed frame components of } {}^R \omega_K) \\
\{\dot{\omega}_{\mathcal{E}(K)}\} &= \begin{bmatrix} {}^R \omega_{\mathcal{E}(K),y} \end{bmatrix} \{\dot{y}\} + \begin{bmatrix} {}^R \dot{\omega}_{\mathcal{E}(K),y} \end{bmatrix} \{y\} \quad (\text{fixed frame components of } {}^R \alpha_{\mathcal{E}(K)}) \\
\{v_{O_{\mathcal{E}(K)}}\} &= \begin{bmatrix} {}^R v_{O_{\mathcal{E}(K)},y} \end{bmatrix} \{y\} \quad (\text{fixed frame components of } {}^R v_{O_{\mathcal{E}(K)}}) \\
\{a_{O_{\mathcal{E}(K)}}\} &= \{\dot{v}_{O_{\mathcal{E}(K)}}\} = \begin{bmatrix} {}^R v_{O_{\mathcal{E}(K)},y} \end{bmatrix} \{\dot{y}\} + \begin{bmatrix} {}^R \dot{v}_{O_{\mathcal{E}(K)},y} \end{bmatrix} \{y\} \quad (\text{fixed frame components of } {}^R a_{O_{\mathcal{E}(K)}})
\end{aligned}$$

Substituting from these results into Equation (88) gives the result.

$$\begin{aligned}
\{a_{O_K}\} &= \{a_{O_{\mathcal{E}(K)}}\} - ([\tilde{q}_K] + [\tilde{s}_K]) \{\dot{\omega}_{\mathcal{E}(K)}\} + [R_{\mathcal{E}(K)}]^T \{\ddot{s}'_K\} \\
&\quad - ([\dot{\tilde{q}}_K] + [\dot{\tilde{s}}_K]) \{\omega_{\mathcal{E}(K)}\} + [\tilde{\omega}_{\mathcal{E}(K)}] [R_{\mathcal{E}(K)}]^T \{\dot{s}'_K\} \\
&= \{a_{O_{\mathcal{E}(K)}}\} + [\tilde{\omega}_{\mathcal{E}(K)}] (\{q_K\} + \{s_K\}) + [R_{\mathcal{E}(K)}]^T \{\ddot{s}'_K\} \\
&\quad + [\tilde{\omega}_{\mathcal{E}(K)}] (\{\dot{q}_K\} + \{\dot{s}_K\}) + [\tilde{\omega}_{\mathcal{E}(K)}] [R_{\mathcal{E}(K)}]^T \{\dot{s}'_K\}
\end{aligned} \tag{89}$$

This result is **identical** to the one shown above in Equation (10).

Using **body frame components** for the **angular velocities** and **relative angular velocities**, the **fixed frame components** of the **acceleration** of O_K can be expanded as follows.

$$\begin{aligned}
\{a_{O_K}\} &= \begin{bmatrix} {}^R v_{O_K,y} \end{bmatrix} \{\dot{y}\} + \begin{bmatrix} {}^R \dot{v}_{O_K,y} \end{bmatrix} \{y\} \\
&= \left(\begin{bmatrix} {}^R v_{O_{\mathcal{E}(K)},y} \end{bmatrix} - [R_{\mathcal{E}(K)}]^T ([\tilde{q}'_K] + [\tilde{s}'_K]) \begin{bmatrix} {}^R \omega'_{\mathcal{E}(K),y} \end{bmatrix} + [R_{\mathcal{E}(K)}]^T [S_{s'_K}] \right) \{\dot{y}\} \\
&\quad + \left(\begin{bmatrix} {}^R \dot{v}_{O_{\mathcal{E}(K)},y} \end{bmatrix} - [R_{\mathcal{E}(K)}]^T ([\tilde{q}'_K] + [\tilde{s}'_K]) \begin{bmatrix} {}^R \dot{\omega}'_{\mathcal{E}(K),y} \end{bmatrix} \right) \{y\} \\
&\quad + \left(-[R_{\mathcal{E}(K)}]^T [\tilde{\omega}'_{\mathcal{E}(K)}] ([\tilde{q}'_K] + [\tilde{s}'_K]) \begin{bmatrix} {}^R \omega'_{\mathcal{E}(K),y} \end{bmatrix} \right) \{y\} \\
&\quad + \left([R_{\mathcal{E}(K)}]^T [\tilde{\omega}'_{\mathcal{E}(K)}] [S_{s'_K}] - [R_{\mathcal{E}(K)}]^T [\tilde{s}'_K] \begin{bmatrix} {}^R \omega'_{\mathcal{E}(K),y} \end{bmatrix} \right) \{y\}
\end{aligned}$$

Or, after rearranging terms

$$\begin{aligned} \{a_{O_K}\} = & \left(\begin{bmatrix} {}^R v_{O_{\mathcal{E}(K)},y} \end{bmatrix} \{\dot{y}\} + \begin{bmatrix} {}^R \dot{v}_{O_{\mathcal{E}(K)},y} \end{bmatrix} \{y\} \right) - \begin{bmatrix} R_{\mathcal{E}(K)} \end{bmatrix}^T \left(\begin{bmatrix} \tilde{q}'_K \end{bmatrix} + \begin{bmatrix} \tilde{s}'_K \end{bmatrix} \right) \left(\begin{bmatrix} {}^R \omega'_{\mathcal{E}(K),y} \end{bmatrix} \{\dot{y}\} + \begin{bmatrix} {}^R \dot{\omega}'_{\mathcal{E}(K),y} \end{bmatrix} \{y\} \right) \\ & - \begin{bmatrix} R_{\mathcal{E}(K)} \end{bmatrix}^T \begin{bmatrix} \tilde{\omega}'_{\mathcal{E}(K)} \end{bmatrix} \left(\begin{bmatrix} \tilde{q}'_K \end{bmatrix} + \begin{bmatrix} \tilde{s}'_K \end{bmatrix} \right) \begin{bmatrix} {}^R \omega'_{\mathcal{E}(K),y} \end{bmatrix} \{y\} - \begin{bmatrix} R_{\mathcal{E}(K)} \end{bmatrix}^T \begin{bmatrix} \tilde{s}'_K \end{bmatrix} \begin{bmatrix} {}^R \omega'_{\mathcal{E}(K),y} \end{bmatrix} \{y\} \\ & + \begin{bmatrix} R_{\mathcal{E}(K)} \end{bmatrix}^T \{\dot{s}'_K\} + \begin{bmatrix} R_{\mathcal{E}(K)} \end{bmatrix}^T \begin{bmatrix} \tilde{\omega}'_{\mathcal{E}(K)} \end{bmatrix} \{\dot{s}'_K\} \end{aligned} \quad (90)$$

To simplify this result, note the following.

$$\{\omega'_{\mathcal{E}(K)}\} = \begin{bmatrix} {}^R \omega'_{\mathcal{E}(K),y} \end{bmatrix} \{y\} \quad (\text{body frame components of } {}^R \omega_{\mathcal{E}(K)})$$

$$\{\dot{\omega}'_{\mathcal{E}(K)}\} = \begin{bmatrix} {}^R \omega'_{\mathcal{E}(K),y} \end{bmatrix} \{\dot{y}\} + \begin{bmatrix} {}^R \dot{\omega}'_{\mathcal{E}(K),y} \end{bmatrix} \{y\} \quad (\text{body frame components of } {}^R \alpha_{\mathcal{E}(K)})$$

$$\{v_{O_{\mathcal{E}(K)}}\} = \begin{bmatrix} {}^R v_{O_{\mathcal{E}(K)},y} \end{bmatrix} \{y\} \quad (\text{fixed frame components of } {}^R v_{O_{\mathcal{E}(K)}})$$

$$\{a_{O_{\mathcal{E}(K)}}\} = \{\dot{v}_{O_{\mathcal{E}(K)}}\} = \begin{bmatrix} {}^R v_{O_{\mathcal{E}(K)},y} \end{bmatrix} \{\dot{y}\} + \begin{bmatrix} {}^R \dot{v}_{O_{\mathcal{E}(K)},y} \end{bmatrix} \{y\} \quad (\text{fixed frame components of } {}^R a_{O_{\mathcal{E}(K)}})$$

Substituting these results into Equation (90) gives the following.

$$\begin{aligned} \{a_{O_K}\} = & \{a_{O_{\mathcal{E}(K)}}\} - \begin{bmatrix} R_{\mathcal{E}(K)} \end{bmatrix}^T \left(\begin{bmatrix} \tilde{q}'_K \end{bmatrix} + \begin{bmatrix} \tilde{s}'_K \end{bmatrix} \right) \{\dot{\omega}'_{\mathcal{E}(K)}\} + \begin{bmatrix} R_{\mathcal{E}(K)} \end{bmatrix}^T \{\dot{s}'_K\} \\ & - \begin{bmatrix} R_{\mathcal{E}(K)} \end{bmatrix}^T \left(\begin{bmatrix} \tilde{\omega}'_{\mathcal{E}(K)} \end{bmatrix} \left(\begin{bmatrix} \tilde{q}'_K \end{bmatrix} + \begin{bmatrix} \tilde{s}'_K \end{bmatrix} \right) + \begin{bmatrix} \dot{s}'_K \end{bmatrix} \right) \{\omega'_{\mathcal{E}(K)}\} + \begin{bmatrix} R_{\mathcal{E}(K)} \end{bmatrix}^T \begin{bmatrix} \tilde{\omega}'_{\mathcal{E}(K)} \end{bmatrix} \{\dot{s}'_K\} \end{aligned} \quad (91)$$

The terms in Equation (91) can be further rearranged to produce the following result.

$$\begin{aligned} \{a_{O_K}\} = & \{a_{O_{\mathcal{E}(K)}}\} - \begin{bmatrix} R_{\mathcal{E}(K)} \end{bmatrix}^T \left(\begin{bmatrix} \tilde{q}'_K \end{bmatrix} + \begin{bmatrix} \tilde{s}'_K \end{bmatrix} \right) \{\dot{\omega}'_{\mathcal{E}(K)}\} + \begin{bmatrix} R_{\mathcal{E}(K)} \end{bmatrix}^T \{\dot{s}'_K\} \\ & - \begin{bmatrix} R_{\mathcal{E}(K)} \end{bmatrix}^T \left(\begin{bmatrix} \tilde{\omega}'_{\mathcal{E}(K)} \end{bmatrix} \left(\begin{bmatrix} \tilde{q}'_K \end{bmatrix} + \begin{bmatrix} \tilde{s}'_K \end{bmatrix} \right) + \begin{bmatrix} \dot{s}'_K \end{bmatrix} \right) \{\omega'_{\mathcal{E}(K)}\} + \begin{bmatrix} R_{\mathcal{E}(K)} \end{bmatrix}^T \begin{bmatrix} \tilde{\omega}'_{\mathcal{E}(K)} \end{bmatrix} \{\dot{s}'_K\} \\ = & \{a_{O_{\mathcal{E}(K)}}\} + \begin{bmatrix} R_{\mathcal{E}(K)} \end{bmatrix}^T \begin{bmatrix} \tilde{\omega}'_{\mathcal{E}(K)} \end{bmatrix} (\{q'_K\} + \{s'_K\}) + \begin{bmatrix} R_{\mathcal{E}(K)} \end{bmatrix}^T \begin{bmatrix} \tilde{\omega}'_{\mathcal{E}(K)} \end{bmatrix} \begin{bmatrix} \tilde{\omega}'_{\mathcal{E}(K)} \end{bmatrix} (\{q'_K\} + \{s'_K\}) \\ & + \begin{bmatrix} R_{\mathcal{E}(K)} \end{bmatrix}^T \{\dot{s}'_K\} + \begin{bmatrix} R_{\mathcal{E}(K)} \end{bmatrix}^T \begin{bmatrix} \tilde{\omega}'_{\mathcal{E}(K)} \end{bmatrix} \{\dot{s}'_K\} + \begin{bmatrix} R_{\mathcal{E}(K)} \end{bmatrix}^T \begin{bmatrix} \tilde{\omega}'_{\mathcal{E}(K)} \end{bmatrix} \{\dot{s}'_K\} \end{aligned}$$

Or,

$$\begin{aligned} \{a_{O_K}\} = & \{a_{O_{\mathcal{E}(K)}}\} + \begin{bmatrix} R_{\mathcal{E}(K)} \end{bmatrix}^T \left(\begin{bmatrix} \tilde{\omega}'_{\mathcal{E}(K)} \end{bmatrix} (\{q'_K\} + \{s'_K\}) + \begin{bmatrix} \tilde{\omega}'_{\mathcal{E}(K)} \end{bmatrix} \begin{bmatrix} \tilde{\omega}'_{\mathcal{E}(K)} \end{bmatrix} (\{q'_K\} + \{s'_K\}) \right. \\ & \left. + \{\dot{s}'_K\} + 2 \begin{bmatrix} \tilde{\omega}'_{\mathcal{E}(K)} \end{bmatrix} \{\dot{s}'_K\} \right) \end{aligned} \quad (92)$$

This result is **identical** to the one shown above in Equation (18).

Body Origins – Body Frame Components

Using **body frame components** for the **angular velocities** and **relative angular velocities**, the terms in Equation (74) for the **body frame components** in of the **acceleration** of O_K can be rearranged as follows.

$$\begin{aligned}
\{a'_{O_K}\} &= [R_K] \{a_{O_1}\}_R + \sum_{r=0}^{u_K-1} \left[{}^{\mathcal{L}^{r+1}(K)} R_K \right] \left\{ a_{O_{\mathcal{L}^r(K)}/O_{\mathcal{L}^{r+1}(K)}} \right\}_{\mathcal{L}^{r+1}(K)} \\
&= [R_K] \{a_{O_1}\}_R + \sum_{r=1}^{u_K-1} \left[{}^{\mathcal{L}^{r+1}(K)} R_K \right] \left\{ a_{O_{\mathcal{L}^r(K)}/O_{\mathcal{L}^{r+1}(K)}} \right\}_{\mathcal{L}^{r+1}(K)} + \left[{}^{\mathcal{L}(K)} R_K \right] \left\{ a_{O_K/O_{\mathcal{L}(K)}} \right\}_{\mathcal{L}(K)} \\
&= \{a_{O_{\mathcal{L}(K)}}\}_K + \left[{}^{\mathcal{L}(K)} R_K \right] \left\{ a_{O_K/O_{\mathcal{L}(K)}} \right\}_{\mathcal{L}(K)} \\
&= \{a_{O_{\mathcal{L}(K)}}\}_K + \{a'_{O_K/O_{\mathcal{L}(K)}}\}
\end{aligned} \tag{93}$$

Expanding the expression in Equation (71), $\{a'_{O_K/O_{\mathcal{L}(K)}}\}$ can be written as follows.

$$\begin{aligned}
\{a'_{O_K/O_{\mathcal{L}(K)}}\} &= \left[{}^{\mathcal{L}(K)} R_K \right] \left(\left[{}^R \hat{v}_{O_K,y} \right]_{\mathcal{L}(K)} \{ \dot{y} \} + \left(\left[{}^R \dot{\hat{v}}_{O_K,y} \right]_{\mathcal{L}(K)} + \left[\omega'_{\mathcal{L}(K)} \right] \left[{}^R \hat{v}_{O_K,y} \right]_{\mathcal{L}(K)} \right) \{ y \} \right) \\
&= \left[{}^{\mathcal{L}(K)} R_K \right] \left(\left[{}^R \hat{v}_{O_K,y} \right]_{\mathcal{L}(K)} \{ \dot{y} \} + \left[{}^R \dot{\hat{v}}_{O_K,y} \right]_{\mathcal{L}(K)} \{ y \} + \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \left[{}^R \hat{v}_{O_K,y} \right]_{\mathcal{L}(K)} \{ y \} \right) \\
&= \left[{}^{\mathcal{L}(K)} R_K \right] \left(\left[S_{s'_K} \right] - ([\tilde{q}'_K] + [\tilde{s}'_K]) \left[{}^R \omega'_{\mathcal{L}(K),y} \right] \right) \{ \dot{y} \} \\
&\quad + \left[{}^{\mathcal{L}(K)} R_K \right] \left(-([\tilde{q}'_K] + [\tilde{s}'_K]) \left[{}^R \dot{\omega}'_{\mathcal{L}(K),y} \right] - [\dot{\tilde{s}}'_K] \left[{}^R \omega'_{\mathcal{L}(K),y} \right] \right) \{ y \} \\
&\quad + \left[{}^{\mathcal{L}(K)} R_K \right] \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \left(\left[S_{s'_K} \right] - ([\tilde{q}'_K] + [\tilde{s}'_K]) \left[{}^R \omega'_{\mathcal{L}(K),y} \right] \right) \{ y \} \\
&= \left[{}^{\mathcal{L}(K)} R_K \right] \left(\left[S_{s'_K} \right] \{ \dot{y} \} - ([\tilde{q}'_K] + [\tilde{s}'_K]) \left(\left[{}^R \omega'_{\mathcal{L}(K),y} \right] \{ \dot{y} \} + \left[{}^R \dot{\omega}'_{\mathcal{L}(K),y} \right] \{ y \} \right) \right) \\
&\quad + \left[{}^{\mathcal{L}(K)} R_K \right] \left(-[\dot{\tilde{s}}'_K] \left[{}^R \omega'_{\mathcal{L}(K),y} \right] \{ y \} + \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \left[S_{s'_K} \right] \{ y \} \right) \\
&\quad - \left[{}^{\mathcal{L}(K)} R_K \right] \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] ([\tilde{q}'_K] + [\tilde{s}'_K]) \left[{}^R \omega'_{\mathcal{L}(K),y} \right] \{ y \} \\
&= \left[{}^{\mathcal{L}(K)} R_K \right] \left(\{ \ddot{s}'_K \} - ([\tilde{q}'_K] + [\tilde{s}'_K]) \{ \dot{\omega}'_{\mathcal{L}(K)} \} \right) + \left[{}^{\mathcal{L}(K)} R_K \right] \left(-[\dot{\tilde{s}}'_K] \{ \omega'_{\mathcal{L}(K)} \} + \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \{ \dot{s}'_K \} \right) \\
&\quad - \left[{}^{\mathcal{L}(K)} R_K \right] \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] ([\tilde{q}'_K] + [\tilde{s}'_K]) \{ \omega'_{\mathcal{L}(K)} \} \\
&= \left[{}^{\mathcal{L}(K)} R_K \right] \left(\{ \ddot{s}'_K \} + \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] (\{ q'_K \} + \{ s'_K \}) \right) + \left[{}^{\mathcal{L}(K)} R_K \right] \left(\left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \{ \dot{s}'_K \} + \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \{ \dot{s}'_K \} \right) \\
&\quad + \left[{}^{\mathcal{L}(K)} R_K \right] \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] (\{ q'_K \} + \{ s'_K \})
\end{aligned}$$

Or, rearranging terms gives

$$\begin{aligned}
\{a'_{O_K/O_{\mathcal{L}(K)}}\} &= \left[{}^{\mathcal{L}(K)} R_K \right] \left(\left[\tilde{\omega}'_{\mathcal{L}(K)} \right] (\{ q'_K \} + \{ s'_K \}) + \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] (\{ q'_K \} + \{ s'_K \}) \right. \\
&\quad \left. + \{ \ddot{s}'_K \} + 2 \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \{ \dot{s}'_K \} \right)
\end{aligned} \tag{94}$$

Substituting this result into Equation (93) gives

$$\begin{aligned}
\{a'_{O_K}\} &= \left[{}^{\mathcal{L}^2(K)} R_K \right] \{a_{O_{\mathcal{L}(K)}}\}_{\mathcal{L}^2(K)} \\
&\quad + \left[{}^{\mathcal{L}(K)} R_K \right] \left(\left[\tilde{\omega}'_{\mathcal{L}(K)} \right] (\{ q'_K \} + \{ s'_K \}) + \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] (\{ q'_K \} + \{ s'_K \}) + \{ \ddot{s}'_K \} + 2 \left[\tilde{\omega}'_{\mathcal{L}(K)} \right] \{ \dot{s}'_K \} \right)
\end{aligned} \tag{95}$$

This result is *identical* to the one shown above in Equation (30).

Body Mass Centers

To **compare** the results of the **methods** used for **fixed frame components**, Equation (81) is **expanded** using results for the partial velocity matrix and its time derivative. Using **base frame components** for the **relative angular velocities**, the **fixed frame components** of the **acceleration** of G_K can be expanded as follows.

$$\begin{aligned}
 \{a_K\} &= [{}^R v_{K,y}] \{\dot{y}\} + [{}^R \dot{v}_{K,y}] \{y\} \\
 &= \left[[{}^R v_{O_K,y}] - [\tilde{r}_K] [{}^R \omega_{K,y}] \right] \{\dot{y}\} + \left[[{}^R \dot{v}_{O_K,y}] - [\tilde{r}_K] [{}^R \dot{\omega}_{K,y}] - [\dot{\tilde{r}}_K] [{}^R \omega_{K,y}] \right] \{y\} \\
 &= \left([{}^R v_{O_K,y}] \{\dot{y}\} + [{}^R \dot{v}_{O_K,y}] \{y\} \right) - [\tilde{r}_K] \left([{}^R \omega_{K,y}] \{\dot{y}\} + [{}^R \dot{\omega}_{K,y}] \{y\} \right) - [\dot{\tilde{r}}_K] [{}^R \omega_{K,y}] \{y\}
 \end{aligned} \tag{96}$$

To simplify this result, note the following.

$$\begin{aligned}
 \{\omega_K\} &= [{}^R \omega_{K,y}] \{y\} \quad (\text{fixed frame components of } {}^R \omega_K) \\
 \{\alpha_K\} &= \{\dot{\omega}_K\} = [{}^R \omega_{K,y}] \{\dot{y}\} + [{}^R \dot{\omega}_{K,y}] \{y\} \quad (\text{fixed frame components of } {}^R \alpha_K) \\
 \{v_{O_K}\} &= [{}^R v_{O_K,y}] \{y\} \quad (\text{fixed frame components of } {}^R v_{O_K}) \\
 \{a_{O_K}\} &= \{\dot{v}_{O_K}\} = [{}^R v_{O_K,y}] \{\dot{y}\} + [{}^R \dot{v}_{O_K,y}] \{y\} \quad (\text{fixed frame components of } {}^R a_{O_K})
 \end{aligned}$$

Substituting these results into Equation (96) gives the following.

$$\{a_K\} = \{a_{O_K}\} - [\tilde{r}_K] \{\dot{\omega}_K\} - [\dot{\tilde{r}}_K] \{\omega_K\} = \{a_{O_K}\} - [\tilde{r}_K] \{\alpha_K\} - [\dot{\tilde{r}}_K] \{\omega_K\} \tag{97}$$

This result is **identical** to the one shown above in Equation (22).

Using **body frame components** for the **relative angular velocities**, the **fixed frame components** of the **acceleration** of G_K can be expanded as follows.

$$\begin{aligned}
 \{a_K\} &= [{}^R v_{K,y}] \{\dot{y}\} + [{}^R \dot{v}_{K,y}] \{y\} \\
 &= \left[[{}^R v_{O_K,y}] - [R_K]^T [\tilde{r}'_K] [{}^R \omega'_{K,y}] \right] \{\dot{y}\} \\
 &\quad + \left[[{}^R \dot{v}_{O_K,y}] - [R_K]^T [\tilde{r}'_K] [{}^R \dot{\omega}'_{K,y}] - [R_K]^T [\tilde{\omega}'_K] [\tilde{r}'_K] [{}^R \omega'_{K,y}] \right] \{y\} \\
 &= \left([{}^R v_{O_K,y}] \{\dot{y}\} + [{}^R \dot{v}_{O_K,y}] \{y\} \right) - [R_K]^T [\tilde{r}'_K] \left([{}^R \omega'_{K,y}] \{\dot{y}\} + [{}^R \dot{\omega}'_{K,y}] \{y\} \right) \\
 &\quad - [R_K]^T [\tilde{\omega}'_K] [\tilde{r}'_K] [{}^R \omega'_{K,y}] \{y\}
 \end{aligned} \tag{98}$$

To simplify this result, note the following.

$$\begin{aligned}
 \{\omega'_K\} &= [{}^R \omega'_{K,y}] \{y\} \quad (\text{body frame components of } {}^R \omega_K) \\
 \{\alpha'_K\} &= \{\dot{\omega}'_K\} = [{}^R \omega'_{K,y}] \{\dot{y}\} + [{}^R \dot{\omega}'_{K,y}] \{y\} \quad (\text{body frame components of } {}^R \alpha_K) \\
 \{v_{O_K}\} &= [{}^R v_{O_K,y}] \{y\} \quad (\text{fixed frame components of } {}^R v_{O_K})
 \end{aligned}$$

$$\{a_{O_K}\} = \{\dot{v}_{O_K}\} = [{}^R v_{O_K, y}] \{\dot{y}\} + [{}^R \dot{v}_{O_K, y}] \{y\} \quad (\text{fixed frame components of } {}^R \underline{a}_{O_K})$$

Substituting these results into Equation (98) gives the following.

$$\begin{aligned} \{a_K\} &= \{a_{O_K}\} - [R_K]^T [\tilde{r}'_K] \{\dot{\omega}'_K\} - [R_K]^T [\tilde{\omega}'_K] [\tilde{r}'_K] \{\omega'_K\} \\ &= \{a_{O_K}\} - [R_K]^T [\tilde{r}'_K] \{\alpha'_K\} - [R_K]^T [\tilde{\omega}'_K] [\tilde{r}'_K] \{\omega'_K\} \end{aligned} \quad (99)$$

This result is **identical** to the one shown above in Equation (25).

To **compare** the results of the **methods** used for **body frame components**, Equation (87) is **expanded** as follows.

$$\begin{aligned} \{a'_K\} &= [R_K] \{a_{O_i}\}_R + \sum_{r=0}^{u_K-1} [{}^{\mathcal{E}^{r+1}(K)} R_K] \left\{ a_{O_{\mathcal{E}^r(K)}/O_{\mathcal{E}^{r+1}(K)}} \right\}_{\mathcal{E}^{r+1}(K)} + [\tilde{\alpha}'_K] \{r'_K\} + [\tilde{\omega}'_K] [\tilde{\omega}'_K] \{r'_K\} \\ &= [R_K] \{a_{O_i}\}_R + \sum_{r=0}^{u_K-1} [{}^{\mathcal{E}^{r+1}(K)} R_K] \left\{ a_{O_{\mathcal{E}^r(K)}/O_{\mathcal{E}^{r+1}(K)}} \right\}_{\mathcal{E}^{r+1}(K)} + [{}^{\mathcal{E}(K)} R_K] \{a_{O_K/O_{\mathcal{E}(K)}}\}_{\mathcal{E}(K)} \\ &\quad + [\tilde{\alpha}'_K] \{r'_K\} + [\tilde{\omega}'_K] [\tilde{\omega}'_K] \{r'_K\} \end{aligned}$$

Or,

$$\begin{aligned} \{a'_K\} &= \{a_{O_{\mathcal{E}(K)}}\}_K + [{}^{\mathcal{E}(K)} R_K] \left([\tilde{\omega}'_{\mathcal{E}(K)}] (\{q'_K\} + \{s'_K\}) + [\tilde{\omega}'_{\mathcal{E}(K)}] [\tilde{\omega}'_{\mathcal{E}(K)}] (\{q'_K\} + \{s'_K\}) \right) \\ &\quad + [{}^{\mathcal{E}(K)} R_K] (\{\ddot{s}'_K\} + 2[\tilde{\omega}'_{\mathcal{E}(K)}] \{\dot{s}'_K\}) + [\tilde{\alpha}'_K] \{r'_K\} + [\tilde{\omega}'_K] [\tilde{\omega}'_K] \{r'_K\} \end{aligned} \quad (100)$$

This result is **identical** to that in Equation (35).

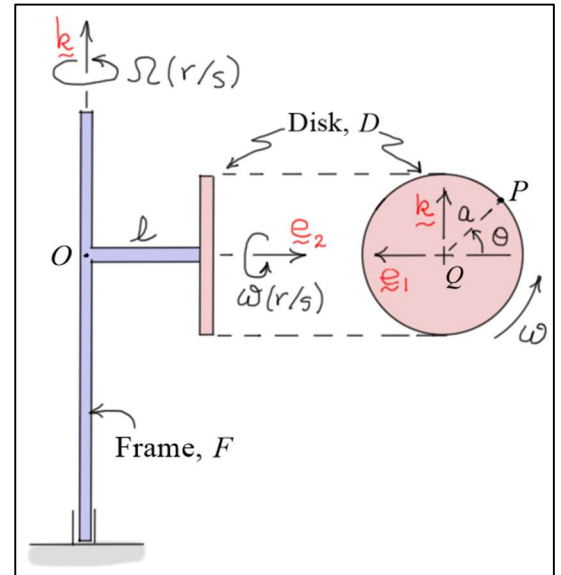
Examples

Example 1

The diagram shows a two-body system with frame F and disk D . Frame F rotates at a rate of $\dot{\phi} = \Omega$ (rad/s) about the fixed vertical direction (annotated by the unit vector $\underline{k}$). Disk D is affixed to and rotates relative to F at a rate of $\dot{\theta} = \omega$ (rad/s) about the horizontal arm of F (direction annotated by the rotating unit vector $\underline{e}_2$).

Reference frames: (all frames align when $\phi = \theta = 0$)

$$\begin{aligned} R : (\underline{i}, \underline{j}, \underline{k}) &\quad (\text{fixed frame}) \\ F : (\underline{e}_1, \underline{e}_2, \underline{k}) &\quad (\text{rotating with frame } F) \\ D : (\underline{n}_1, \underline{e}_2, \underline{n}_3) &\quad (\text{rotating with disk } D) \end{aligned}$$



Let $\{\hat{\omega}_F\}$ and $\{\hat{\omega}_D\}$ be the **base frame components** and let $\{\hat{\omega}'_F\}$ and $\{\hat{\omega}'_D\}$ be the **body frame components** of the **relative angular velocities** ${}^R \underline{\omega}_F$ and ${}^F \underline{\omega}_D$. Find the **fixed frame components** of the **time derivatives** of the

partial velocity matrices and **acceleration** of point P a) using **base frame** components and b) using **body frame** components of the relative angular velocity vectors. c) Find the **body frame components** of the **time derivatives** of the **partial velocity matrices** and **acceleration** of point P associated with the **body frame** components of the relative angular velocity vectors. Use results from Example 1 of Unit 4 for the angular velocities, partial angular velocity matrices, velocities, and partial velocity matrices.

Solution:

a) Results from Unit 4:

The **base frame components** of the **relative angular velocities** are

$$\{\hat{\omega}_F\} = [0 \ 0 \ \Omega]^T \quad \{\hat{\omega}_D\} = [0 \ \omega \ 0]^T$$

Using these results, the **fixed frame components** of the **angular velocities** of the bodies are

$$\{\omega_F\} = \{\hat{\omega}_F\} \quad \{\omega_D\} = \{\hat{\omega}_F\} + [R_F]^T \{\hat{\omega}_D\}$$

Defining the column vector of **generalized speeds** to be $\{y\} \triangleq \begin{Bmatrix} \{\hat{\omega}_F\} \\ \{\hat{\omega}_D\} \end{Bmatrix}$, the **fixed frame components** of the **partial angular velocity matrices** are

$$[{}^R\omega_{F,y}]_{3 \times 6} = \begin{bmatrix} [{}^R\omega_{F,\hat{\omega}_F}] & [{}^R\omega_{F,\hat{\omega}_D}] \end{bmatrix} = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$[{}^R\omega_{D,y}]_{3 \times 6} = \begin{bmatrix} [{}^R\omega_{D,\hat{\omega}_F}] & [{}^R\omega_{D,\hat{\omega}_D}] \end{bmatrix} = \begin{bmatrix} [I]_{3 \times 3} & [R_F]^T \end{bmatrix}$$

The **fixed frame components** of ${}^Rv_P$ and the partial velocity matrices are

$$\{v_P\} = [{}^Rv_{P,\hat{\omega}_F}] \{\hat{\omega}_F\} + [{}^Rv_{P,\hat{\omega}_D}] \{\hat{\omega}_D\}$$

$$[{}^Rv_{P,\hat{\omega}_F}] = -\left([{}^R\tilde{r}_{Q/O}] + [{}^R\tilde{r}_{P/Q}]\right) \quad (101)$$

$$[{}^Rv_{P,\hat{\omega}_D}] = -[{}^R\tilde{r}_{P/Q}][R_F]^T \quad (102)$$

$$[R_F]^T = \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \{r_{Q/O}\} = [R_F]^T \begin{Bmatrix} 0 \\ \ell \\ 0 \end{Bmatrix} = \begin{Bmatrix} -\ell S_\phi \\ \ell C_\phi \\ 0 \end{Bmatrix} \quad [{}^R\tilde{r}_{Q/O}] = \begin{bmatrix} 0 & 0 & \ell C_\phi \\ 0 & 0 & \ell S_\phi \\ -\ell C_\phi & -\ell S_\phi & 0 \end{bmatrix} \quad (103)$$

$$[R_D]^T = \begin{bmatrix} C_\phi C_\theta & -S_\phi & C_\phi S_\theta \\ S_\phi C_\theta & C_\phi & S_\phi S_\theta \\ -S_\theta & 0 & C_\theta \end{bmatrix} \quad \{r_{P/Q}\} = [R_D]^T \begin{Bmatrix} -a \\ 0 \\ 0 \end{Bmatrix} = -a \begin{Bmatrix} C_\phi C_\theta \\ S_\phi C_\theta \\ -S_\theta \end{Bmatrix} \quad (104)$$

$$[{}^R\tilde{r}_{P/Q}] = \begin{bmatrix} 0 & -aS_\theta & -aS_\phi C_\theta \\ aS_\theta & 0 & aC_\phi C_\theta \\ aS_\phi C_\theta & -aC_\phi C_\theta & 0 \end{bmatrix} \quad (105)$$

$$\begin{bmatrix} {}^R v_{P, \hat{\omega}_F} \end{bmatrix} = \begin{bmatrix} 0 & aS_\theta & -\ell C_\phi + aS_\phi C_\theta \\ -aS_\theta & 0 & -(\ell S_\phi + aC_\phi C_\theta) \\ \ell C_\phi - aS_\phi C_\theta & \ell S_\phi + aC_\phi C_\theta & 0 \end{bmatrix} \quad (106)$$

$$\begin{bmatrix} {}^R v_{P, \hat{\omega}_D} \end{bmatrix} = \begin{bmatrix} aS_\phi S_\theta & aC_\phi S_\theta & aS_\phi C_\theta \\ -aC_\phi S_\theta & aS_\phi S_\theta & -aC_\phi C_\theta \\ 0 & aC_\theta & 0 \end{bmatrix} \quad (107)$$

Acceleration:

The **fixed frame components** of ${}^R a_P$ can be calculated by **differentiating** the fixed frame components of ${}^R v_P$ as follows.

$$\{a_P\} = \begin{bmatrix} {}^R v_{P, \hat{\omega}_F} \end{bmatrix} \{\dot{\hat{\omega}}_F\} + \begin{bmatrix} {}^R v_{P, \hat{\omega}_D} \end{bmatrix} \{\dot{\hat{\omega}}_D\} + \begin{bmatrix} {}^R \dot{v}_{P, \hat{\omega}_F} \end{bmatrix} \{\hat{\omega}_F\} + \begin{bmatrix} {}^R \dot{v}_{P, \hat{\omega}_D} \end{bmatrix} \{\hat{\omega}_D\} \quad (108)$$

The **time derivatives** of the **partial velocity matrices** are found by **differentiating** Equations (101) and (102).

$$\begin{bmatrix} {}^R \dot{v}_{P, \hat{\omega}_F} \end{bmatrix} = -\left(\begin{bmatrix} \dot{\tilde{r}}_{Q/O} \end{bmatrix} + \begin{bmatrix} \dot{\tilde{r}}_{P/Q} \end{bmatrix} \right) \quad (109)$$

$$\begin{bmatrix} {}^R \dot{v}_{P, \hat{\omega}_D} \end{bmatrix} = -\begin{bmatrix} \dot{\tilde{r}}_{P/Q} \end{bmatrix} [R_F]^T - \begin{bmatrix} \tilde{r}_{P/Q} \end{bmatrix} [\dot{R}_F]^T = -\begin{bmatrix} \dot{\tilde{r}}_{P/Q} \end{bmatrix} [R_F]^T - \begin{bmatrix} \tilde{r}_{P/Q} \end{bmatrix} [\tilde{\omega}_F] [R_F]^T \quad (110)$$

The **time derivatives** of the **two skew symmetric matrices** are built using the **time derivatives** of the **fixed frame components** of the **position vectors** $\tilde{r}_{Q/O}$ and $\tilde{r}_{P/Q}$. Differentiating the results presented in Equations (103) and (104) gives

$$\begin{aligned} \{\dot{r}_{Q/O}\} &= [\dot{R}_F]^T \begin{Bmatrix} 0 \\ \ell \\ 0 \end{Bmatrix} = [\tilde{\omega}_F] [R_F]^T \begin{Bmatrix} 0 \\ \ell \\ 0 \end{Bmatrix} = \begin{bmatrix} 0 & -\Omega & 0 \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ \ell \\ 0 \end{Bmatrix} = \begin{bmatrix} -\Omega S_\phi & -\Omega C_\phi & 0 \\ \Omega C_\phi & -\Omega S_\phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ \ell \\ 0 \end{Bmatrix} \\ \Rightarrow \{\dot{r}_{Q/O}\} &= \begin{Bmatrix} -\ell \Omega C_\phi \\ -\ell \Omega S_\phi \\ 0 \end{Bmatrix} \Rightarrow \begin{bmatrix} \dot{\tilde{r}}_{Q/O} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\ell \Omega S_\phi \\ 0 & 0 & \ell \Omega C_\phi \\ \ell \Omega S_\phi & -\ell \Omega C_\phi & 0 \end{bmatrix} \end{aligned} \quad (111)$$

$$\{\omega_D\} = \{\hat{\omega}_F\} + [R_F]^T \{\hat{\omega}_D\} = \begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix} + \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ \omega \\ 0 \end{Bmatrix} = \begin{Bmatrix} -\omega S_\phi \\ \omega C_\phi \\ \Omega \end{Bmatrix}$$

$$\begin{aligned}
\{\dot{r}_{P/Q}\} &= [\dot{R}_D]^T \begin{Bmatrix} -a \\ 0 \\ 0 \end{Bmatrix} = [\tilde{\omega}_D][R_D]^T \begin{Bmatrix} -a \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 0 & -\Omega & \omega C_\phi \\ \Omega & 0 & \omega S_\phi \\ -\omega C_\phi & -\omega S_\phi & 0 \end{bmatrix} \begin{bmatrix} C_\phi C_\theta & -S_\phi & C_\phi S_\theta \\ S_\phi C_\theta & C_\phi & S_\phi S_\theta \\ -S_\theta & 0 & C_\theta \end{bmatrix} \begin{Bmatrix} -a \\ 0 \\ 0 \end{Bmatrix} \\
&= \begin{bmatrix} 0 & -\Omega & \omega C_\phi \\ \Omega & 0 & \omega S_\phi \\ -\omega C_\phi & -\omega S_\phi & 0 \end{bmatrix} \begin{Bmatrix} -a C_\phi C_\theta \\ -a S_\phi C_\theta \\ a S_\theta \end{Bmatrix} \\
\Rightarrow \{\dot{r}_{P/Q}\} &= a \begin{Bmatrix} \Omega S_\phi C_\theta + \omega C_\phi S_\theta \\ -\Omega C_\phi C_\theta + \omega S_\phi S_\theta \\ \omega C_\theta \end{Bmatrix} \\
\Rightarrow [\dot{\tilde{r}}_{P/Q}] &= a \begin{bmatrix} 0 & -\omega C_\theta & -\Omega C_\phi C_\theta + \omega S_\phi S_\theta \\ \omega C_\theta & 0 & -(\Omega S_\phi C_\theta + \omega C_\phi S_\theta) \\ \Omega C_\phi C_\theta - \omega S_\phi S_\theta & \Omega S_\phi C_\theta + \omega C_\phi S_\theta & 0 \end{bmatrix}
\end{aligned}$$

Substituting these results into Equations (109) and (110) for the **time derivatives** of the **partial velocity matrices** gives the following.

$$\begin{aligned}
[{}^R\dot{v}_{P,\hat{\omega}_F}] &= \ell \begin{bmatrix} 0 & 0 & \Omega S_\phi \\ 0 & 0 & -\Omega C_\phi \\ -\Omega S_\phi & \Omega C_\phi & 0 \end{bmatrix} + a \begin{bmatrix} 0 & \omega C_\theta & \Omega C_\phi C_\theta - \omega S_\phi S_\theta \\ -\omega C_\theta & 0 & \Omega S_\phi C_\theta + \omega C_\phi S_\theta \\ -\Omega C_\phi C_\theta + \omega S_\phi S_\theta & -(\Omega S_\phi C_\theta + \omega C_\phi S_\theta) & 0 \end{bmatrix} \\
[{}^R\dot{v}_{P,\hat{\omega}_F}] &= \begin{bmatrix} 0 & a\omega C_\theta & \ell\Omega S_\phi + a(\Omega C_\phi C_\theta - \omega S_\phi S_\theta) \\ -a\omega C_\theta & 0 & -\ell\Omega C_\phi + a(\Omega S_\phi C_\theta + \omega C_\phi S_\theta) \\ -\ell\Omega S_\phi - a(\Omega C_\phi C_\theta - \omega S_\phi S_\theta) & \ell\Omega C_\phi - a(\Omega S_\phi C_\theta + \omega C_\phi S_\theta) & 0 \end{bmatrix} \quad (112)
\end{aligned}$$

$$\begin{aligned}
[{}^R\dot{v}_{P,\hat{\omega}_D}] &= -[\dot{\tilde{r}}_{P/Q}][R_F]^T - [\tilde{r}_{P/Q}][\tilde{\omega}_F][R_F]^T \\
&= -a \begin{bmatrix} 0 & -\omega C_\theta & -\Omega C_\phi C_\theta + \omega S_\phi S_\theta \\ \omega C_\theta & 0 & -(\Omega S_\phi C_\theta + \omega C_\phi S_\theta) \\ \Omega C_\phi C_\theta - \omega S_\phi S_\theta & \Omega S_\phi C_\theta + \omega C_\phi S_\theta & 0 \end{bmatrix} \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&\quad - \begin{bmatrix} 0 & -aS_\theta & -aS_\phi C_\theta \\ aS_\theta & 0 & aC_\phi C_\theta \\ aS_\phi C_\theta & -aC_\phi C_\theta & 0 \end{bmatrix} \begin{bmatrix} 0 & -\Omega & 0 \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= -a \begin{bmatrix} -\omega C_\theta S_\phi & -\omega C_\theta C_\phi & -\Omega C_\phi C_\theta + \omega S_\phi S_\theta \\ \omega C_\theta C_\phi & -\omega C_\theta S_\phi & -(\Omega S_\phi C_\theta + \omega C_\phi S_\theta) \\ \Omega C_\theta & \omega S_\theta & 0 \end{bmatrix} \begin{bmatrix} -a\Omega S_\theta & 0 & 0 \\ 0 & -a\Omega S_\theta & 0 \\ -a\Omega C_\phi C_\theta & -a\Omega S_\phi C_\theta & 0 \end{bmatrix} \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
& \vdots \\
& = \begin{bmatrix} a\omega C_\theta S_\phi & a\omega C_\theta C_\phi & a(\Omega C_\phi C_\theta - \omega S_\phi S_\theta) \\ -a\omega C_\theta C_\phi & a\omega C_\theta S_\phi & a(\Omega S_\phi C_\theta + \omega C_\phi S_\theta) \\ -a\Omega C_\theta & -a\omega S_\theta & 0 \end{bmatrix} + \begin{bmatrix} a\Omega C_\phi S_\theta & -a\Omega S_\phi S_\theta & 0 \\ a\Omega S_\phi S_\theta & a\Omega C_\phi S_\theta & 0 \\ a\Omega C_\theta & 0 & 0 \end{bmatrix} \\
& = \begin{bmatrix} a\omega C_\theta S_\phi + a\Omega C_\phi S_\theta & a\omega C_\theta C_\phi - a\Omega S_\phi S_\theta & a(\Omega C_\phi C_\theta - \omega S_\phi S_\theta) \\ -a\omega C_\theta C_\phi + a\Omega S_\phi S_\theta & a\omega C_\theta S_\phi + a\Omega C_\phi S_\theta & a(\Omega S_\phi C_\theta + \omega C_\phi S_\theta) \\ 0 & -a\omega S_\theta & 0 \end{bmatrix} \\
\Rightarrow \left[{}^R \dot{v}_{P, \hat{\omega}_D} \right] &= a \begin{bmatrix} (\omega C_\theta S_\phi + \Omega C_\phi S_\theta) & (\omega C_\theta C_\phi - \Omega S_\phi S_\theta) & (\Omega C_\phi C_\theta - \omega S_\phi S_\theta) \\ (-\omega C_\theta C_\phi + \Omega S_\phi S_\theta) & (\omega C_\theta S_\phi + \Omega C_\phi S_\theta) & (\Omega S_\phi C_\theta + \omega C_\phi S_\theta) \\ 0 & -\omega S_\theta & 0 \end{bmatrix} \quad (113)
\end{aligned}$$

Substituting from Equations (106), (107), (112), and (113) into Equation (108) gives the **fixed frame components** of ${}^R a_P$ the acceleration of point P .

$$\begin{aligned}
\{a_P\} &= \left[{}^R v_{P, \hat{\omega}_F} \right] \left\{ \dot{\hat{\omega}}_F \right\} + \left[{}^R v_{P, \hat{\omega}_D} \right] \left\{ \dot{\hat{\omega}}_D \right\} + \left[{}^R \dot{v}_{P, \hat{\omega}_F} \right] \left\{ \hat{\omega}_F \right\} + \left[{}^R \dot{v}_{P, \hat{\omega}_D} \right] \left\{ \hat{\omega}_D \right\} \\
&= \begin{bmatrix} 0 & aS_\theta & -\ell C_\phi + aS_\phi C_\theta \\ -aS_\theta & 0 & -(\ell S_\phi + aC_\phi C_\theta) \\ \ell C_\phi - aS_\phi C_\theta & \ell S_\phi + aC_\phi C_\theta & 0 \end{bmatrix} \left\{ \begin{bmatrix} 0 \\ 0 \\ \dot{\Omega} \end{bmatrix} \right\} + \begin{bmatrix} aS_\phi S_\theta & aC_\phi S_\theta & aS_\phi C_\theta \\ -aC_\phi S_\theta & aS_\phi S_\theta & -aC_\phi C_\theta \\ 0 & aC_\theta & 0 \end{bmatrix} \left\{ \begin{bmatrix} 0 \\ \dot{\omega} \\ 0 \end{bmatrix} \right\} \\
&+ \begin{bmatrix} 0 & a\omega C_\theta & \ell \Omega S_\phi + a(\Omega C_\phi C_\theta - \omega S_\phi S_\theta) \\ -a\omega C_\theta & 0 & -\ell \Omega C_\phi + a(\Omega S_\phi C_\theta + \omega C_\phi S_\theta) \\ -\ell \Omega S_\phi - a(\Omega C_\phi C_\theta - \omega S_\phi S_\theta) & \ell \Omega C_\phi - a(\Omega S_\phi C_\theta + \omega C_\phi S_\theta) & 0 \end{bmatrix} \left\{ \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} \right\} \\
&+ a \begin{bmatrix} (\omega C_\theta S_\phi + \Omega C_\phi S_\theta) & (\omega C_\theta C_\phi - \Omega S_\phi S_\theta) & (\Omega C_\phi C_\theta - \omega S_\phi S_\theta) \\ (-\omega C_\theta C_\phi + \Omega S_\phi S_\theta) & (\omega C_\theta S_\phi + \Omega C_\phi S_\theta) & (\Omega S_\phi C_\theta + \omega C_\phi S_\theta) \\ 0 & -\omega S_\theta & 0 \end{bmatrix} \left\{ \begin{bmatrix} 0 \\ \omega \\ 0 \end{bmatrix} \right\} \\
&= \left\{ \begin{bmatrix} -\ell C_\phi + aS_\phi C_\theta \\ -(\ell S_\phi + aC_\phi C_\theta) \\ 0 \end{bmatrix} \right\} \dot{\Omega} + \left\{ \begin{bmatrix} aC_\phi S_\theta \\ aS_\phi S_\theta \\ aC_\theta \end{bmatrix} \right\} \dot{\omega} + \left\{ \begin{bmatrix} \ell \Omega^2 S_\phi + a(\Omega^2 C_\phi C_\theta - \Omega \omega S_\phi S_\theta) + a(\omega^2 C_\theta C_\phi - \Omega \omega S_\phi S_\theta) \\ -\ell \Omega^2 C_\phi + a(\Omega^2 S_\phi C_\theta + \Omega \omega C_\phi S_\theta) + a(\omega^2 C_\theta S_\phi + \Omega \omega C_\phi S_\theta) \\ -a\omega^2 S_\theta \end{bmatrix} \right\} \\
&= \left\{ \begin{bmatrix} -\ell C_\phi + aS_\phi C_\theta \\ -(\ell S_\phi + aC_\phi C_\theta) \\ 0 \end{bmatrix} \right\} \dot{\Omega} + \left\{ \begin{bmatrix} aC_\phi S_\theta \\ aS_\phi S_\theta \\ aC_\theta \end{bmatrix} \right\} \dot{\omega} + \left\{ \begin{bmatrix} \ell \Omega^2 S_\phi + aC_\phi C_\theta (\Omega^2 + \omega^2) - 2a\Omega \omega S_\phi S_\theta \\ -\ell \Omega^2 C_\phi + aS_\phi C_\theta (\Omega^2 + \omega^2) + 2a\Omega \omega C_\phi S_\theta \\ -a\omega^2 S_\theta \end{bmatrix} \right\}
\end{aligned}$$

$$\Rightarrow \{a_P\} = \begin{Bmatrix} -\ell C_\phi + aS_\phi C_\theta \\ -(\ell S_\phi + aC_\phi C_\theta) \\ 0 \end{Bmatrix} \dot{\Omega} + \begin{Bmatrix} aC_\phi S_\theta \\ aS_\phi S_\theta \\ aC_\theta \end{Bmatrix} \dot{\omega} + \begin{Bmatrix} \ell \Omega^2 S_\phi + aC_\phi C_\theta (\Omega^2 + \omega^2) - 2a\Omega \omega S_\phi S_\theta \\ -\ell \Omega^2 C_\phi + aS_\phi C_\theta (\Omega^2 + \omega^2) + 2a\Omega \omega C_\phi S_\theta \\ -a\omega^2 S_\theta \end{Bmatrix} \quad (114)$$

b) Results from Unit 4:

The **body frame components** of the **relative angular velocities** are

$$\{\hat{\omega}'_F\} = [0 \ 0 \ \Omega]^T \quad \{\hat{\omega}'_D\} = [0 \ \omega \ 0]^T$$

Using these results, the **body frame components** of the **angular velocities** of the bodies are

$$\{\omega'_F\} = \{\hat{\omega}'_F\} \quad \{\omega'_D\} = [{}^F R_D] \{\hat{\omega}'_F\} + \{\hat{\omega}'_D\}$$

Defining the column vector of **generalized speeds** to be $\{y\} \triangleq \begin{Bmatrix} \{\hat{\omega}'_F\} \\ \{\hat{\omega}'_D\} \end{Bmatrix}$, the **body frame components** of the **partial angular velocity matrices** are observed to be

$$[{}^R \omega'_{F,y}]_{3 \times 6} = \begin{bmatrix} [{}^R \omega'_{F,\hat{\omega}'_F}] & [{}^R \omega'_{F,\hat{\omega}'_D}] \end{bmatrix} = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$[{}^R \omega'_{D,y}]_{3 \times 6} = \begin{bmatrix} [{}^R \omega'_{D,\hat{\omega}'_F}] & [{}^R \omega'_{D,\hat{\omega}'_D}] \end{bmatrix} = \begin{bmatrix} [{}^F R_D] & [I]_{3 \times 3} \end{bmatrix}$$

The **fixed frame components** of ${}^R v_P$ and the partial velocity matrices are

$$\{v_P\} = [{}^R v_{P,\hat{\omega}'_F}] \{\hat{\omega}'_F\} + [{}^R v_{P,\hat{\omega}'_D}] \{\hat{\omega}'_D\}$$

$$[{}^R v_{P,\hat{\omega}'_F}] = -\left([R_F]^T [\tilde{r}'_{Q/O}] + [R_D]^T [\tilde{r}'_{P/Q}] [{}^F R_D] \right) \quad (115)$$

$$[{}^R v_{P,\hat{\omega}'_D}] = -[R_D]^T [\tilde{r}'_{P/Q}] \quad (116)$$

$$[{}^R v_{P,\hat{\omega}'_F}] = \begin{bmatrix} aS_\phi S_\theta & aC_\phi S_\theta & -\ell C_\phi + aS_\phi C_\theta \\ -aC_\phi S_\theta & aS_\phi S_\theta & -\ell S_\phi - aC_\phi C_\theta \\ \ell & aC_\theta & 0 \end{bmatrix} \quad [{}^R v_{P,\hat{\omega}'_D}] = \begin{bmatrix} 0 & aC_\phi S_\theta & aS_\phi \\ 0 & aS_\phi S_\theta & -aC_\phi \\ 0 & aC_\theta & 0 \end{bmatrix} \quad (117)$$

The transformation matrices $[R_F]^T$ and $[R_D]^T$ are given in Equations (103) and (104), and $[{}^F R_D]$ is as follows.

$$[{}^F R_D] = \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix} \quad (118)$$

Using this result, $\{\omega'_D\}$ the body frame components of ${}^R \omega_D$ can be written as follows.

$$\{\omega'_D\} = [{}^F R_D] \{\hat{\omega}'_F\} + \{\hat{\omega}'_D\} = \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix} + \begin{Bmatrix} 0 \\ \omega \\ 0 \end{Bmatrix} = \begin{Bmatrix} -\Omega S_\theta \\ \omega \\ \Omega C_\theta \end{Bmatrix} \quad (119)$$

The 3×3 skew symmetric matrices are

$$\begin{bmatrix} \tilde{r}'_{Q/O} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \ell \\ 0 & 0 & 0 \\ -\ell & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \tilde{r}'_{P/Q} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{bmatrix} \quad (120)$$

Acceleration:

The **fixed frame components** of ${}^R a_P$ can be calculated by **differentiating** the fixed frame components of ${}^R v_P$ as follows.

$$\{a_P\} = \{\dot{v}_P\} = [{}^R v_{P,\hat{\omega}'_F}] \{\dot{\hat{\omega}'_F}\} + [{}^R v_{P,\hat{\omega}'_D}] \{\dot{\hat{\omega}'_D}\} + [{}^R \dot{v}_{P,\hat{\omega}'_F}] \{\hat{\omega}'_F\} + [{}^R \dot{v}_{P,\hat{\omega}'_D}] \{\hat{\omega}'_D\} \quad (121)$$

The **time derivatives** of the **partial velocity matrices** are found by **differentiating** Equations (115) and (116). Differentiating Equation (115) gives the following.

$$\begin{aligned} [{}^R \dot{v}_{P,\hat{\omega}'_F}] &= -[\dot{R}_F]^T [\tilde{r}'_{Q/O}] - [\dot{R}_D]^T [\tilde{r}'_{P/Q}] [{}^F R_D] - [R_D]^T [\tilde{r}'_{P/Q}] [{}^F \dot{R}_D] \\ &= -[R_F]^T [\dot{\tilde{\omega}'_F}] [\tilde{r}'_{Q/O}] - [R_D]^T [\dot{\tilde{\omega}'_D}] [\tilde{r}'_{P/Q}] [{}^F R_D] - [R_D]^T [\tilde{r}'_{P/Q}] [{}^F \dot{\tilde{\omega}'_D}]^T [{}^F R_D] \end{aligned} \quad (122)$$

The **first term** on the **right side** can be **simplified** as follows.

$$\begin{aligned} -[R_F]^T [\dot{\tilde{\omega}'_F}] [\tilde{r}'_{Q/O}] &= -\begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\Omega & 0 \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \ell \\ 0 & 0 & 0 \\ -\ell & 0 & 0 \end{bmatrix} = -\begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \ell\Omega \\ 0 & 0 & 0 \end{bmatrix} \\ \Rightarrow -[R_F]^T [\dot{\tilde{\omega}'_F}] [\tilde{r}'_{Q/O}] &= \begin{bmatrix} 0 & 0 & \ell\Omega S_\phi \\ 0 & 0 & -\ell\Omega C_\phi \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

The **second term** on the **right side** can be **simplified** as follows.

$$\begin{aligned} -[R_D]^T [\dot{\tilde{\omega}'_D}] [\tilde{r}'_{P/Q}] [{}^F R_D] &= -\begin{bmatrix} C_\phi C_\theta & -S_\phi & C_\phi S_\theta \\ S_\phi C_\theta & C_\phi & S_\phi S_\theta \\ -S_\theta & 0 & C_\theta \end{bmatrix} \begin{bmatrix} 0 & -\Omega C_\theta & \omega \\ \Omega C_\theta & 0 & \Omega S_\theta \\ -\omega & -\Omega S_\theta & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{bmatrix} \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix} \\ &= -\begin{bmatrix} C_\phi C_\theta & -S_\phi & C_\phi S_\theta \\ S_\phi C_\theta & C_\phi & S_\phi S_\theta \\ -S_\theta & 0 & C_\theta \end{bmatrix} \begin{bmatrix} 0 & -a\omega & -a\Omega C_\theta \\ 0 & -a\Omega S_\theta & 0 \\ 0 & 0 & -a\Omega S_\theta \end{bmatrix} \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix} \\ &= -\begin{bmatrix} C_\phi C_\theta & -S_\phi & C_\phi S_\theta \\ S_\phi C_\theta & C_\phi & S_\phi S_\theta \\ -S_\theta & 0 & C_\theta \end{bmatrix} \begin{bmatrix} -a\Omega S_\theta C_\theta & -a\omega & -a\Omega C_\theta^2 \\ 0 & -a\Omega S_\theta & 0 \\ -a\Omega S_\theta^2 & 0 & -a\Omega S_\theta C_\theta \end{bmatrix} \\ &= -\begin{bmatrix} -a\Omega C_\phi S_\theta & -a\omega C_\phi C_\theta + a\Omega S_\phi S_\theta & -a\Omega C_\phi C_\theta \\ -a\Omega S_\phi S_\theta & -a\omega S_\phi C_\theta - a\Omega C_\phi S_\theta & -a\Omega S_\phi C_\theta \\ 0 & a\omega S_\theta & 0 \end{bmatrix} \end{aligned}$$

$$\Rightarrow -[R_D]^T [\tilde{\omega}'_D] [\tilde{r}'_{P/Q}] [{}^F R_D] = \begin{bmatrix} a\Omega C_\phi S_\theta & a\omega C_\phi C_\theta - a\Omega S_\phi S_\theta & a\Omega C_\phi C_\theta \\ a\Omega S_\phi S_\theta & a\omega S_\phi C_\theta + a\Omega C_\phi S_\theta & a\Omega S_\phi C_\theta \\ 0 & -a\omega S_\theta & 0 \end{bmatrix}$$

The **third term** on the **right side** can be **simplified** as follows.

$$\begin{aligned} -[R_D]^T [\tilde{r}'_{P/Q}] [{}^F \tilde{\omega}'_D]^T [{}^F R_D] &= - \begin{bmatrix} C_\phi C_\theta & -S_\phi & C_\phi S_\theta \\ S_\phi C_\theta & C_\phi & S_\phi S_\theta \\ -S_\theta & 0 & C_\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -\omega \\ 0 & 0 & 0 \\ \omega & 0 & 0 \end{bmatrix} \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix} \\ &= - \begin{bmatrix} C_\phi C_\theta & -S_\phi & C_\phi S_\theta \\ S_\phi C_\theta & C_\phi & S_\phi S_\theta \\ -S_\theta & 0 & C_\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ a\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix} \\ &= - \begin{bmatrix} -a\omega S_\phi & 0 & 0 \\ a\omega C_\phi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix} \\ \Rightarrow -[R_D]^T [\tilde{r}'_{P/Q}] [{}^F \tilde{\omega}'_D]^T [{}^F R_D] &= \begin{bmatrix} a\omega S_\phi C_\theta & 0 & -a\omega S_\phi S_\theta \\ -a\omega C_\phi C_\theta & 0 & a\omega C_\phi S_\theta \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Substituting these results into Equation (122) gives the following.

$$\begin{aligned} [{}^R \dot{V}_{P,\tilde{\omega}'_F}] &= -[R_F]^T [\tilde{\omega}'_F] [\tilde{r}'_{Q/O}] - [R_D]^T [\tilde{\omega}'_D] [\tilde{r}'_{P/Q}] [{}^F R_D] - [R_D]^T [\tilde{r}'_{P/Q}] [\tilde{\omega}'_D] [{}^F R_D] \\ &= \begin{bmatrix} 0 & 0 & \ell\Omega S_\phi \\ 0 & 0 & -\ell\Omega C_\phi \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} a\Omega C_\phi S_\theta & a\omega C_\phi C_\theta - a\Omega S_\phi S_\theta & a\Omega C_\phi C_\theta \\ a\Omega S_\phi S_\theta & a\omega S_\phi C_\theta + a\Omega C_\phi S_\theta & a\Omega S_\phi C_\theta \\ 0 & -a\omega S_\theta & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} a\omega S_\phi C_\theta & 0 & -a\omega S_\phi S_\theta \\ -a\omega C_\phi C_\theta & 0 & a\omega C_\phi S_\theta \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{123}$$

Differentiating Equation (116) gives the following.

$$\begin{aligned} [{}^R \dot{V}_{P,\tilde{\omega}'_D}] &= -[\dot{R}_D]^T [\tilde{r}'_{P/Q}] = -[R_D]^T [\tilde{\omega}'_D] [\tilde{r}'_{P/Q}] \\ &= - \begin{bmatrix} C_\phi C_\theta & -S_\phi & C_\phi S_\theta \\ S_\phi C_\theta & C_\phi & S_\phi S_\theta \\ -S_\theta & 0 & C_\theta \end{bmatrix} \begin{bmatrix} 0 & -\Omega C_\theta & \omega \\ \Omega C_\theta & 0 & \Omega S_\theta \\ -\omega & -\Omega S_\theta & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{bmatrix} \\ &\quad \vdots \end{aligned}$$

$$\begin{aligned}
& \vdots \\
& = - \begin{bmatrix} C_\phi C_\theta & -S_\phi & C_\phi S_\theta \\ S_\phi C_\theta & C_\phi & S_\phi S_\theta \\ -S_\theta & 0 & C_\theta \end{bmatrix} \begin{bmatrix} 0 & -a\omega & -a\Omega C_\theta \\ 0 & -a\Omega S_\theta & 0 \\ 0 & 0 & -a\Omega S_\theta \end{bmatrix} \\
& = - \begin{bmatrix} 0 & -a\omega C_\phi C_\theta + a\Omega S_\phi S_\theta & -a\Omega C_\phi \\ 0 & -a\omega S_\phi C_\theta - a\Omega C_\phi S_\theta & -a\Omega S_\phi \\ 0 & a\omega S_\theta & 0 \end{bmatrix} \\
& \Rightarrow \boxed{{}^R \dot{\mathbf{v}}_{P, \hat{\omega}'_D} = \begin{bmatrix} 0 & a\omega C_\phi C_\theta - a\Omega S_\phi S_\theta & a\Omega C_\phi \\ 0 & a\omega S_\phi C_\theta + a\Omega C_\phi S_\theta & a\Omega S_\phi \\ 0 & -a\omega S_\theta & 0 \end{bmatrix}} \quad (124)
\end{aligned}$$

Substituting from Equations (117), (123), and (124) into Equation (121) gives the **fixed frame components** of ${}^R \mathbf{a}_P$ the acceleration of P in R .

$$\begin{aligned}
\{a_P\} &= [{}^R \mathbf{v}_{P, \hat{\omega}'_F}] \{\dot{\hat{\omega}'_F}\} + [{}^R \mathbf{v}_{P, \hat{\omega}'_D}] \{\dot{\hat{\omega}'_D}\} + [{}^R \dot{\mathbf{v}}_{P, \hat{\omega}'_F}] \{\hat{\omega}'_F\} + [{}^R \dot{\mathbf{v}}_{P, \hat{\omega}'_D}] \{\hat{\omega}'_D\} \\
&= \begin{bmatrix} aS_\phi S_\theta & aC_\phi S_\theta & -\ell C_\phi + aS_\phi C_\theta \\ -aC_\phi S_\theta & aS_\phi S_\theta & -\ell S_\phi - aC_\phi C_\theta \\ \ell & aC_\theta & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\Omega} \end{bmatrix} + \begin{bmatrix} 0 & aC_\phi S_\theta & aS_\phi \\ 0 & aS_\phi S_\theta & -aC_\phi \\ 0 & aC_\theta & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\omega} \\ 0 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 & \ell \Omega S_\phi \\ 0 & 0 & -\ell \Omega C_\phi \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} + \begin{bmatrix} a\Omega C_\phi S_\theta & a\omega C_\phi C_\theta - a\Omega S_\phi S_\theta & a\Omega C_\phi C_\theta \\ a\Omega S_\phi S_\theta & a\omega S_\phi C_\theta + a\Omega C_\phi S_\theta & a\Omega S_\phi C_\theta \\ 0 & -a\omega S_\theta & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} \\
&+ \begin{bmatrix} a\omega S_\phi C_\theta & 0 & -a\omega S_\phi S_\theta \\ -a\omega C_\phi C_\theta & 0 & a\omega C_\phi S_\theta \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} + \begin{bmatrix} 0 & a\omega C_\phi C_\theta - a\Omega S_\phi S_\theta & a\Omega C_\phi \\ 0 & a\omega S_\phi C_\theta + a\Omega C_\phi S_\theta & a\Omega S_\phi \\ 0 & -a\omega S_\theta & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \omega \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -\ell C_\phi + aS_\phi C_\theta \\ -\ell S_\phi - aC_\phi C_\theta \\ 0 \end{bmatrix} \dot{\Omega} + \begin{bmatrix} aC_\phi S_\theta \\ aS_\phi S_\theta \\ aC_\theta \end{bmatrix} \dot{\omega} + \begin{bmatrix} \ell \Omega^2 S_\phi + a\Omega^2 C_\phi C_\theta - a\omega \Omega S_\phi S_\theta \\ -\ell \Omega^2 C_\phi + a\Omega^2 S_\phi C_\theta + a\omega \Omega C_\phi S_\theta \\ 0 \end{bmatrix} + \begin{bmatrix} a\omega^2 C_\phi C_\theta - a\omega \Omega S_\phi S_\theta \\ a\omega^2 S_\phi C_\theta + a\omega \Omega C_\phi S_\theta \\ -a\omega^2 S_\theta \end{bmatrix} \\
&\Rightarrow \boxed{\{a_P\} = \begin{bmatrix} -\ell C_\phi + aS_\phi C_\theta \\ -(\ell S_\phi + aC_\phi C_\theta) \\ 0 \end{bmatrix} \dot{\Omega} + \begin{bmatrix} aC_\phi S_\theta \\ aS_\phi S_\theta \\ aC_\theta \end{bmatrix} \dot{\omega} + \begin{bmatrix} \ell \Omega^2 S_\phi + aC_\phi C_\theta (\Omega^2 + \omega^2) - 2a\omega \Omega S_\phi S_\theta \\ -\ell \Omega^2 C_\phi + aS_\phi C_\theta (\Omega^2 + \omega^2) + 2a\omega \Omega C_\phi S_\theta \\ -a\omega^2 S_\theta \end{bmatrix}} \quad (125)
\end{aligned}$$

As expected, this result is **identical** to that of part (a) in Equation (114).

c) Results from Unit 4:

The **body frame components** of the **relative angular velocities** are

$$\boxed{\{\hat{\omega}'_F\} = [0 \quad 0 \quad \Omega]^T} \quad \boxed{\{\hat{\omega}'_D\} = [0 \quad \omega \quad 0]^T}$$

Using these results, the **body frame components** of the **angular velocities** of the bodies are

$$\{\omega'_F\} = \{\hat{\omega}'_F\} \quad \{\omega'_D\} = \left[{}^F R_D \right] \{\hat{\omega}'_F\} + \{\hat{\omega}'_D\}$$

Defining the column vector of **generalized speeds** to be $\{y\} \triangleq \begin{Bmatrix} \{\hat{\omega}'_F\} \\ \{\hat{\omega}'_D\} \end{Bmatrix}$, the **body frame components** of the **partial angular velocity matrices** are observed to be

$$\begin{aligned} \left[{}^R \omega'_{F,y} \right]_{3 \times 6} &= \begin{bmatrix} \left[{}^R \omega'_{F,\hat{\omega}'_F} \right] & \left[{}^R \omega'_{F,\hat{\omega}'_D} \right] \end{bmatrix} = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix} \\ \left[{}^R \omega'_{D,y} \right]_{3 \times 6} &= \begin{bmatrix} \left[{}^R \omega'_{D,\hat{\omega}'_F} \right] & \left[{}^R \omega'_{D,\hat{\omega}'_D} \right] \end{bmatrix} = \begin{bmatrix} \left[{}^F R_D \right] & [I]_{3 \times 3} \end{bmatrix} \end{aligned}$$

The body frame components of the velocity of P are

$$\{v'_P\} = -\left(\left[{}^F R_D \right] \left[\tilde{r}'_{Q/O} \right] + \left[\tilde{r}'_{P/Q} \right] \left[{}^F R_D \right] \right) \{\hat{\omega}'_F\} - \left[\tilde{r}'_{P/Q} \right] \{\hat{\omega}'_D\} = \left[{}^R v'_{P,\hat{\omega}'_F} \right] \{\hat{\omega}'_F\} + \left[{}^R v'_{P,\hat{\omega}'_D} \right] \{\hat{\omega}'_D\}$$

Here,

$$\begin{aligned} \left[\tilde{r}'_{Q/O} \right] &= \begin{bmatrix} 0 & 0 & \ell \\ 0 & 0 & 0 \\ -\ell & 0 & 0 \end{bmatrix} & \left[\tilde{r}'_{P/Q} \right] &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{bmatrix} & \left[{}^F R_D \right] &= \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix} \\ \left[{}^R v'_{P,\hat{\omega}'_F} \right] &= \begin{bmatrix} -\ell S_\theta & 0 & -\ell C_\theta \\ -a S_\theta & 0 & -a C_\theta \\ \ell C_\theta & a & -\ell S_\theta \end{bmatrix} & \left[{}^R v'_{P,\hat{\omega}'_D} \right] &= -\left[\tilde{r}'_{P/Q} \right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & a & 0 \end{bmatrix} & \{v'_P\} &= \begin{Bmatrix} -\ell C_\theta \Omega \\ -a C_\theta \Omega \\ -\ell S_\theta \Omega + a \omega \end{Bmatrix} \end{aligned}$$

Acceleration:

The body frame components of ${}^R \underline{a}_P$ can be found by applying the **derivative rule** as follows.

$${}^R \underline{a}_P = \frac{{}^R d}{{}^R dt} \left({}^R \underline{v}_P \right) = \frac{{}^D d}{{}^D dt} \left({}^R \underline{v}_P \right) + {}^R \underline{\omega}_D \times {}^R \underline{v}_P$$

Using the body frame components of the angular velocity components, the scalar form of this equation is

$$\begin{aligned} \{a'_P\} &= \{v'_P\} + [\tilde{\omega}'_D] \{v'_P\} \\ &= \left[{}^R v'_{P,\hat{\omega}'_F} \right] \{\dot{\hat{\omega}}'_F\} + \left[{}^R v'_{P,\hat{\omega}'_D} \right] \{\dot{\hat{\omega}}'_D\} + \left[{}^R \dot{v}'_{P,\hat{\omega}'_F} \right] \{\hat{\omega}'_F\} + \left[{}^R \dot{v}'_{P,\hat{\omega}'_D} \right] \{\hat{\omega}'_D\} + [\tilde{\omega}'_D] \{v'_P\} \end{aligned} \quad (126)$$

Here,

$$\begin{aligned} \left[{}^R \dot{v}'_{P,\hat{\omega}'_D} \right] &= -\left[\tilde{r}'_{P/Q} \right] = [0]_{3 \times 3} \\ \{ \omega'_D \} &= \left[{}^F R_D \right] \{ \hat{\omega}'_F \} + \{ \hat{\omega}'_D \} = \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix} + \begin{Bmatrix} 0 \\ \omega \\ 0 \end{Bmatrix} = \begin{Bmatrix} -\Omega S_\theta \\ \omega \\ \Omega C_\theta \end{Bmatrix} \\ \left[{}^R \dot{v}'_{P,\hat{\omega}'_F} \right] &= \frac{d}{{}^R dt} \begin{bmatrix} -\ell S_\theta & 0 & -\ell C_\theta \\ -a S_\theta & 0 & -a C_\theta \\ \ell C_\theta & a & -\ell S_\theta \end{bmatrix} = \begin{bmatrix} -\ell \dot{\theta} C_\theta & 0 & \ell \dot{\theta} S_\theta \\ -a \dot{\theta} C_\theta & 0 & a \dot{\theta} S_\theta \\ -\ell \dot{\theta} S_\theta & 0 & -\ell \dot{\theta} C_\theta \end{bmatrix} = \omega \begin{bmatrix} -\ell C_\theta & 0 & \ell S_\theta \\ -a C_\theta & 0 & a S_\theta \\ -\ell S_\theta & 0 & -\ell C_\theta \end{bmatrix} \end{aligned}$$

This final can also be calculated by **differentiating** the general equation for $\begin{bmatrix} {}^R \mathbf{v}'_{P, \dot{\omega}'_F} \end{bmatrix}$ as follows.

$$\begin{bmatrix} {}^R \mathbf{v}'_{P, \dot{\omega}'_F} \end{bmatrix} = - \left(\begin{bmatrix} {}^F \dot{R}_D \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{r}}'_{Q/O} \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{r}}'_{P/Q} \end{bmatrix} \begin{bmatrix} {}^F \dot{R}_D \end{bmatrix} \right) = - \left(\begin{bmatrix} \tilde{\dot{\omega}}'_D \end{bmatrix}^T \begin{bmatrix} {}^F R_D \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{r}}'_{Q/O} \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{r}}'_{P/Q} \end{bmatrix} \begin{bmatrix} \tilde{\dot{\omega}}'_D \end{bmatrix}^T \begin{bmatrix} {}^F R_D \end{bmatrix} \right)$$

Expanding this equation gives the same result.

Substituting into Equation (126) and simplifying gives the following.

$$\begin{aligned} \{a'_P\} &= \begin{bmatrix} {}^R \mathbf{v}'_{P, \dot{\omega}'_F} \end{bmatrix} \left\{ \dot{\omega}'_F \right\} + \begin{bmatrix} {}^R \mathbf{v}'_{P, \dot{\omega}'_D} \end{bmatrix} \left\{ \dot{\omega}'_D \right\} + \begin{bmatrix} {}^R \mathbf{v}'_{P, \dot{\omega}'_F} \end{bmatrix} \left\{ \dot{\omega}'_F \right\} + \underbrace{\begin{bmatrix} {}^R \mathbf{v}'_{P, \dot{\omega}'_D} \end{bmatrix} \left\{ \dot{\omega}'_D \right\}}_{\text{zero}} + \begin{bmatrix} \dot{\omega}'_D \end{bmatrix} \left\{ v'_P \right\} \\ &= \begin{bmatrix} -\ell S_\theta & 0 & -\ell C_\theta \\ -a S_\theta & 0 & -a C_\theta \\ \ell C_\theta & a & -\ell S_\theta \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\Omega} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & a & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ \dot{\omega} \\ 0 \end{Bmatrix} + \omega \begin{bmatrix} -\ell C_\theta & 0 & \ell S_\theta \\ -a C_\theta & 0 & a S_\theta \\ -\ell S_\theta & 0 & -\ell C_\theta \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix} \\ &\quad + \begin{bmatrix} 0 & -\Omega C_\theta & \omega \\ \Omega C_\theta & 0 & \Omega S_\theta \\ -\omega & -\Omega S_\theta & 0 \end{bmatrix} \begin{Bmatrix} -\ell C_\theta \Omega \\ -a C_\theta \Omega \\ -\ell S_\theta \Omega + a \omega \end{Bmatrix} \\ &= \begin{Bmatrix} -\ell C_\theta \\ -a C_\theta \\ -\ell S_\theta \end{Bmatrix} \dot{\Omega} + \begin{Bmatrix} 0 \\ 0 \\ a \end{Bmatrix} \dot{\omega} + \begin{Bmatrix} \ell S_\theta \\ a S_\theta \\ -\ell C_\theta \end{Bmatrix} \omega \Omega + \begin{Bmatrix} a \Omega^2 C_\theta^2 - \ell S_\theta \Omega \omega + a \omega^2 \\ -\ell \Omega^2 (C_\theta^2 + S_\theta^2) + a S_\theta \Omega \omega \\ \ell C_\theta \Omega \omega + a S_\theta C_\theta \Omega^2 \end{Bmatrix} \\ &= \begin{Bmatrix} -\ell C_\theta \\ -a C_\theta \\ -\ell S_\theta \end{Bmatrix} \dot{\Omega} + \begin{Bmatrix} 0 \\ 0 \\ a \end{Bmatrix} \dot{\omega} + \begin{Bmatrix} \ell S_\theta \\ a S_\theta \\ -\ell C_\theta \end{Bmatrix} \omega \Omega + \begin{Bmatrix} -\ell S_\theta \\ a S_\theta \\ \ell C_\theta \end{Bmatrix} \omega \Omega + \begin{Bmatrix} a C_\theta^2 \\ -\ell \\ a S_\theta C_\theta \end{Bmatrix} \Omega^2 + \begin{Bmatrix} a \\ 0 \\ 0 \end{Bmatrix} \omega^2 \\ \{a'_P\} &= \begin{Bmatrix} -\ell C_\theta \\ -a C_\theta \\ -\ell S_\theta \end{Bmatrix} \dot{\Omega} + \begin{Bmatrix} 0 \\ 0 \\ a \end{Bmatrix} \dot{\omega} + \begin{Bmatrix} 0 \\ 2a S_\theta \\ 0 \end{Bmatrix} \omega \Omega + \begin{Bmatrix} a C_\theta^2 \\ -\ell \\ a S_\theta C_\theta \end{Bmatrix} \Omega^2 + \begin{Bmatrix} a \\ 0 \\ 0 \end{Bmatrix} \omega^2 \end{aligned} \tag{127}$$

Comparison with Previous Results: (Volume I, Examples in Units 2-4)

In Units 2-4 of Volume I, ${}^R \underline{a}_P$ the acceleration of P in R was expressed in frame F as follows.

$${}^R \underline{a}_P = \begin{bmatrix} a \dot{\omega} S_\theta - \ell \dot{\Omega} + a C_\theta (\omega^2 + \Omega^2) \\ -a \dot{\Omega} C_\theta + 2a \omega \Omega S_\theta - \ell \Omega^2 \\ a \dot{\omega} C_\theta - a \omega^2 S_\theta \end{bmatrix} \underline{e}_1 + \begin{bmatrix} -a \dot{\Omega} C_\theta + 2a \omega \Omega S_\theta - \ell \Omega^2 \\ a \dot{\omega} C_\theta - a \omega^2 S_\theta \end{bmatrix} \underline{e}_2 + \begin{bmatrix} a \dot{\omega} C_\theta - a \omega^2 S_\theta \end{bmatrix} \underline{k}$$

This previous result can be converted into **fixed frame components** as follows.

$$\begin{aligned} \{a_P\} &= [R_F]^T \{a_P\}_F = \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} a \dot{\omega} S_\theta - \ell \dot{\Omega} + a C_\theta (\omega^2 + \Omega^2) \\ -a \dot{\Omega} C_\theta + 2a \omega \Omega S_\theta - \ell \Omega^2 \\ a \dot{\omega} C_\theta - a \omega^2 S_\theta \end{Bmatrix} \\ &\vdots \end{aligned}$$

$$\begin{aligned}
& \vdots \\
& = \begin{Bmatrix} a \dot{\omega} C_{\phi} S_{\theta} - \ell C_{\phi} \dot{\Omega} + a C_{\phi} C_{\theta} (\omega^2 + \Omega^2) + a \dot{\Omega} S_{\phi} C_{\theta} - 2a \omega \Omega S_{\phi} S_{\theta} + \ell S_{\phi} \Omega^2 \\ a \dot{\omega} S_{\phi} S_{\theta} - \ell S_{\phi} \dot{\Omega} + a S_{\phi} C_{\theta} (\omega^2 + \Omega^2) - a \dot{\Omega} C_{\phi} C_{\theta} + 2a \omega \Omega C_{\phi} S_{\theta} - \ell C_{\phi} \Omega^2 \\ a \dot{\omega} C_{\theta} - a \omega^2 S_{\theta} \end{Bmatrix} \\
& = \begin{Bmatrix} -\ell C_{\phi} + a S_{\phi} C_{\theta} \\ -\ell S_{\phi} - a C_{\phi} C_{\theta} \\ 0 \end{Bmatrix} \dot{\Omega} + \begin{Bmatrix} a C_{\phi} S_{\theta} \\ a S_{\phi} S_{\theta} \\ a C_{\theta} \end{Bmatrix} \dot{\omega} + \begin{Bmatrix} \ell S_{\phi} \Omega^2 + a C_{\phi} C_{\theta} (\omega^2 + \Omega^2) - 2a \omega \Omega S_{\phi} S_{\theta} \\ -\ell C_{\phi} \Omega^2 + a S_{\phi} C_{\theta} (\omega^2 + \Omega^2) + 2a \omega \Omega C_{\phi} S_{\theta} \\ -a \omega^2 S_{\theta} \end{Bmatrix} \\
& \Rightarrow \boxed{\{a_P\} = \begin{Bmatrix} -\ell C_{\phi} + a S_{\phi} C_{\theta} \\ -\ell S_{\phi} - a C_{\phi} C_{\theta} \\ 0 \end{Bmatrix} \dot{\Omega} + \begin{Bmatrix} a C_{\phi} S_{\theta} \\ a S_{\phi} S_{\theta} \\ a C_{\theta} \end{Bmatrix} \dot{\omega} + \begin{Bmatrix} \ell S_{\phi} \Omega^2 + a C_{\phi} C_{\theta} (\omega^2 + \Omega^2) - 2a \omega \Omega S_{\phi} S_{\theta} \\ -\ell C_{\phi} \Omega^2 + a S_{\phi} C_{\theta} (\omega^2 + \Omega^2) + 2a \omega \Omega C_{\phi} S_{\theta} \\ -a \omega^2 S_{\theta} \end{Bmatrix}} \quad \checkmark
\end{aligned}$$

These results are **identical** the those found above.

The previous result can also be converted into **body frame components** as follows.

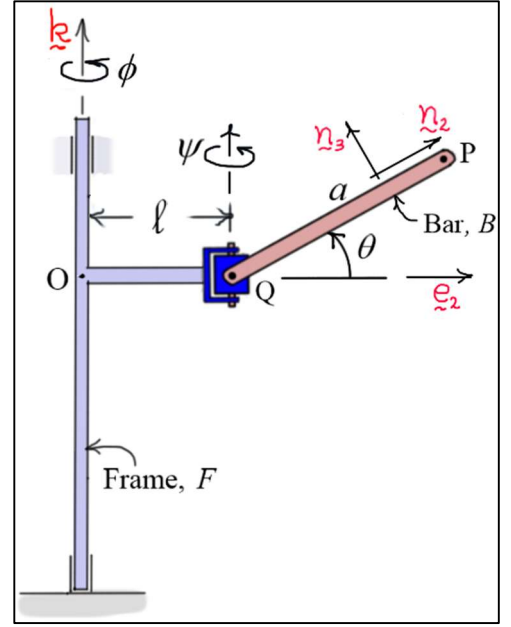
$$\begin{aligned}
\{a'_P\} &= \begin{bmatrix} C_{\theta} & 0 & -S_{\theta} \\ 0 & 1 & 0 \\ S_{\theta} & 0 & C_{\theta} \end{bmatrix} \begin{Bmatrix} a \dot{\omega} S_{\theta} - \ell \dot{\Omega} + a C_{\theta} (\omega^2 + \Omega^2) \\ -a \dot{\Omega} C_{\theta} + 2a \omega \Omega S_{\theta} - \ell \Omega^2 \\ a \dot{\omega} C_{\theta} - a \omega^2 S_{\theta} \end{Bmatrix} \\
&= \begin{Bmatrix} a \dot{\omega} S_{\theta} C_{\theta} - \ell C_{\theta} \dot{\Omega} + a C_{\theta}^2 (\omega^2 + \Omega^2) - a \dot{\omega} S_{\theta} C_{\theta} + a \omega^2 S_{\theta}^2 \\ -a \dot{\Omega} C_{\theta} + 2a \omega \Omega S_{\theta} - \ell \Omega^2 \\ a \dot{\omega} S_{\theta}^2 - \ell S_{\theta} \dot{\Omega} + a S_{\theta} C_{\theta} (\omega^2 + \Omega^2) + a \dot{\omega} C_{\theta}^2 - a \omega^2 S_{\theta} C_{\theta} \end{Bmatrix} \\
&\Rightarrow \boxed{\{a'_P\} = \begin{Bmatrix} -\ell C_{\theta} \\ -a C_{\theta} \\ -\ell S_{\theta} \end{Bmatrix} \dot{\Omega} + \begin{Bmatrix} 0 \\ 0 \\ a \end{Bmatrix} \dot{\omega} + \begin{Bmatrix} 0 \\ 2a S_{\theta} \\ 0 \end{Bmatrix} \omega \Omega + \begin{Bmatrix} a C_{\theta}^2 \\ -\ell \\ a S_{\theta} C_{\theta} \end{Bmatrix} \Omega^2 + \begin{Bmatrix} a \\ 0 \\ 0 \end{Bmatrix} \omega^2} \quad \checkmark
\end{aligned}$$

This result is **identical** to that in Equation (127).

Example 2

The diagram shows a **two-body** system with **frame** F and **bar** B . Frame F rotates ($\dot{\phi} = \Omega$) about a fixed vertical direction annotated by the unit vector $\underline{\hat{k}}$, and bar B is attached to F with a joint allowing rotations about **two axes**. Rotation about the first axis is measured by the angle ψ , and rotation about the second axis is measured by the angle θ . The angles ϕ and ψ are both **zero** in the position shown. B is oriented with respect to F using a 3-1 rotation sequence. Three reference frames are defined for the system as follows.

$$\begin{aligned} R: (\underline{\hat{i}}, \underline{\hat{j}}, \underline{\hat{k}}) & \quad \text{fixed frame} \\ F: (\underline{\hat{e}}_1, \underline{\hat{e}}_2, \underline{\hat{k}}) & \quad \text{rotating with Frame, } F \\ B: (\underline{\hat{n}}_1, \underline{\hat{n}}_2, \underline{\hat{n}}_3) & \quad \text{rotating with Bar, } B \end{aligned}$$



Let $\{\hat{\omega}_F\}$ and $\{\hat{\omega}_B\}$ be the **base frame components** and let $\{\hat{\omega}'_F\}$ and $\{\hat{\omega}'_B\}$ be the **body frame components** of the **relative angular velocities** ${}^R\omega_F$ and ${}^F\omega_B$. a) Find the **fixed frame components** of the **time derivatives** of the **partial velocity matrices** and of the **acceleration** of point P using **base frame** components or the relative angular velocity vectors. b) Find the **body frame components** of the **time derivatives** of the **partial velocity matrices** and **acceleration** of point P associated with the **body frame** components of the relative angular velocity vectors. Use results from Example 2 of Unit 4 for the angular velocities, partial angular velocity matrices, velocities, and partial velocity matrices.

Published results of the **derivations** of the **results** of this example using the **Symbolic Math Toolbox** of **MATLAB** are shown in **Addendum A**.

Solution:

a) Results from Unit 4:

The **base frame components** of the **relative angular velocities** can be written as follows.

$$\{\hat{\omega}_F\} = \begin{bmatrix} 0 & 0 & \dot{\phi} \end{bmatrix}^T$$

$${}^F\omega_B = \dot{\theta} \underline{\hat{n}}_1 + \dot{\psi} \underline{\hat{k}} = \dot{\theta} (C_\psi \underline{\hat{e}}_1 + S_\psi \underline{\hat{e}}_2) + \dot{\psi} \underline{\hat{k}} \Rightarrow \{\hat{\omega}_B\} = \begin{bmatrix} \dot{\theta} C_\psi & \dot{\theta} S_\psi & \dot{\psi} \end{bmatrix}^T$$

Using these results, the **fixed frame components** of the **angular velocities** of the bodies can be written as

$$\{\omega_F\} = \{\hat{\omega}_F\} \quad \{\omega_B\} = \{\hat{\omega}_F\} + [R_F]^T \{\hat{\omega}_B\}$$

Defining the column vector of **generalized speeds** to be $\{y\} \triangleq \begin{Bmatrix} \{\hat{\omega}_F\} \\ \{\hat{\omega}_B\} \end{Bmatrix}$, the **fixed frame components** of the **partial angular velocity matrices** are observed to be

$$\begin{bmatrix} {}^R\omega_{F,y} \end{bmatrix}_{3 \times 6} = \begin{bmatrix} {}^R\omega_{F,\hat{\omega}_F} & {}^R\omega_{F,\hat{\omega}_B} \end{bmatrix} = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\omega_{B,y} \end{bmatrix}_{3 \times 6} = \begin{bmatrix} {}^R\omega_{B,\hat{\omega}_F} & {}^R\omega_{B,\hat{\omega}_B} \end{bmatrix} = \begin{bmatrix} [I]_{3 \times 3} & [R_F]^T \end{bmatrix}$$

The **fixed frame components** of ${}^Rv_P$ and the partial velocity matrices are

$$\{v_P\} = \begin{bmatrix} {}^Rv_{P,\hat{\omega}_F} \end{bmatrix} \{\hat{\omega}_F\} + \begin{bmatrix} {}^Rv_{P,\hat{\omega}_B} \end{bmatrix} \{\hat{\omega}_B\} \quad (128)$$

$$\begin{bmatrix} {}^Rv_{P,\hat{\omega}_F} \end{bmatrix} = -\begin{bmatrix} \tilde{r}_{Q/O} \end{bmatrix} - \begin{bmatrix} \tilde{r}_{P/Q} \end{bmatrix} \quad (129)$$

$$\begin{bmatrix} {}^Rv_{P,\hat{\omega}_B} \end{bmatrix} = -\begin{bmatrix} \tilde{r}_{P/Q} \end{bmatrix} [R_F]^T \quad (130)$$

$$\begin{bmatrix} [R_F]^T \end{bmatrix} = \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \{r_{Q/O}\} = [R_F]^T \begin{Bmatrix} 0 \\ \ell \\ 0 \end{Bmatrix} = \begin{Bmatrix} -\ell S_\phi \\ \ell C_\phi \\ 0 \end{Bmatrix} \quad (131)$$

$$\begin{bmatrix} [R_B]^T \end{bmatrix} = \begin{bmatrix} C_{\phi+\psi} & -S_{\phi+\psi}C_\theta & S_{\phi+\psi}S_\theta \\ S_{\phi+\psi} & C_{\phi+\psi}C_\theta & -C_{\phi+\psi}S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \quad \{r_{P/Q}\} = [R_B]^T \begin{Bmatrix} 0 \\ a \\ 0 \end{Bmatrix} = \begin{Bmatrix} -aS_{\phi+\psi}C_\theta \\ aC_{\phi+\psi}C_\theta \\ aS_\theta \end{Bmatrix} \quad (132)$$

$$\begin{bmatrix} \tilde{r}_{Q/O} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \ell C_\phi \\ 0 & 0 & \ell S_\phi \\ -\ell C_\phi & -\ell S_\phi & 0 \end{bmatrix} \quad \begin{bmatrix} \tilde{r}_{P/Q} \end{bmatrix} = \begin{bmatrix} 0 & -aS_\theta & aC_{\phi+\psi}C_\theta \\ aS_\theta & 0 & aS_{\phi+\psi}C_\theta \\ -aC_{\phi+\psi}C_\theta & -aS_{\phi+\psi}C_\theta & 0 \end{bmatrix} \quad (133)$$

$$\{\omega_B\} = \{\hat{\omega}_F\} + [R_F]^T \{\hat{\omega}_B\} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\phi} \end{Bmatrix} + \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\theta}C_\psi \\ \dot{\theta}S_\psi \\ \dot{\psi} \end{Bmatrix} = \begin{Bmatrix} \dot{\theta}(C_\phi C_\psi - S_\phi S_\psi) \\ \dot{\theta}(S_\phi C_\psi + C_\phi S_\psi) \\ \dot{\phi} + \dot{\psi} \end{Bmatrix} = \begin{Bmatrix} \dot{\theta}C_{\phi+\psi} \\ \dot{\theta}S_{\phi+\psi} \\ \dot{\phi} + \dot{\psi} \end{Bmatrix} \quad (134)$$

$$\begin{bmatrix} {}^Rv_{P,\hat{\omega}_F} \end{bmatrix} = \begin{bmatrix} 0 & aS_\theta & -(\ell C_\phi + aC_{\phi+\psi}C_\theta) \\ -aS_\theta & 0 & -(\ell S_\phi + aS_{\phi+\psi}C_\theta) \\ \ell C_\phi + aC_{\phi+\psi}C_\theta & \ell S_\phi + aS_{\phi+\psi}C_\theta & 0 \end{bmatrix} \quad (135)$$

$$\begin{bmatrix} {}^Rv_{P,\hat{\omega}_B} \end{bmatrix} = \begin{bmatrix} aS_\phi S_\theta & aC_\phi S_\theta & -aC_{\phi+\psi}C_\theta \\ -aC_\phi S_\theta & aS_\phi S_\theta & -aS_{\phi+\psi}C_\theta \\ aC_\theta C_\psi & aC_\theta S_\psi & 0 \end{bmatrix} \quad \{v_P\} = \begin{Bmatrix} -\ell \dot{\phi}C_\phi + a\dot{\theta}S_{\phi+\psi}S_\theta - (\dot{\phi} + \dot{\psi})aC_{\phi+\psi}C_\theta \\ -\ell \dot{\phi}S_\phi - a\dot{\theta}C_{\phi+\psi}S_\theta - (\dot{\phi} + \dot{\psi})aS_{\phi+\psi}C_\theta \\ a\dot{\theta}C_\theta \end{Bmatrix} \quad (136)$$

Acceleration:

The **fixed frame components** of ${}^R\mathbf{a}_P$ the **acceleration** of P can be calculated by **differentiating** the fixed frame components of ${}^R\mathbf{v}_P$ as follows.

$$\{a_P\} = \{\dot{v}_P\} = \begin{bmatrix} {}^R\mathbf{v}_{P,\hat{\omega}_F} \end{bmatrix} \{\dot{\hat{\omega}}_F\} + \begin{bmatrix} {}^R\mathbf{v}_{P,\hat{\omega}_B} \end{bmatrix} \{\dot{\hat{\omega}}_B\} + \begin{bmatrix} {}^R\dot{\mathbf{v}}_{P,\hat{\omega}_F} \end{bmatrix} \{\hat{\omega}_F\} + \begin{bmatrix} {}^R\dot{\mathbf{v}}_{P,\hat{\omega}_B} \end{bmatrix} \{\hat{\omega}_B\} \quad (137)$$

The **time derivatives** of the **partial velocity matrices** are found by **differentiating** Equations (129) and (130).

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{P,\hat{\omega}_F} \end{bmatrix} = -\begin{bmatrix} \dot{\tilde{\mathbf{r}}}_{Q/O} \end{bmatrix} - \begin{bmatrix} \dot{\tilde{\mathbf{r}}}_{P/Q} \end{bmatrix} \quad (138)$$

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{P,\hat{\omega}_B} \end{bmatrix} = -\begin{bmatrix} \dot{\tilde{\mathbf{r}}}_{P/Q} \end{bmatrix} [R_F]^T - \begin{bmatrix} \tilde{\mathbf{r}}_{P/Q} \end{bmatrix} [\dot{R}_F]^T = -\begin{bmatrix} \dot{\tilde{\mathbf{r}}}_{P/Q} \end{bmatrix} [R_F]^T - \begin{bmatrix} \tilde{\mathbf{r}}_{P/Q} \end{bmatrix} [\tilde{\omega}_F] [R_F]^T \quad (139)$$

The **time derivatives** of the **two skew symmetric matrices** are built using the **time derivatives** of the **fixed frame components** of the **position vectors** $\mathbf{r}_{Q/O}$ and $\mathbf{r}_{P/Q}$. **Differentiating** the results presented in Equations (131) and (132) gives

$$\begin{aligned} \{\dot{\mathbf{r}}_{Q/O}\} &= [\dot{R}_F]^T \begin{Bmatrix} 0 \\ \ell \\ 0 \end{Bmatrix} = [\tilde{\omega}_F] [R_F]^T \begin{Bmatrix} 0 \\ \ell \\ 0 \end{Bmatrix} = \begin{bmatrix} 0 & -\dot{\phi} & 0 \\ \dot{\phi} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ \ell \\ 0 \end{Bmatrix} = \begin{bmatrix} -\dot{\phi}S_\phi & -\dot{\phi}C_\phi & 0 \\ \dot{\phi}C_\phi & -\dot{\phi}S_\phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ \ell \\ 0 \end{Bmatrix} \\ \Rightarrow \{\dot{\mathbf{r}}_{Q/O}\} &= \begin{Bmatrix} -\ell\dot{\phi}C_\phi \\ -\ell\dot{\phi}S_\phi \\ 0 \end{Bmatrix} \quad \begin{bmatrix} \dot{\tilde{\mathbf{r}}}_{Q/O} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\ell\dot{\phi}S_\phi \\ 0 & 0 & \ell\dot{\phi}C_\phi \\ \ell\dot{\phi}S_\phi & -\ell\dot{\phi}C_\phi & 0 \end{bmatrix} \end{aligned} \quad (140)$$

$$\begin{aligned} \{\dot{\mathbf{r}}_{P/Q}\} &= [\dot{R}_B]^T \begin{Bmatrix} 0 \\ a \\ 0 \end{Bmatrix} = [\tilde{\omega}_B] [R_B]^T \begin{Bmatrix} 0 \\ a \\ 0 \end{Bmatrix} = \begin{bmatrix} 0 & -(\dot{\phi}+\dot{\psi}) & \dot{\theta}S_{\phi+\psi} \\ (\dot{\phi}+\dot{\psi}) & 0 & -\dot{\theta}C_{\phi+\psi} \\ -\dot{\theta}S_{\phi+\psi} & \dot{\theta}C_{\phi+\psi} & 0 \end{bmatrix} \begin{bmatrix} C_{\phi+\psi} & -S_{\phi+\psi}C_\theta & S_{\phi+\psi}S_\theta \\ S_{\phi+\psi} & C_{\phi+\psi}C_\theta & -C_{\phi+\psi}S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \begin{Bmatrix} 0 \\ a \\ 0 \end{Bmatrix} \\ &= \begin{bmatrix} 0 & -(\dot{\phi}+\dot{\psi}) & \dot{\theta}S_{\phi+\psi} \\ (\dot{\phi}+\dot{\psi}) & 0 & -\dot{\theta}C_{\phi+\psi} \\ -\dot{\theta}S_{\phi+\psi} & \dot{\theta}C_{\phi+\psi} & 0 \end{bmatrix} \begin{Bmatrix} -aS_{\phi+\psi}C_\theta \\ aC_{\phi+\psi}C_\theta \\ aS_\theta \end{Bmatrix} \\ \Rightarrow \{\dot{\mathbf{r}}_{P/Q}\} &= \begin{Bmatrix} -aC_{\phi+\psi}C_\theta(\dot{\phi}+\dot{\psi}) + aS_\theta S_{\phi+\psi}\dot{\theta} \\ -aS_{\phi+\psi}C_\theta(\dot{\phi}+\dot{\psi}) - aS_\theta C_{\phi+\psi}\dot{\theta} \\ aC_\theta\dot{\theta} \end{Bmatrix} \end{aligned}$$

$$\begin{bmatrix} \dot{\tilde{\mathbf{r}}}_{P/Q} \end{bmatrix} = \begin{bmatrix} 0 & -aC_\theta\dot{\theta} & -aS_{\phi+\psi}C_\theta(\dot{\phi}+\dot{\psi}) - aS_\theta C_{\phi+\psi}\dot{\theta} \\ aC_\theta\dot{\theta} & 0 & aC_{\phi+\psi}C_\theta(\dot{\phi}+\dot{\psi}) - aS_\theta S_{\phi+\psi}\dot{\theta} \\ aS_{\phi+\psi}C_\theta(\dot{\phi}+\dot{\psi}) + aS_\theta C_{\phi+\psi}\dot{\theta} & -aC_{\phi+\psi}C_\theta(\dot{\phi}+\dot{\psi}) + aS_\theta S_{\phi+\psi}\dot{\theta} & 0 \end{bmatrix} \quad (141)$$

Substituting the results from Equations (140) and (141) into Equation (138) and **simplifying** gives the following.

$$\left[{}^R\dot{V}_{P,\hat{\omega}_F} \right] = -\left[\dot{\tilde{r}}_{Q/O} \right] - \left[\dot{\tilde{r}}_{P/Q} \right] = \begin{bmatrix} 0 & 0 & \ell\dot{\phi}S_\phi \\ 0 & 0 & -\ell\dot{\phi}C_\phi \\ -\ell\dot{\phi}S_\phi & \ell\dot{\phi}C_\phi & 0 \end{bmatrix} + \begin{bmatrix} 0 & aC_\theta\dot{\theta} & -\left[\dot{\tilde{r}}_{P/Q} \right]_{13} \\ -aC_\theta\dot{\theta} & 0 & -\left[\dot{\tilde{r}}_{P/Q} \right]_{23} \\ \left[\dot{\tilde{r}}_{P/Q} \right]_{13} & \left[\dot{\tilde{r}}_{P/Q} \right]_{23} & 0 \end{bmatrix} \quad (142)$$

Here,

$$\left[\dot{\tilde{r}}_{P/Q} \right]_{13} = -aS_{\phi+\psi}C_\theta(\dot{\phi}+\dot{\psi}) - aS_\theta C_{\phi+\psi}\dot{\theta} \quad \text{and} \quad \left[\dot{\tilde{r}}_{P/Q} \right]_{23} = aC_{\phi+\psi}C_\theta(\dot{\phi}+\dot{\psi}) - aS_\theta S_{\phi+\psi}\dot{\theta} \quad (143)$$

Substituting from Equation (141) with other results from Unit 4 give the **time derivative** of the **second partial velocity matrix**.

$$\begin{aligned} \left[{}^R\dot{V}_{P,\hat{\omega}_B} \right] &= -\left[\dot{\tilde{r}}_{P/Q} \right] \left[R_F \right]^T - \left[\tilde{r}_{P/Q} \right] \left[\tilde{\omega}_F \right] \left[R_F \right]^T \\ &= \begin{bmatrix} 0 & aC_\theta\dot{\theta} & -\left[\dot{\tilde{r}}_{P/Q} \right]_{13} \\ -aC_\theta\dot{\theta} & 0 & -\left[\dot{\tilde{r}}_{P/Q} \right]_{23} \\ \left[\dot{\tilde{r}}_{P/Q} \right]_{13} & \left[\dot{\tilde{r}}_{P/Q} \right]_{23} & 0 \end{bmatrix} \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &\quad - \begin{bmatrix} 0 & -aS_\theta & aC_{\phi+\psi}C_\theta \\ aS_\theta & 0 & aS_{\phi+\psi}C_\theta \\ -aC_{\phi+\psi}C_\theta & -aS_{\phi+\psi}C_\theta & 0 \end{bmatrix} \begin{bmatrix} 0 & -\dot{\phi} & 0 \\ \dot{\phi} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} aS_\phi C_\theta\dot{\theta} & aC_\phi C_\theta\dot{\theta} & -\left[\dot{\tilde{r}}_{P/Q} \right]_{13} \\ -aC_\phi C_\theta\dot{\theta} & aS_\phi C_\theta\dot{\theta} & -\left[\dot{\tilde{r}}_{P/Q} \right]_{23} \\ \left[\dot{\tilde{r}}_{P/Q} \right]_{13} C_\phi + \left[\dot{\tilde{r}}_{P/Q} \right]_{23} S_\phi & -\left[\dot{\tilde{r}}_{P/Q} \right]_{13} S_\phi + \left[\dot{\tilde{r}}_{P/Q} \right]_{23} C_\phi & 0 \end{bmatrix} \\ &\quad - \begin{bmatrix} -aS_\theta\dot{\phi} & 0 & 0 \\ 0 & -aS_\theta\dot{\phi} & 0 \\ -aS_{\phi+\psi}C_\theta\dot{\phi} & aC_{\phi+\psi}C_\theta\dot{\phi} & 0 \end{bmatrix} \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} aS_\phi C_\theta\dot{\theta} & aC_\phi C_\theta\dot{\theta} & -\left[\dot{\tilde{r}}_{P/Q} \right]_{13} \\ -aC_\phi C_\theta\dot{\theta} & aS_\phi C_\theta\dot{\theta} & -\left[\dot{\tilde{r}}_{P/Q} \right]_{23} \\ \left[\dot{\tilde{r}}_{P/Q} \right]_{13} C_\phi + \left[\dot{\tilde{r}}_{P/Q} \right]_{23} S_\phi & -\left[\dot{\tilde{r}}_{P/Q} \right]_{13} S_\phi + \left[\dot{\tilde{r}}_{P/Q} \right]_{23} C_\phi & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} aC_\phi S_\theta\dot{\phi} & -aS_\phi S_\theta\dot{\phi} & 0 \\ aS_\phi S_\theta\dot{\phi} & aC_\phi S_\theta\dot{\phi} & 0 \\ aC_\phi S_{\phi+\psi}C_\theta\dot{\phi} - aS_\phi C_{\phi+\psi}C_\theta\dot{\phi} & -aS_\phi S_{\phi+\psi}C_\theta\dot{\phi} - aC_\phi C_{\phi+\psi}C_\theta\dot{\phi} & 0 \end{bmatrix} \end{aligned}$$

Simplifying terms in the two matrices gives the following.

$$\begin{aligned}
& \left[\dot{\tilde{r}}_{P/Q} \right]_{13} C_\phi + \left[\dot{\tilde{r}}_{P/Q} \right]_{23} S_\phi = \left(-aS_{\phi+\psi} C_\theta (\dot{\phi} + \dot{\psi}) - aS_\theta C_{\phi+\psi} \dot{\theta} \right) C_\phi + \left(aC_{\phi+\psi} C_\theta (\dot{\phi} + \dot{\psi}) - aS_\theta S_{\phi+\psi} \dot{\theta} \right) S_\phi \\
& = \left(-aC_\phi S_{\phi+\psi} C_\theta + aS_\phi C_{\phi+\psi} C_\theta \right) (\dot{\phi} + \dot{\psi}) - \left(aS_\theta C_\phi C_{\phi+\psi} + aS_\theta S_\phi S_{\phi+\psi} \right) \dot{\theta} \\
& = -aC_\theta \left(C_\phi S_{\phi+\psi} - S_\phi C_{\phi+\psi} \right) (\dot{\phi} + \dot{\psi}) - aS_\theta \left(C_\phi C_{\phi+\psi} + S_\phi S_{\phi+\psi} \right) \dot{\theta} \\
& \Rightarrow \left[\dot{\tilde{r}}_{P/Q} \right]_{13} C_\phi + \left[\dot{\tilde{r}}_{P/Q} \right]_{23} S_\phi = -aC_\theta S_\psi (\dot{\phi} + \dot{\psi}) - aS_\theta C_\psi \dot{\theta} \\
& - \left[\dot{\tilde{r}}_{P/Q} \right]_{13} S_\phi + \left[\dot{\tilde{r}}_{P/Q} \right]_{23} C_\phi = \left(aS_{\phi+\psi} C_\theta (\dot{\phi} + \dot{\psi}) + aS_\theta C_{\phi+\psi} \dot{\theta} \right) S_\phi + \left(aC_{\phi+\psi} C_\theta (\dot{\phi} + \dot{\psi}) - aS_\theta S_{\phi+\psi} \dot{\theta} \right) C_\phi \\
& = aC_\theta \left(S_{\phi+\psi} S_\phi + C_{\phi+\psi} C_\phi \right) (\dot{\phi} + \dot{\psi}) + aS_\theta \left(C_{\phi+\psi} S_\phi - S_{\phi+\psi} C_\phi \right) \dot{\theta} \\
& = aC_\theta C_\psi (\dot{\phi} + \dot{\psi}) - aS_\theta S_\psi \dot{\theta} \\
& \Rightarrow - \left[\dot{\tilde{r}}_{P/Q} \right]_{13} S_\phi + \left[\dot{\tilde{r}}_{P/Q} \right]_{23} C_\phi = aC_\theta C_\psi (\dot{\phi} + \dot{\psi}) - aS_\theta S_\psi \dot{\theta} \\
& aC_\phi S_{\phi+\psi} C_\theta \dot{\phi} - aS_\phi C_{\phi+\psi} C_\theta \dot{\phi} = aC_\theta \left(C_\phi S_{\phi+\psi} - S_\phi C_{\phi+\psi} \right) \dot{\phi} = aC_\theta S_\psi \dot{\phi} \\
& -aS_\phi S_{\phi+\psi} C_\theta \dot{\phi} - aC_\phi C_{\phi+\psi} C_\theta \dot{\phi} = -aC_\theta \left(S_\phi S_{\phi+\psi} + C_\phi C_{\phi+\psi} \right) \dot{\phi} = -aC_\theta C_\psi \dot{\phi}
\end{aligned}$$

Substituting into the two matrices gives the following result.

$$\left[{}^R \dot{v}_{P, \hat{\omega}_B} \right] = \begin{bmatrix} a \left(S_\phi C_\theta \dot{\theta} + C_\phi S_\theta \dot{\phi} \right) & a \left(C_\phi C_\theta \dot{\theta} - S_\phi S_\theta \dot{\phi} \right) & a \left(S_{\phi+\psi} C_\theta (\dot{\phi} + \dot{\psi}) + S_\theta C_{\phi+\psi} \dot{\theta} \right) \\ a \left(S_\phi S_\theta \dot{\theta} - C_\phi C_\theta \dot{\phi} \right) & a \left(S_\phi C_\theta \dot{\theta} + C_\phi S_\theta \dot{\phi} \right) & a \left(S_\theta S_{\phi+\psi} \dot{\theta} - C_{\phi+\psi} C_\theta (\dot{\phi} + \dot{\psi}) \right) \\ -a \left(C_\theta S_\psi \dot{\psi} + S_\theta C_\psi \dot{\theta} \right) & a \left(C_\theta C_\psi \dot{\psi} - S_\theta S_\psi \dot{\theta} \right) & 0 \end{bmatrix} \quad (144)$$

Substituting from Equations (135), (136), (142), (143), and (144) into Equation (137) gives the **fixed frame components** of ${}^R a_P$ the **acceleration** of P in R .

$$\{a_P\} = \left[{}^R v_{P, \hat{\omega}_F} \right] \{ \dot{\hat{\omega}}_F \} + \left[{}^R v_{P, \hat{\omega}_B} \right] \{ \dot{\hat{\omega}}_B \} + \left[{}^R \dot{v}_{P, \hat{\omega}_F} \right] \{ \hat{\omega}_F \} + \left[{}^R \dot{v}_{P, \hat{\omega}_B} \right] \{ \hat{\omega}_B \}$$

Each of the terms is expanded and simplified below.

$$\left[{}^R v_{P, \hat{\omega}_F} \right] \{ \dot{\hat{\omega}}_F \} = \begin{bmatrix} 0 & aS_\theta & -(\ell C_\phi + aC_{\phi+\psi} C_\theta) \\ -aS_\theta & 0 & -(\ell S_\phi + aS_{\phi+\psi} C_\theta) \\ \ell C_\phi + aC_{\phi+\psi} C_\theta & \ell S_\phi + aS_{\phi+\psi} C_\theta & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \ddot{\phi} \end{Bmatrix} = \begin{bmatrix} -\ell C_\phi - aC_{\phi+\psi} C_\theta \\ -\ell S_\phi - aS_{\phi+\psi} C_\theta \\ 0 \end{bmatrix} \ddot{\phi}$$

$$\begin{aligned}
\left[{}^R v_{P, \hat{\omega}_B} \right] \{ \dot{\hat{\omega}}_B \} &= \begin{bmatrix} aS_\phi S_\theta & aC_\phi S_\theta & -aC_{\phi+\psi} C_\theta \\ -aC_\phi S_\theta & aS_\phi S_\theta & -aS_{\phi+\psi} C_\theta \\ aC_\theta C_\psi & aC_\theta S_\psi & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\theta} C_\psi - \dot{\theta} \dot{\psi} S_\psi \\ \ddot{\theta} S_\psi + \dot{\theta} \dot{\psi} C_\psi \\ \ddot{\psi} \end{Bmatrix} \\
&= \begin{bmatrix} aS_\phi S_\theta & aC_\phi S_\theta & -aC_{\phi+\psi} C_\theta \\ -aC_\phi S_\theta & aS_\phi S_\theta & -aS_{\phi+\psi} C_\theta \\ aC_\theta C_\psi & aC_\theta S_\psi & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\theta} C_\psi \\ \ddot{\theta} S_\psi \\ \ddot{\psi} \end{Bmatrix} + \begin{bmatrix} aS_\phi S_\theta & aC_\phi S_\theta & -aC_{\phi+\psi} C_\theta \\ -aC_\phi S_\theta & aS_\phi S_\theta & -aS_{\phi+\psi} C_\theta \\ aC_\theta C_\psi & aC_\theta S_\psi & 0 \end{bmatrix} \begin{Bmatrix} -\dot{\theta} \dot{\psi} S_\psi \\ \dot{\theta} \dot{\psi} C_\psi \\ 0 \end{Bmatrix} \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
& \vdots \\
& = \begin{Bmatrix} aS_{\theta}(S_{\phi}C_{\psi} + C_{\phi}S_{\psi})\ddot{\theta} - aC_{\phi+\psi}C_{\theta}\ddot{\psi} \\ aS_{\theta}(-C_{\phi}C_{\psi} + S_{\phi}S_{\psi})\ddot{\theta} - aS_{\phi+\psi}C_{\theta}\ddot{\psi} \\ aC_{\theta}(C_{\psi}C_{\psi} + S_{\psi}S_{\psi})\ddot{\theta} \end{Bmatrix} + \begin{Bmatrix} aS_{\theta}(-S_{\phi}S_{\psi} + C_{\phi}C_{\psi}) \\ aS_{\theta}(C_{\phi}S_{\psi} + S_{\phi}C_{\psi}) \\ aC_{\theta}(-C_{\psi}S_{\psi} + S_{\psi}C_{\psi}) \end{Bmatrix} \dot{\theta}\dot{\psi} \\
& = \begin{Bmatrix} aS_{\theta}S_{\phi+\psi}\ddot{\theta} - aC_{\phi+\psi}C_{\theta}\ddot{\psi} \\ -aS_{\theta}C_{\phi+\psi}\ddot{\theta} - aS_{\phi+\psi}C_{\theta}\ddot{\psi} \\ aC_{\theta}\ddot{\theta} \end{Bmatrix} + \begin{Bmatrix} aS_{\theta}C_{\phi+\psi} \\ aS_{\theta}S_{\phi+\psi} \\ 0 \end{Bmatrix} \dot{\theta}\dot{\psi} \\
& \Rightarrow \left[{}^R\dot{v}_{P,\hat{\omega}_B} \right] \{ \dot{\omega}_B \} = \begin{Bmatrix} -aC_{\phi+\psi}C_{\theta} \\ -aS_{\phi+\psi}C_{\theta} \\ 0 \end{Bmatrix} \ddot{\psi} + \begin{Bmatrix} aS_{\theta}S_{\phi+\psi} \\ -aS_{\theta}C_{\phi+\psi} \\ aC_{\theta} \end{Bmatrix} \ddot{\theta} + \begin{Bmatrix} aS_{\theta}C_{\phi+\psi} \\ aS_{\theta}S_{\phi+\psi} \\ 0 \end{Bmatrix} \dot{\theta}\dot{\psi} \\
& \left[{}^R\dot{v}_{P,\hat{\omega}_F} \right] \{ \dot{\omega}_F \} = \begin{bmatrix} 0 & 0 & \ell\dot{\phi}S_{\phi} \\ 0 & 0 & -\ell\dot{\phi}C_{\phi} \\ -\ell\dot{\phi}S_{\phi} & \ell\dot{\phi}C_{\phi} & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\phi} \end{Bmatrix} \\
& + \begin{bmatrix} 0 & aC_{\theta}\dot{\theta} & aS_{\phi+\psi}C_{\theta}(\dot{\phi}+\dot{\psi}) + aS_{\theta}C_{\phi+\psi}\dot{\theta} \\ -aC_{\theta}\dot{\theta} & 0 & -aC_{\phi+\psi}C_{\theta}(\dot{\phi}+\dot{\psi}) + aS_{\theta}S_{\phi+\psi}\dot{\theta} \\ -aS_{\phi+\psi}C_{\theta}(\dot{\phi}+\dot{\psi}) - aS_{\theta}C_{\phi+\psi}\dot{\theta} & aC_{\phi+\psi}C_{\theta}(\dot{\phi}+\dot{\psi}) - aS_{\theta}S_{\phi+\psi}\dot{\theta} & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\phi} \end{Bmatrix} \\
& = \begin{Bmatrix} \ell S_{\phi} \\ -\ell C_{\phi} \\ 0 \end{Bmatrix} \dot{\phi}^2 + \begin{Bmatrix} aS_{\phi+\psi}C_{\theta}(\dot{\phi}+\dot{\psi}) + aS_{\theta}C_{\phi+\psi}\dot{\theta} \\ -aC_{\phi+\psi}C_{\theta}(\dot{\phi}+\dot{\psi}) + aS_{\theta}S_{\phi+\psi}\dot{\theta} \\ 0 \end{Bmatrix} \dot{\phi} \\
& \Rightarrow \left[{}^R\dot{v}_{P,\hat{\omega}_F} \right] \{ \dot{\omega}_F \} = \begin{Bmatrix} \ell S_{\phi} + aS_{\phi+\psi}C_{\theta} \\ -\ell C_{\phi} - aC_{\phi+\psi}C_{\theta} \\ 0 \end{Bmatrix} \dot{\phi}^2 + \begin{Bmatrix} aS_{\phi+\psi}C_{\theta} \\ -aC_{\phi+\psi}C_{\theta} \\ 0 \end{Bmatrix} \dot{\phi}\dot{\psi} + \begin{Bmatrix} aS_{\theta}C_{\phi+\psi} \\ aS_{\theta}S_{\phi+\psi} \\ 0 \end{Bmatrix} \dot{\phi}\dot{\theta} \\
& \left[{}^R\dot{v}_{P,\hat{\omega}_B} \right] \{ \dot{\omega}_B \} \\
& = \begin{bmatrix} a(S_{\phi}C_{\theta}\dot{\theta} + C_{\phi}S_{\theta}\dot{\phi}) & a(C_{\phi}C_{\theta}\dot{\theta} - S_{\phi}S_{\theta}\dot{\phi}) & a(S_{\phi+\psi}C_{\theta}(\dot{\phi}+\dot{\psi}) + S_{\theta}C_{\phi+\psi}\dot{\theta}) \\ a(S_{\phi}S_{\theta}\dot{\phi} - C_{\phi}C_{\theta}\dot{\theta}) & a(S_{\phi}C_{\theta}\dot{\theta} + C_{\phi}S_{\theta}\dot{\phi}) & a(S_{\theta}S_{\phi+\psi}\dot{\theta} - C_{\phi+\psi}C_{\theta}(\dot{\phi}+\dot{\psi})) \\ -a(C_{\theta}S_{\psi}\dot{\psi} + S_{\theta}C_{\psi}\dot{\theta}) & a(C_{\theta}C_{\psi}\dot{\psi} - S_{\theta}S_{\psi}\dot{\theta}) & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta}C_{\psi} \\ \dot{\theta}S_{\psi} \\ \dot{\psi} \end{Bmatrix} \\
& = \begin{Bmatrix} a(S_{\phi}C_{\theta}\dot{\theta} + C_{\phi}S_{\theta}\dot{\phi})\dot{\theta}C_{\psi} + a(C_{\phi}C_{\theta}\dot{\theta} - S_{\phi}S_{\theta}\dot{\phi})\dot{\theta}S_{\psi} + a(S_{\phi+\psi}C_{\theta}(\dot{\phi}+\dot{\psi}) + S_{\theta}C_{\phi+\psi}\dot{\theta})\dot{\psi} \\ a(S_{\phi}S_{\theta}\dot{\phi} - C_{\phi}C_{\theta}\dot{\theta})\dot{\theta}C_{\psi} + a(S_{\phi}C_{\theta}\dot{\theta} + C_{\phi}S_{\theta}\dot{\phi})\dot{\theta}S_{\psi} + a(S_{\theta}S_{\phi+\psi}\dot{\theta} - C_{\phi+\psi}C_{\theta}(\dot{\phi}+\dot{\psi}))\dot{\psi} \\ -a(C_{\theta}S_{\psi}\dot{\psi} + S_{\theta}C_{\psi}\dot{\theta})\dot{\theta}C_{\psi} + a(C_{\theta}C_{\psi}\dot{\psi} - S_{\theta}S_{\psi}\dot{\theta})\dot{\theta}S_{\psi} \end{Bmatrix} \\
& \vdots
\end{aligned}$$

$$\begin{aligned}
& \vdots \\
& = a \left\{ \begin{aligned} & C_\theta (S_\phi C_\psi + C_\phi S_\psi) \dot{\theta}^2 + (S_{\phi+\psi} C_\theta) \dot{\psi}^2 + (S_{\phi+\psi} C_\theta) \dot{\phi} \dot{\psi} + (S_\theta C_{\phi+\psi}) \dot{\psi} \dot{\theta} + S_\theta (C_\phi C_\psi - S_\phi S_\psi) \dot{\phi} \dot{\theta} \\ & C_\theta (-C_\phi C_\psi + S_\phi S_\psi) \dot{\theta}^2 + (-C_{\phi+\psi} C_\theta) \dot{\psi}^2 + (-C_{\phi+\psi} C_\theta) \dot{\phi} \dot{\psi} + (S_\theta S_{\phi+\psi}) \dot{\psi} \dot{\theta} + S_\theta (S_\phi C_\psi + C_\phi S_\psi) \dot{\phi} \dot{\theta} \\ & -S_\theta (C_\psi^2 + S_\psi^2) \dot{\theta}^2 \end{aligned} \right\} \\
& = a \left\{ \begin{aligned} & (C_\theta S_{\phi+\psi}) \dot{\theta}^2 + (S_{\phi+\psi} C_\theta) \dot{\psi}^2 + (S_{\phi+\psi} C_\theta) \dot{\phi} \dot{\psi} + (S_\theta C_{\phi+\psi}) \dot{\psi} \dot{\theta} + (S_\theta C_{\phi+\psi}) \dot{\phi} \dot{\theta} \\ & (-C_\theta C_{\phi+\psi}) \dot{\theta}^2 + (-C_{\phi+\psi} C_\theta) \dot{\psi}^2 + (-C_{\phi+\psi} C_\theta) \dot{\phi} \dot{\psi} + (S_\theta S_{\phi+\psi}) \dot{\psi} \dot{\theta} + (S_\theta S_{\phi+\psi}) \dot{\phi} \dot{\theta} \\ & (-S_\theta) \dot{\theta}^2 \end{aligned} \right\} \\
& \Rightarrow \left[{}^R \dot{v}_{P, \hat{\omega}_B} \right] \{ \hat{\omega}_B \} = \begin{Bmatrix} a C_\theta S_{\phi+\psi} \\ -a C_\theta C_{\phi+\psi} \\ -a S_\theta \end{Bmatrix} \dot{\theta}^2 + \begin{Bmatrix} a S_{\phi+\psi} C_\theta \\ -a C_{\phi+\psi} C_\theta \\ 0 \end{Bmatrix} \dot{\psi}^2 + \begin{Bmatrix} a S_{\phi+\psi} C_\theta \\ -a C_{\phi+\psi} C_\theta \\ 0 \end{Bmatrix} \dot{\phi} \dot{\psi} + \begin{Bmatrix} a S_\theta C_{\phi+\psi} \\ a S_\theta S_{\phi+\psi} \\ 0 \end{Bmatrix} \dot{\psi} \dot{\theta} + \begin{Bmatrix} a S_\theta C_{\phi+\psi} \\ a S_\theta S_{\phi+\psi} \\ 0 \end{Bmatrix} \dot{\phi} \dot{\theta}
\end{aligned}$$

Combining results for the four terms gives the result.

$$\begin{aligned}
\{a_P\} &= \left[{}^R v_{P, \hat{\omega}_F} \right] \{ \dot{\hat{\omega}}_F \} + \left[{}^R v_{P, \hat{\omega}_B} \right] \{ \dot{\hat{\omega}}_B \} + \left[{}^R \dot{v}_{P, \hat{\omega}_F} \right] \{ \hat{\omega}_F \} + \left[{}^R \dot{v}_{P, \hat{\omega}_B} \right] \{ \hat{\omega}_B \} \\
&= \begin{Bmatrix} -\ell C_\phi - a C_{\phi+\psi} C_\theta \\ -\ell S_\phi - a S_{\phi+\psi} C_\theta \\ 0 \end{Bmatrix} \ddot{\phi} + \begin{Bmatrix} -a C_{\phi+\psi} C_\theta \\ -a S_{\phi+\psi} C_\theta \\ 0 \end{Bmatrix} \ddot{\psi} + \begin{Bmatrix} a S_\theta S_{\phi+\psi} \\ -a S_\theta C_{\phi+\psi} \\ a C_\theta \end{Bmatrix} \ddot{\theta} + \begin{Bmatrix} a S_\theta C_{\phi+\psi} \\ a S_\theta S_{\phi+\psi} \\ 0 \end{Bmatrix} \dot{\theta} \dot{\psi} \\
&+ \begin{Bmatrix} \ell S_\phi + a S_{\phi+\psi} C_\theta \\ -\ell C_\phi - a C_{\phi+\psi} C_\theta \\ 0 \end{Bmatrix} \dot{\phi}^2 + \begin{Bmatrix} a S_{\phi+\psi} C_\theta \\ -a C_{\phi+\psi} C_\theta \\ 0 \end{Bmatrix} \dot{\phi} \dot{\psi} + \begin{Bmatrix} a S_\theta C_{\phi+\psi} \\ a S_\theta S_{\phi+\psi} \\ 0 \end{Bmatrix} \dot{\phi} \dot{\theta} \\
&+ \begin{Bmatrix} a C_\theta S_{\phi+\psi} \\ -a C_\theta C_{\phi+\psi} \\ -a S_\theta \end{Bmatrix} \dot{\theta}^2 + \begin{Bmatrix} a S_{\phi+\psi} C_\theta \\ -a C_{\phi+\psi} C_\theta \\ 0 \end{Bmatrix} \dot{\psi}^2 + \begin{Bmatrix} a S_{\phi+\psi} C_\theta \\ -a C_{\phi+\psi} C_\theta \\ 0 \end{Bmatrix} \dot{\phi} \dot{\psi} + \begin{Bmatrix} a S_\theta C_{\phi+\psi} \\ a S_\theta S_{\phi+\psi} \\ 0 \end{Bmatrix} \dot{\psi} \dot{\theta} + \begin{Bmatrix} a S_\theta C_{\phi+\psi} \\ a S_\theta S_{\phi+\psi} \\ 0 \end{Bmatrix} \dot{\phi} \dot{\theta} \\
&\Rightarrow \left[{}^R v_{P, \hat{\omega}_F} \right] \{ \dot{\hat{\omega}}_F \} + \left[{}^R v_{P, \hat{\omega}_B} \right] \{ \dot{\hat{\omega}}_B \} + \left[{}^R \dot{v}_{P, \hat{\omega}_F} \right] \{ \hat{\omega}_F \} + \left[{}^R \dot{v}_{P, \hat{\omega}_B} \right] \{ \hat{\omega}_B \} \\
&= \begin{Bmatrix} -\ell C_\phi - a C_{\phi+\psi} C_\theta \\ -\ell S_\phi - a S_{\phi+\psi} C_\theta \\ 0 \end{Bmatrix} \ddot{\phi} + \begin{Bmatrix} -a C_{\phi+\psi} C_\theta \\ -a S_{\phi+\psi} C_\theta \\ 0 \end{Bmatrix} \ddot{\psi} + \begin{Bmatrix} a S_\theta S_{\phi+\psi} \\ -a S_\theta C_{\phi+\psi} \\ a C_\theta \end{Bmatrix} \ddot{\theta} + \begin{Bmatrix} \ell S_\phi + a S_{\phi+\psi} C_\theta \\ -\ell C_\phi - a C_{\phi+\psi} C_\theta \\ 0 \end{Bmatrix} \dot{\phi}^2 \\
&+ \begin{Bmatrix} a S_{\phi+\psi} C_\theta \\ -a C_{\phi+\psi} C_\theta \\ 0 \end{Bmatrix} \dot{\psi}^2 + \begin{Bmatrix} a C_\theta S_{\phi+\psi} \\ -a C_\theta C_{\phi+\psi} \\ -a S_\theta \end{Bmatrix} \dot{\theta}^2 + \begin{Bmatrix} 2a S_{\phi+\psi} C_\theta \\ -2a C_{\phi+\psi} C_\theta \\ 0 \end{Bmatrix} \dot{\phi} \dot{\psi} + \begin{Bmatrix} 2a S_\theta C_{\phi+\psi} \\ 2a S_\theta S_{\phi+\psi} \\ 0 \end{Bmatrix} \dot{\phi} \dot{\theta} + \begin{Bmatrix} 2a S_\theta C_{\phi+\psi} \\ 2a S_\theta S_{\phi+\psi} \\ 0 \end{Bmatrix} \dot{\psi} \dot{\theta}
\end{aligned} \tag{145}$$

This result **agrees** with results generated using the concept of **relative motion**. It can also be verified using the concept of **direct differentiation**.

b) Results from Unit 4:

As in the previous example, the **base frame** and **body frame components** of ${}^R\omega_F$ are **identical**.

$$\{\hat{\omega}'_F\} = \{\hat{\omega}_F\} = \begin{bmatrix} 0 & 0 & \dot{\phi} \end{bmatrix}^T$$

The angular velocity of B relative to F (3-1 rotation sequence) can be written in the **body frame** as follows.

$${}^F\omega_B = \dot{\theta}\hat{n}_1 + \dot{\psi}\hat{k} = \dot{\theta}\hat{n}_1 + \dot{\psi}(S_\theta\hat{n}_2 + C_\theta\hat{n}_3) \Rightarrow \{\hat{\omega}'_B\} = \begin{bmatrix} \dot{\theta} & \dot{\psi}S_\theta & \dot{\psi}C_\theta \end{bmatrix}^T$$

Using these results, the **body frame components** of the **angular velocities** of the bodies can be written as

$$\{\omega'_F\} = \{\hat{\omega}'_F\} \quad \{\omega'_B\} = [{}^F R_B] \{\hat{\omega}'_F\} + \{\hat{\omega}'_B\}$$

Defining the column vector of **generalized speeds** to be $\{y\} \triangleq \begin{Bmatrix} \{\hat{\omega}'_F\} \\ \{\hat{\omega}'_B\} \end{Bmatrix}$, the **body frame components** of the

partial angular velocity matrices are observed to be

$$\begin{aligned} [{}^R\omega'_{F,y}]_{3 \times 6} &= \begin{bmatrix} [{}^R\omega'_{F,\hat{\omega}'_F}] & [{}^R\omega'_{F,\hat{\omega}'_B}] \end{bmatrix} = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix} \\ [{}^R\omega'_{B,y}]_{3 \times 6} &= \begin{bmatrix} [{}^R\omega'_{B,\hat{\omega}'_F}] & [{}^R\omega'_{B,\hat{\omega}'_B}] \end{bmatrix} = \begin{bmatrix} [{}^F R_B] & [I]_{3 \times 3} \end{bmatrix} \end{aligned}$$

The **body frame components** of ${}^Rv_P$ and the partial velocity matrices are

$$\begin{aligned} \{v'_P\} &= -\left([{}^F R_B][\tilde{r}'_{Q/O}] + [\tilde{r}'_{P/Q}][{}^F R_B]\right)\{\hat{\omega}'_F\} - [\tilde{r}'_{P/Q}]\{\hat{\omega}'_B\} \\ [v'_{P,\hat{\omega}'_F}] &= -[{}^F R_B][\tilde{r}'_{Q/O}] - [\tilde{r}'_{P/Q}][{}^F R_B] \quad [{}^R v'_{P,\hat{\omega}'_B}] = -[\tilde{r}'_{P/Q}] \end{aligned} \tag{146}$$

$$\begin{aligned} [\tilde{r}'_{Q/O}] &= \begin{bmatrix} 0 & 0 & \ell \\ 0 & 0 & 0 \\ -\ell & 0 & 0 \end{bmatrix} \quad [\tilde{r}'_{P/Q}] = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ -a & 0 & 0 \end{bmatrix} \quad [{}^F R_B] = \begin{bmatrix} C_\psi & S_\psi & 0 \\ -S_\psi C_\theta & C_\psi C_\theta & S_\theta \\ S_\psi S_\theta & -C_\psi S_\theta & C_\theta \end{bmatrix} \end{aligned}$$

$$\begin{aligned} [v'_{P,\hat{\omega}'_F}] &= \begin{bmatrix} -aS_\psi S_\theta & aC_\psi S_\theta & -(\ell C_\psi + aC_\theta) \\ \ell S_\theta & 0 & \ell S_\psi C_\theta \\ \ell C_\theta + aC_\psi & aS_\psi & -\ell S_\psi S_\theta \end{bmatrix} \quad [{}^R v'_{P,\hat{\omega}'_B}] = -[\tilde{r}'_{P/Q}] = \begin{bmatrix} 0 & 0 & -a \\ 0 & 0 & 0 \\ a & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\{v'_P\} = \begin{Bmatrix} -(\ell C_\psi + aC_\theta)\dot{\phi} - aC_\theta\dot{\psi} \\ \ell S_\psi C_\theta\dot{\phi} \\ -\ell S_\psi S_\theta\dot{\phi} + a\dot{\theta} \end{Bmatrix}$$

Acceleration:

The **body frame components** of ${}^R a_P$ can be found by applying the **derivative rule** as follows.

$${}^R a_P = \frac{{}^R d}{{}^R dt}({}^R v_P) = \frac{{}^D d}{{}^D dt}({}^R v_P) + {}^R \omega_D \times {}^R v_P$$

Using the **body frame components** of the **angular velocity components**, the **scalar** form of this equation is

$$\begin{aligned} \{a'_P\} &= \{\dot{v}'_P\} + [\tilde{\omega}'_B]\{v'_P\} \\ &= [{}^R v'_{P,\dot{\omega}'_F}]\{\dot{\omega}'_F\} + [{}^R v'_{P,\dot{\omega}'_B}]\{\dot{\omega}'_B\} + [{}^R \dot{v}'_{P,\dot{\omega}'_F}]\{\hat{\omega}'_F\} + [{}^R \dot{v}'_{P,\dot{\omega}'_B}]\{\hat{\omega}'_B\} + [\tilde{\omega}'_B]\{v'_P\} \end{aligned} \quad (147)$$

Here,

$$[{}^R \dot{v}'_{P,\dot{\omega}'_F}] = \begin{bmatrix} -a(C_\psi S_\theta \dot{\psi} + S_\psi C_\theta \dot{\theta}) & a(-S_\psi S_\theta \dot{\psi} + C_\psi C_\theta \dot{\theta}) & (\ell S_\psi \dot{\psi} + a S_\theta \dot{\theta}) \\ \ell C_\theta \dot{\theta} & 0 & \ell(C_\psi C_\theta \dot{\psi} - S_\psi S_\theta \dot{\theta}) \\ -(\ell S_\theta \dot{\theta} + a S_\psi \dot{\psi}) & a C_\psi \dot{\psi} & -\ell(C_\psi S_\theta \dot{\psi} + S_\psi C_\theta \dot{\theta}) \end{bmatrix} \quad [{}^R \dot{v}'_{P,\dot{\omega}'_B}] = [0]_{3 \times 3}$$

$$\{\omega'_B\} = [{}^F R_B]\{\hat{\omega}'_F\} + \{\hat{\omega}'_B\} = \begin{bmatrix} C_\psi & S_\psi & 0 \\ -S_\psi C_\theta & C_\psi C_\theta & S_\theta \\ S_\psi S_\theta & -C_\psi S_\theta & C_\theta \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\phi} \end{Bmatrix} + \begin{Bmatrix} \dot{\theta} \\ \dot{\psi} S_\theta \\ \dot{\psi} C_\theta \end{Bmatrix} = \begin{Bmatrix} \dot{\theta} \\ S_\theta(\dot{\phi} + \dot{\psi}) \\ C_\theta(\dot{\phi} + \dot{\psi}) \end{Bmatrix}$$

$$[\tilde{\omega}'_B] = \begin{bmatrix} 0 & -C_\theta(\dot{\phi} + \dot{\psi}) & S_\theta(\dot{\phi} + \dot{\psi}) \\ C_\theta(\dot{\phi} + \dot{\psi}) & 0 & -\dot{\theta} \\ -S_\theta(\dot{\phi} + \dot{\psi}) & \dot{\theta} & 0 \end{bmatrix}$$

Substituting into Equation (147) gives the **body frame components** of ${}^R a_P$.

$$\begin{aligned} \{a'_P\} &= [{}^R \dot{v}'_{P,\dot{\omega}'_F}]\{\dot{\omega}'_F\} + [{}^R \dot{v}'_{P,\dot{\omega}'_B}]\{\dot{\omega}'_B\} + [{}^R \dot{v}'_{P,\dot{\omega}'_F}]\{\hat{\omega}'_F\} + [{}^R \dot{v}'_{P,\dot{\omega}'_B}]\{\hat{\omega}'_B\} + [\tilde{\omega}'_B]\{v'_P\} \\ &= \begin{bmatrix} -a S_\psi S_\theta & a C_\psi S_\theta & -(\ell C_\psi + a C_\theta) \\ \ell S_\theta & 0 & \ell S_\psi C_\theta \\ \ell C_\theta + a C_\psi & a S_\psi & -\ell S_\psi S_\theta \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \ddot{\phi} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & -a \\ 0 & 0 & 0 \\ a & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{\psi} S_\theta + \dot{\psi} \dot{\theta} C_\theta \\ \ddot{\psi} C_\theta - \dot{\psi} \dot{\theta} S_\theta \end{Bmatrix} \\ &+ \begin{bmatrix} -a(C_\psi S_\theta \dot{\psi} + S_\psi C_\theta \dot{\theta}) & a(-S_\psi S_\theta \dot{\psi} + C_\psi C_\theta \dot{\theta}) & (\ell S_\psi \dot{\psi} + a S_\theta \dot{\theta}) \\ \ell C_\theta \dot{\theta} & 0 & \ell(C_\psi C_\theta \dot{\psi} - S_\psi S_\theta \dot{\theta}) \\ -(\ell S_\theta \dot{\theta} + a S_\psi \dot{\psi}) & a C_\psi \dot{\psi} & -\ell(C_\psi S_\theta \dot{\psi} + S_\psi C_\theta \dot{\theta}) \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\phi} \end{Bmatrix} + \underbrace{[{}^R \dot{v}'_{P,\dot{\omega}'_B}]}_{\text{zero}} \{\hat{\omega}'_B\} \\ &+ \begin{bmatrix} 0 & -C_\theta(\dot{\phi} + \dot{\psi}) & S_\theta(\dot{\phi} + \dot{\psi}) \\ C_\theta(\dot{\phi} + \dot{\psi}) & 0 & -\dot{\theta} \\ -S_\theta(\dot{\phi} + \dot{\psi}) & \dot{\theta} & 0 \end{bmatrix} \begin{Bmatrix} -(\ell C_\psi + a C_\theta)\dot{\phi} - a C_\theta \dot{\psi} \\ \ell S_\psi C_\theta \dot{\phi} \\ -\ell S_\psi S_\theta \dot{\phi} + a \dot{\theta} \end{Bmatrix} \\ &\vdots \end{aligned}$$

$$\begin{aligned}
& \vdots \\
& = \begin{Bmatrix} -(\ell C_\psi + a C_\theta) \\ \ell S_\psi C_\theta \\ -\ell S_\psi S_\theta \end{Bmatrix} \ddot{\phi} + \begin{Bmatrix} -a(\ddot{\psi} C_\theta - \dot{\psi} \dot{\theta} S_\theta) \\ 0 \\ a \ddot{\theta} \end{Bmatrix} + \begin{Bmatrix} \ell S_\psi \dot{\psi} + a S_\theta \dot{\theta} \\ \ell(C_\psi C_\theta \dot{\psi} - S_\psi S_\theta \dot{\theta}) \\ -\ell(C_\psi S_\theta \dot{\psi} + S_\psi C_\theta \dot{\theta}) \end{Bmatrix} \dot{\phi} \\
& + \begin{Bmatrix} -C_\theta(\dot{\phi} + \dot{\psi}) \ell S_\psi C_\theta \dot{\phi} + S_\theta(\dot{\phi} + \dot{\psi})(-\ell S_\psi S_\theta \dot{\phi} + a \dot{\theta}) \\ -C_\theta(\dot{\phi} + \dot{\psi})((\ell C_\psi + a C_\theta) \dot{\phi} + a C_\theta \dot{\psi}) - \dot{\theta}(-\ell S_\psi S_\theta \dot{\phi} + a \dot{\theta}) \\ S_\theta(\dot{\phi} + \dot{\psi})((\ell C_\psi + a C_\theta) \dot{\phi} + a C_\theta \dot{\psi}) + \ell S_\psi C_\theta \dot{\phi} \dot{\theta} \end{Bmatrix} \\
& = \begin{Bmatrix} -(\ell C_\psi + a C_\theta) \\ \ell S_\psi C_\theta \\ -\ell S_\psi S_\theta \end{Bmatrix} \ddot{\phi} + \begin{Bmatrix} -a C_\theta \\ 0 \\ 0 \end{Bmatrix} \ddot{\psi} + \begin{Bmatrix} 0 \\ 0 \\ a \ddot{\theta} \end{Bmatrix} \ddot{\theta} + \begin{Bmatrix} -\ell S_\psi (C_\theta^2 + S_\theta^2) \\ -C_\theta(\ell C_\psi + a C_\theta) \\ S_\theta(\ell C_\psi + a C_\theta) \end{Bmatrix} \dot{\phi}^2 + \begin{Bmatrix} 0 \\ -a C_\theta^2 \\ a S_\theta C_\theta \end{Bmatrix} \dot{\psi}^2 + \begin{Bmatrix} 0 \\ -a \\ 0 \end{Bmatrix} \dot{\theta}^2 \\
& + \begin{Bmatrix} \ell S_\psi - \ell S_\psi (C_\theta^2 + S_\theta^2) \\ \ell C_\psi C_\theta - a C_\theta^2 - C_\theta(\ell C_\psi + a C_\theta) \\ -\ell C_\psi S_\theta + a S_\theta C_\theta + S_\theta(\ell C_\psi + a C_\theta) \end{Bmatrix} \dot{\phi} \dot{\psi} + \begin{Bmatrix} 2a S_\theta \\ -\ell S_\psi S_\theta + \ell S_\psi S_\theta \\ -\ell S_\psi C_\theta + \ell S_\psi C_\theta \end{Bmatrix} \dot{\phi} \dot{\theta} + \begin{Bmatrix} 2a S_\theta \\ 0 \\ 0 \end{Bmatrix} \dot{\psi} \dot{\theta}
\end{aligned}$$

Simplifying further gives the result.

$$\begin{aligned}
\{a'_P\} = & \begin{Bmatrix} -(\ell C_\psi + a C_\theta) \\ \ell S_\psi C_\theta \\ -\ell S_\psi S_\theta \end{Bmatrix} \ddot{\phi} + \begin{Bmatrix} -a C_\theta \\ 0 \\ 0 \end{Bmatrix} \ddot{\psi} + \begin{Bmatrix} 0 \\ 0 \\ a \end{Bmatrix} \ddot{\theta} + \begin{Bmatrix} -\ell S_\psi \\ -C_\theta(\ell C_\psi + a C_\theta) \\ S_\theta(\ell C_\psi + a C_\theta) \end{Bmatrix} \dot{\phi}^2 + \begin{Bmatrix} 0 \\ -a C_\theta^2 \\ a S_\theta C_\theta \end{Bmatrix} \dot{\psi}^2 + \begin{Bmatrix} 0 \\ -a \\ 0 \end{Bmatrix} \dot{\theta}^2 \\
& + \begin{Bmatrix} 0 \\ -2a C_\theta^2 \\ 2a S_\theta C_\theta \end{Bmatrix} \dot{\phi} \dot{\psi} + \begin{Bmatrix} 2a S_\theta \\ 0 \\ 0 \end{Bmatrix} \dot{\phi} \dot{\theta} + \begin{Bmatrix} 2a S_\theta \\ 0 \\ 0 \end{Bmatrix} \dot{\psi} \dot{\theta}
\end{aligned} \tag{148}$$

This result **agrees** with results generated using the concept of **relative motion**.

Note that the **time derivative** of the **partial velocity matrix** ${}^R \mathbf{v}'_{P, \hat{\omega}'_F}$ can also be calculated by **differentiating** the **expression** in Equation (146). That is,

$$\begin{aligned}
[\dot{\mathbf{v}}'_{P, \hat{\omega}'_F}] &= -[{}^F \dot{\mathbf{R}}_B][\tilde{\mathbf{r}}'_{Q/O}] - [\tilde{\mathbf{r}}'_{P/Q}][{}^F \dot{\mathbf{R}}_B] \\
&= -[{}^F \tilde{\boldsymbol{\omega}}'_B]^T [{}^F \mathbf{R}_B][\tilde{\mathbf{r}}'_{Q/O}] - [\tilde{\mathbf{r}}'_{P/Q}][{}^F \tilde{\boldsymbol{\omega}}'_B]^T [{}^F \mathbf{R}_B]
\end{aligned} \tag{149}$$

Here,

$$[{}^F \tilde{\boldsymbol{\omega}}'_B] = \begin{bmatrix} 0 & -\dot{\psi} C_\theta & \dot{\psi} S_\theta \\ \dot{\psi} C_\theta & 0 & -\dot{\theta} \\ -\dot{\psi} S_\theta & \dot{\theta} & 0 \end{bmatrix}$$

Expanding Equation (149) gives the **same result** as that found above by **directly differentiating** its elements.

Example 3

Write MATLAB scripts to *numerically* evaluate the *matrix equations* you derived above in Example 2 using the data below. First calculate the *partial angular velocity matrices* and the *angular velocity components*. Then calculate the *partial velocity matrices* and the *velocity components* of point *P*. Use the results of Unit 4 as a basis for these calculations. Finally, calculate the *time derivatives* of the *partial velocity matrices* and the *acceleration components* of point *P*.

Data:

$$\begin{array}{lll} \ell = 19 \text{ (in)} & a = 11 \text{ (in)} & \\ \phi = 0 & \psi = 20 \text{ (deg)} & \theta = 30 \text{ (deg)} \\ \dot{\phi} = 1.5 \text{ (rad/s)} & \dot{\psi} = 2 \text{ (rad/s)} & \dot{\theta} = -0.5 \text{ (rad/s)} \\ \ddot{\phi} = 3 \text{ (rad/s}^2\text{)} & \ddot{\psi} = 5 \text{ (rad/s}^2\text{)} & \ddot{\theta} = -1.5 \text{ (rad/s}^2\text{)} \end{array}$$

Solution:

The MATLAB scripts are shown in the highlighted sections below. The scripts use procedures presented in Units 2, 4, and 5. They calculate some results *twice* using different methods to provide a *check* on those results.

The scripts use the following three functions

1. calculateTransformationMatrixFromAngleX
2. calculateTransformationMatrixFromAngleZ
3. buildTildeMatrix

The first two functions construct *transformation matrices* associated with single angle rotations about the X and Z axes, respectively. The matrices were first presented in Unit 5 of Volume I. The third function accepts a 3×1 vector and constructs a 3×3 *skew-symmetric matrix* for computing *vector cross products*. This function was first discussed in Exercise 1.1 of Unit 1 of this volume. The *contents* of the *three functions* are shown in *Addendum B*.

MATLAB Script and Results for Example 3(a): (Equations of Example 2(a))

```
%  
% Solution for Volume III, Unit 5, Example 3(a) - Rotating Arm with 2 DOF  
% joint  
%  
% Calculation of fixed frame components of partial angular velocity,  
% angular velocity, partial velocity, velocity,  
% partial velocity derivatives, acceleration of point P  
%  
% These calculations follow the equations developed in Unit 5, Example 2  
% which uses base frame components of relative angular velocity vectors  
%  
clearvars  
  
% Lengths and Position Vectors  
lengthOQ = 19/12; % (feet)  
lengthQP = 11/12; % (feet)
```

```

% Position vectors in local coordinates
positionVectorQwrtOPrime = [0; lengthOQ; 0]; % Coordinates in frame F
positionVectorPwrtQPrime = [0; lengthQP; 0]; % Coordinates in frame B

% Angles and their derivatives
phiDeg = 0.0; % angle phi in degrees
psiDeg = 20.0; % angle psi in degrees
thetaDeg = 30.0; % angle theta in degrees

phiDot = 1.5; % phi dot in rad/s
psiDot = 2.0; % psi dot in rad/s
thetaDot = -0.5; % theta dot in rad/s

phiDoubleDot = 3.0; % phi double dot in rad/s^2
psiDoubleDot = 5.0; % psi double dot in rad/s^2
thetaDoubleDot = -1.5; % theta double dot in rad/s^2

% Convert to radians and calculate trigonometric functions
phiRad = deg2rad(phiDeg);
psiRad = deg2rad(psiDeg);
thetaRad = deg2rad(thetaDeg);

sinPhi = sin(phiRad); cosPhi = cos(phiRad);
sinPsi = sin(psiRad); cosPsi = cos(psiRad);
sinTheta = sin(thetaRad); cosTheta = cos(thetaRad);

sinPhiPlusPsi = sin(phiRad+psiRad);
cosPhiPlusPsi = cos(phiRad+psiRad);

% Transformation matrices - FF is the fixed frame
transformationMatrixFwrtFF = ...
    calculateTransformationMatrixFromAngleZ(phiRad);
transformationMatrixBwrtF = ...
    calculateTransformationMatrixFromAngleX(thetaRad)* ...
    calculateTransformationMatrixFromAngleZ(psiRad);
transformationMatrixBwrtFF = ...
    transformationMatrixBwrtF*transformationMatrixFwrtFF;

% Check of transformation matrix of B with respect to F
transformationMatrixBwrtFFCheck = ...
    [cosPsi, sinPsi, 0; ...
    -sinPsi*cosTheta, cosPsi*cosTheta, sinTheta; ...
    sinPsi*sinTheta, -cosPsi*sinTheta, cosTheta];

transformationMatrixBwrtFFCheck = ...
    [cosPhiPlusPsi, sinPhiPlusPsi, 0; ...
    -sinPhiPlusPsi*cosTheta, cosPhiPlusPsi*cosTheta, sinTheta; ...
    sinPhiPlusPsi*sinTheta, -cosPhiPlusPsi*sinTheta, cosTheta];

% Using base frame components of relative angular velocity vectors
% build the state vector y.
omegaHatF = [0; 0; phiDot];
omegaHatB = [thetaDot*cosPsi; thetaDot*sinPsi; psiDot];

y = [omegaHatF; omegaHatB]; % 6x1 state vector

```

```

% Build yDot, the time derivative of the state vector y
omegaHatFDot = [0; 0; phiDoubleDot];
omegaHatBDot = ...
    [(thetaDoubleDot*cosPsi) - (thetaDot*sinPsi*psiDot); ...
    (thetaDoubleDot*sinPsi) + (thetaDot*cosPsi*psiDot); ...
    psiDoubleDot];

yDot = [omegaHatFDot;omegaHatBDot];

% Partial Angular Velocity Matrices - Fixed frame components
% (as given in Example 2 of Unit 4)
partialOmegaFwrtY = zeros(3,6);
partialOmegaBwrtY = zeros(3,6);

partialOmegaFwrtY(:,1:3) = eye(3,3);

partialOmegaBwrtY(:,1:3) = eye(3,3);
partialOmegaBwrtY(:,4:6) = transpose(transformationMatrixFwrtFF);

% Display header
disp(' ')
disp('-----')
disp('                Results for Example 3(a)                ')
disp('-----')
disp(' ')

% Display the partial angular velocity matrices of the bodies
disp(' ');
disp('Partial angular velocity matrix of body F with respect to')
disp(' state vector y (unitless) - Fixed frame components')
disp(' ');
disp(partialOmegaFwrtY);

disp(' ');
disp('Partial angular velocity matrix of body B with respect to')
disp(' state vector y (unitless) - Fixed frame components')
disp(' ');
disp(partialOmegaBwrtY);

% Angular Velocity Vectors - Fixed Frame Components
omegaF = partialOmegaFwrtY*y;
omegaB = partialOmegaBwrtY*y;

% Check for omegaB
omegaBCheck = omegaHatF + ...
    (transpose(transformationMatrixFwrtFF)*omegaHatB);

% Display fixed frame components of the angular velocities of the bodies
disp(' ');
disp('Angular velocity of body F (rad/s) - Fixed frame components')
disp(' ');
disp(omegaF);

disp(' ');
disp('Angular velocity of body B (rad/s) - Fixed frame components')
disp(' ');
disp(omegaB);

```

```

% Display the fixed frame components of the angular velocity of body B - Check
disp(' ');
disp('Fixed frame components of the angular velocity of body B (rad/s) - Check')
disp(' ');
disp(omegaBCheck);

% Partial Velocity Matrix of Point P - Fixed frame Components
% (as calculated in Example 2 of Unit 4)
partialVelocityPwrtY = zeros(3,6);

% Fixed frame components of relative position vectors
positionVectorQwrtO = ...
    transpose(transformationMatrixFwrtFF)*positionVectorQwrtOPrime;
positionVectorPwrtQ = ...
    transpose(transformationMatrixBwrtFF)*positionVectorPwrtQPrime;

% Build skew symmetric matrices using fixed frame components
rTildeQwrtO = buildTildeMatrix(positionVectorQwrtO);
rTildePwrtQ = buildTildeMatrix(positionVectorPwrtQ);

% Calculate fixed frame components of partial velocity matrix
partialVelocityPwrtY(:,1:3) = -rTildeQwrtO - rTildePwrtQ;
partialVelocityPwrtY(:,4:6) = ...
    -rTildePwrtQ*transpose(transformationMatrixFwrtFF);

% Display the partial velocity matrix for Point P
% (Fixed frame components)
disp(' ');
disp('Fixed frame components of the partial velocity matrix of point P')
disp(' with respect to state vector y (ft/s)')
disp(' ');
disp(partialVelocityPwrtY);

% Velocity of Point P - Fixed Frame Components
velocityP = partialVelocityPwrtY*y;

% Display the Velocity of Point P - Fixed frame Components
disp(' ');
disp('Fixed frame components of the velocity of point P (ft/s)')
disp(' ');
disp(velocityP);

% Velocity of Point P - Fixed frame components - Check
v1 = (-lengthOQ*phiDot*cosPhi) + ...
    ( lengthQP*thetaDot*sinPhiPlusPsi*sinTheta) + ...
    (-(phiDot+psiDot)*lengthQP*cosPhiPlusPsi*cosTheta);

v2 = (-lengthOQ*phiDot*sinPhi) + ...
    (-lengthQP*thetaDot*cosPhiPlusPsi*sinTheta) + ...
    (-(phiDot+psiDot)*lengthQP*sinPhiPlusPsi*cosTheta);

v3 = lengthQP*thetaDot*cosTheta;

velocityPCheck = [v1; v2; v3];

% Display the velocity of point P - Fixed frame components - Check
disp(' ');
disp('Velocity of point P (ft/s) - Fixed frame components - Check')

```

```

disp(' ');
disp(velocityPCheck);

% Time Derivatives of Partial Velocity Matrix of Point P
% (Fixed Frame Components - as calculated in Example 2 of Unit 5)
omegaTildeFwrtFF = buildTildeMatrix(omegaF);
positionVectorQwrtODot = omegaTildeFwrtFF*positionVectorQwrtO;
rTildeQwrtODot = buildTildeMatrix(positionVectorQwrtODot);

omegaTildeBwrtFF = buildTildeMatrix(omegaB);
positionVectorPwrtQDot = omegaTildeBwrtFF*positionVectorPwrtQ;
rTildePwrtQDot = buildTildeMatrix(positionVectorPwrtQDot);

timeDerivativePartialVelocityPwrtY(:,1:3) = ...
    -rTildeQwrtODot - rTildePwrtQDot;
timeDerivativePartialVelocityPwrtY(:,4:6) = ...
    -(rTildePwrtQDot+(rTildePwrtQ*omegaTildeFwrtFF))* ...
    transpose(transformationMatrixFwrtFF);

% Display the Time Derivative of Partial Velocity Matrix of Point P
% (Fixed Frame Components)
disp(' ');
disp('Time derivative of the partial velocity matrix of point P with')
disp(' respect to state vector y (ft/s) - Fixed frame components')
disp(' ');
disp(timeDerivativePartialVelocityPwrtY);

% Acceleration of Point P - Fixed Frame Components
accelerationP = (partialVelocityPwrtY*yDot) + ...
    (timeDerivativePartialVelocityPwrtY*y);

% Display the Acceleration of Point P - Fixed Frame Components
disp(' ');
disp('Acceleration of point P (ft/s^2) - Fixed frame components')
disp(' ');
disp(accelerationP);

% Acceleration of Point P - Fixed Frame Components - Check
% (results from a relative motion analysis)
tempVector = zeros(3,1);

tempVector(1,1) = -(lengthOQ*cosPhi) - (lengthQP*cosTheta*cosPhiPlusPsi);
tempVector(2,1) = -(lengthOQ*sinPhi) - (lengthQP*cosTheta*sinPhiPlusPsi);
tempVector(3,1) = 0;
accelerationPCheck = tempVector.*phiDoubleDot;

tempVector(1,1) = -lengthQP*cosPhiPlusPsi*cosTheta;
tempVector(2,1) = -lengthQP*sinPhiPlusPsi*cosTheta;
tempVector(3,1) = 0;
accelerationPCheck = accelerationPCheck + (tempVector.*psiDoubleDot);

tempVector(1,1) = lengthQP*sinPhiPlusPsi*sinTheta;
tempVector(2,1) = -lengthQP*cosPhiPlusPsi*sinTheta;
tempVector(3,1) = lengthQP*cosTheta;
accelerationPCheck = accelerationPCheck + (tempVector.*thetaDoubleDot);

tempVector(1,1) = (lengthOQ*sinPhi) + (lengthQP*sinPhiPlusPsi*cosTheta);
tempVector(2,1) = (-lengthOQ*cosPhi) - (lengthQP*cosPhiPlusPsi*cosTheta);

```

```

tempVector(3,1) = 0;
accelerationPCheck = accelerationPCheck + (tempVector.*(phiDot*phiDot));

tempVector(1,1) = lengthQP*sinPhiPlusPsi*cosTheta;
tempVector(2,1) = -lengthQP*cosPhiPlusPsi*cosTheta;
tempVector(3,1) = 0;
accelerationPCheck = accelerationPCheck + (tempVector.*(psiDot*psiDot));

tempVector(1,1) = lengthQP*sinPhiPlusPsi*cosTheta;
tempVector(2,1) = -lengthQP*cosPhiPlusPsi*cosTheta;
tempVector(3,1) = -lengthQP*sinTheta;
accelerationPCheck = accelerationPCheck + (tempVector.*(thetaDot*thetaDot));

tempVector(1,1) = 2*lengthQP*sinPhiPlusPsi*cosTheta;
tempVector(2,1) = -2*lengthQP*cosPhiPlusPsi*cosTheta;
tempVector(3,1) = 0;
accelerationPCheck = accelerationPCheck + (tempVector.*(phiDot*psiDot));

tempVector(1,1) = 2*lengthQP*cosPhiPlusPsi*sinTheta;
tempVector(2,1) = 2*lengthQP*sinPhiPlusPsi*sinTheta;
tempVector(3,1) = 0;
accelerationPCheck = accelerationPCheck + (tempVector.*(phiDot*thetaDot));

tempVector(1,1) = 2*lengthQP*cosPhiPlusPsi*sinTheta;
tempVector(2,1) = 2*lengthQP*sinPhiPlusPsi*sinTheta;
tempVector(3,1) = 0;
accelerationPCheck = accelerationPCheck + (tempVector.*(psiDot*thetaDot));

% Display the Acceleration of Point P - Fixed Frame Components - Check
% (results from relative motion analysis)
disp(' ');
disp('Acceleration of point P (ft/s^2) - Fixed frame Components - Check')
disp(' ');
disp(accelerationPCheck);

```

Results for Example 3(a)

Partial angular velocity matrix of body F with respect to
state vector y (unitless) - Fixed frame components

1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0

Partial angular velocity matrix of body B with respect to
state vector y (unitless) - Fixed frame components

1	0	0	1	0	0
0	1	0	0	1	0
0	0	1	0	0	1

Angular velocity of body F (rad/s) - Fixed frame components

0
0
1.5000000000000000

Angular velocity of body B (rad/s) - Fixed frame components

-0.469846310392954
-0.171010071662834
3.5000000000000000

Fixed frame components of the angular velocity of body B (rad/s) - Check

-0.469846310392954
-0.171010071662834
3.500000000000000

Fixed frame components of the partial velocity matrix of point P
with respect to state vector y (ft)

Columns 1 through 3

0	0.458333333333333	-2.329314541236926
-0.458333333333333	0	-0.271514954998855
2.329314541236926	0.271514954998855	0

Columns 4 through 6

0	0.458333333333333	-0.745981207903593
-0.458333333333333	0	-0.271514954998855
0.745981207903593	0.271514954998855	0

Fixed frame components of the velocity of point P (ft/s)

-5.064313843841372
-0.734956116899223
-0.396928310067868

Velocity of point P (ft/s) - Fixed frame components - Check

-5.064313843841373
-0.734956116899222
-0.396928310067868

Time derivative of the partial velocity matrix of point P with
respect to state vector y (ft/s) - Fixed frame components

Columns 1 through 3

0	-0.396928310067868	0.734956116899223
0.396928310067868	0	-5.064313843841373
-0.734956116899223	5.064313843841373	0

Columns 4 through 6

0.687500000000000	-0.396928310067868	0.734956116899223
0.396928310067868	0.687500000000000	-2.689313843841373
-0.327683684400940	1.570342031985984	0

Acceleration of point P (ft/s²) - Fixed frame components

-9.066475153456842
-14.962003375247033
-1.305368263536937

Acceleration of point P (ft/s²) - Fixed frame Components - Check

-9.066475153456842
-14.962003375247029
-1.305368263536936

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MATLAB Script and Results for Example 3(b): (Equations of Example 2(b))

```
%
% Solution for Volume III, Unit 5, Example 3(b) - Rotating Arm with 2 DOF
% joint
%
% Calculation of body frame components of partial angular velocity,
% angular velocity, partial velocity, velocity,
% partial velocity derivatives, acceleration of point P
%
% These calculations follow the equations developed in Unit 5, Example 2
% which uses body frame components of relative angular velocity vectors;
%
clearvars

% Lengths and Position Vectors
lengthOQ = 19/12;    % (L, feet)
lengthQP = 11/12;    % (a, feet)

% Position vectors in local coordinates
positionVectorQwrtOPrime = [0; lengthOQ; 0]; % Coordinates in frame F
positionVectorPwrtQPrime = [0; lengthQP; 0]; % Coordinates in frame B

% Angles and their derivatives
phiDeg = 0.0; % angle phi in degrees
psiDeg = 20.0; % angle psi in degrees
thetaDeg = 30.0; % angle theta in degrees

phiDot = 1.5; % phi dot in rad/s
psiDot = 2.0; % psi dot in rad/s
thetaDot = -0.5; % theta dot in rad/s

phiDoubleDot = 3.0; % phi double dot in rad/s^2
psiDoubleDot = 5.0; % psi double dot in rad/s^2
thetaDoubleDot = -1.5; % theta double dot in rad/s^2

% Convert to radians and calculate trigonometric functions
phiRad = deg2rad(phiDeg);
psiRad = deg2rad(psiDeg);
thetaRad = deg2rad(thetaDeg);

sinPhi = sin(phiRad);    cosPhi = cos(phiRad);
sinPsi = sin(psiRad);    cosPsi = cos(psiRad);
sinTheta = sin(thetaRad);    cosTheta = cos(thetaRad);

sinPhiPlusPsi = sin(phiRad+psiRad);
cosPhiPlusPsi = cos(phiRad+psiRad);

% Transformation matrices - Bar B w.r.t. Frame F
transformationMatrixBwrtF = ...
    calculateTransformationMatrixFromAngleX(thetaRad)* ...
    calculateTransformationMatrixFromAngleZ(psiRad);

% Check of transformation matrix of B with respect to F
transformationMatrixBwrtFCheck = ...
    [cosPsi, sinPsi, 0; ...
    -sinPsi*cosTheta, cosPsi*cosTheta, sinTheta; ...
    sinPsi*sinTheta, -cosPsi*sinTheta, cosTheta];
```

```

% Using body frame components of relative angular velocity
% components build the state vector y.
omegaHatFPrime = [0; 0; phiDot];
omegaHatBPrime = [thetaDot; psiDot*sinTheta; psiDot*cosTheta];

yPrime = [omegaHatFPrime; omegaHatBPrime];    % 6x1 state vector

% Build yDot, the time derivative of the state vector y
omegaHatFPrimeDot = [0; 0; phiDoubleDot];
omegaHatBPrimeDot = ...
    [thetaDoubleDot; ...
     (psiDoubleDot*sinTheta) + (psiDot*thetaDot*cosTheta); ...
     (psiDoubleDot*cosTheta) - (psiDot*thetaDot*sinTheta)];

yPrimeDot = [omegaHatFPrimeDot; omegaHatBPrimeDot];

% Partial Angular Velocity Matrices - Body frame components
% (as calculated in Unit 4)
partialOmegaFPrimewrtYPrime = zeros(3,6);
partialOmegaBPrimewrtYPrime = zeros(3,6);

partialOmegaFPrimewrtYPrime(:,1:3) = eye(3,3);

partialOmegaBPrimewrtYPrime(:,1:3) = transformationMatrixBwrtF;
partialOmegaBPrimewrtYPrime(:,4:6) = eye(3,3);

% Display header
disp(' ')
disp('-----')
disp('                Results for Example 3(b)                ')
disp('-----')
disp(' ')

% Display the partial angular velocity matrices of the bodies
disp(' ');
disp('Partial angular velocity matrix of body F with respect to')
disp(' state vector y (unitless) - Body frame components')
disp(' ');
disp(partialOmegaFPrimewrtYPrime);

disp(' ');
disp('Partial angular velocity matrix of body B with respect to')
disp(' state vector y (unitless) - Body frame components')
disp(' ');
disp(partialOmegaBPrimewrtYPrime);

% Angular Velocity Vectors - Body Frame Components
omegaFPrime = partialOmegaFPrimewrtYPrime*yPrime;
omegaBPrime = partialOmegaBPrimewrtYPrime*yPrime;

% Check for omegaBPrime
omegaBPrimeCheck = (transformationMatrixBwrtF*omegaHatFPrime) + ...
    (omegaHatBPrime);

% Display the body frame components of the angular velocities of the bodies
disp(' ');
disp('Angular velocity of body F (rad/s) - Body frame components')

```

```

disp(' ');
disp(omegaFPrime);

disp(' ');
disp('Angular velocity of body B (rad/s) - Body frame components')
disp(' ');
disp(omegaBPrime);

% Display the body frame components of the angular velocity of body B - Check
disp(' ');
disp('Body frame components of the angular velocity of body B (rad/s) - Check')
disp(' ');
disp(omegaBPrimeCheck);

% Partial Velocity Matrix of Point P - Body frame Components
% (as calculated in Example 2 of Unit 4)
partialVelocityPPrimewrtYPrime = zeros(3,6);

rTildeQwrtOPrime = buildTildeMatrix(positionVectorQwrtOPrime);
rTildePwrtQPrime = buildTildeMatrix(positionVectorPwrtQPrime);

partialVelocityPPrimewrtYPrime(:,1:3) = ...
    -(transformationMatrixBwrtF*rTildeQwrtOPrime) ...
    -(rTildePwrtQPrime*transformationMatrixBwrtF);

partialVelocityPPrimewrtYPrime(:,4:6) = -rTildePwrtQPrime;

% Display the partial velocity matrix for Point P
% (Body frame components)
disp(' ');
disp('Body frame components of the partial velocity matrix of point P')
disp(' with respect to state vector y (ft)')
disp(' ');
disp(partialVelocityPPrimewrtYPrime);

% Velocity of Point P - Body Frame Components
velocityPPrime = partialVelocityPPrimewrtYPrime*yPrime;

% Display the Velocity of Point P - Body frame Components
disp(' ');
disp('Body frame components of the velocity of point P (ft/s)')
disp(' ');
disp(velocityPPrime);

% Velocity of Point P - Body frame components - Check
v1 = -(lengthOQ*cosPsi*phiDot) - (lengthQP*cosTheta*phiDot) ...
    - (lengthQP*cosTheta*psiDot);

v2 = lengthOQ*sinPsi*cosTheta*phiDot;

v3 = (-lengthOQ*sinPsi*sinTheta*phiDot) + (lengthQP*thetaDot);

velocityPPrimeCheck = [v1; v2; v3];

% Display the velocity of point P - Body frame components - Check
disp(' ');
disp('Velocity of point P (ft/s) - Body frame components - Check')
disp(' ');

```

```

disp(velocityPPrimeCheck);

% Time Derivative of Partial Velocity Matrix of Point P
%   (Body Frame Components - as calculated in Example 2 of Unit 5)

timeDerivativePartialVelocityPPrimewrtYPrime = zeros(3,6);

omegaTildeBwrtFPrime = buildTildeMatrix(omegaHatBPrime);
tempMatrix = transpose(omegaTildeBwrtFPrime);

timeDerivativePartialVelocityPPrimewrtYPrime(:,1:3) = ...
    - (tempMatrix*transformationMatrixBwrtF*rTildeQwrtOPrime) - ...
    (rTildePwrtQPrime*tempMatrix*transformationMatrixBwrtF);

% Display the Time Derivative of Partial Velocity Matrix of Point P
%   (Body Frame Components)
disp(' ');
disp('Time derivative of the partial velocity matrix of point P with')
disp(' respect to state vector y (ft/s) - Body frame components')
disp(' ');
disp(timeDerivativePartialVelocityPPrimewrtYPrime);

% Acceleration of Point P - Body Frame Components

omegaTildeBwrtFFPrime = buildTildeMatrix(omegaBPrime);

accelerationPPrime = (partialVelocityPPrimewrtYPrime*yPrimeDot) + ...
    (timeDerivativePartialVelocityPPrimewrtYPrime*yPrime) + ...
    (omegaTildeBwrtFFPrime*velocityPPrime);

% Display the Acceleration of Point P - Body Frame Components
disp(' ');
disp('Acceleration of point P (ft/s^2) - Body frame components')
disp(' ');
disp(accelerationPPrime);

% Acceleration of Point P - Body Frame Components - Check
%   (results from relative motion analysis)
tempVector = zeros(3,1);

tempVector(1,1) = -(lengthOQ*cosPsi) - (lengthQP*cosTheta);
tempVector(2,1) = lengthOQ*sinPsi*cosTheta;
tempVector(3,1) = -lengthOQ*sinPsi*sinTheta;
accelerationPPrimeCheck = tempVector.*phiDoubleDot;

tempVector(1,1) = -lengthQP*cosTheta;
tempVector(2,1) = 0;
tempVector(3,1) = 0;
accelerationPPrimeCheck = accelerationPPrimeCheck + (tempVector.*psiDoubleDot);

tempVector(1,1) = 0;
tempVector(2,1) = 0;
tempVector(3,1) = lengthQP;
accelerationPPrimeCheck = accelerationPPrimeCheck + (tempVector.*thetaDoubleDot);

tempVector(1,1) = -lengthOQ*sinPsi;
tempVector(2,1) = (-lengthOQ*cosPsi*cosTheta) - (lengthQP*cosTheta*cosTheta);
tempVector(3,1) = (lengthOQ*cosPsi*sinTheta) + (lengthQP*sinTheta*cosTheta);

```

```

accelerationPPrimeCheck = accelerationPPrimeCheck + (tempVector.*(phiDot*phiDot));

tempVector(1,1) = 0;
tempVector(2,1) = -lengthQP*cosTheta*cosTheta;
tempVector(3,1) = lengthQP*sinTheta*cosTheta;
accelerationPPrimeCheck = accelerationPPrimeCheck + (tempVector.*(psiDot*psiDot));

tempVector(1,1) = 0;
tempVector(2,1) = -lengthQP;
tempVector(3,1) = 0;
accelerationPPrimeCheck = accelerationPPrimeCheck + (tempVector.*(thetaDot*thetaDot));

tempVector(1,1) = 0;
tempVector(2,1) = -2*lengthQP*cosTheta*cosTheta;
tempVector(3,1) = 2*lengthQP*sinTheta*cosTheta;
accelerationPPrimeCheck = accelerationPPrimeCheck + (tempVector.*(phiDot*psiDot));

tempVector(1,1) = 2*lengthQP*sinTheta;
tempVector(2,1) = 0;
tempVector(3,1) = 0;
accelerationPPrimeCheck = accelerationPPrimeCheck + (tempVector.*(phiDot*thetaDot));

tempVector(1,1) = 2*lengthQP*sinTheta;
tempVector(2,1) = 0;
tempVector(3,1) = 0;
accelerationPPrimeCheck = accelerationPPrimeCheck + (tempVector.*(psiDot*thetaDot));

% Display the Acceleration of Point P - Body Frame Components - Check
% (results from relative motion analysis)
disp(' ');
disp('Acceleration of point P (ft/s^2) - Body frame Components - Check')
disp(' ');
disp(accelerationPPrimeCheck);

```

Results for Example 3(b)

Partial angular velocity matrix of body F with respect to
state vector y (unitless) - Body frame components

1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0

Partial angular velocity matrix of body B with respect to
state vector y (unitless) - Body frame components

Columns 1 through 3

0.939692620785908	0.342020143325669	0
-0.296198132726024	0.813797681349374	0.500000000000000
0.171010071662834	-0.469846310392954	0.866025403784439

Columns 4 through 6

1.000000000000000	0	0
0	1.000000000000000	0
0	0	1.000000000000000

Angular velocity of body F (rad/s) - Body frame components

0
0
1.500000000000000

Angular velocity of body B (rad/s) - Body frame components

-0.500000000000000
1.750000000000000
3.031088913245536

Body frame components of the angular velocity of body B (rad/s) - Check

-0.500000000000000
1.750000000000000
3.031088913245536

Body frame components of the partial velocity matrix of point P
with respect to state vector y (ft)

Columns 1 through 3

-0.156759232357598	0.430692451193541	-2.281703269713424
0.791666666666667	0	0.468980376816204
2.232591791712444	0.313518464715196	-0.270765946799488

Columns 4 through 6

0	0	-0.916666666666667
0	0	0
0.916666666666667	0	0

Body frame components of the velocity of point P (ft/s)

-5.010268144841607
0.703470565224307
-0.864482253532565

Velocity of point P (ft/s) - Body frame components - Check

-5.010268144841606
0.703470565224307
-0.864482253532565

Time derivative of the partial velocity matrix of point P with
respect to state vector y (ft/s) - Body frame components

Columns 1 through 3

-0.725627424887655	-0.686509068666993	0.853897120531285
-0.685603444662681	0	2.712408964339427
-0.231203596097059	1.722769804774166	-1.253356461169586

Columns 4 through 6

0	0	0
0	0	0
0	0	0

Acceleration of point P (ft/s^2) - Body frame components

-13.637006337083310
-10.143254776025202
4.348901438707816

Acceleration of point P (ft/s^2) - Body frame Components - Check

-13.637006337083310
-10.143254776025199
4.348901438707815

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Example 4

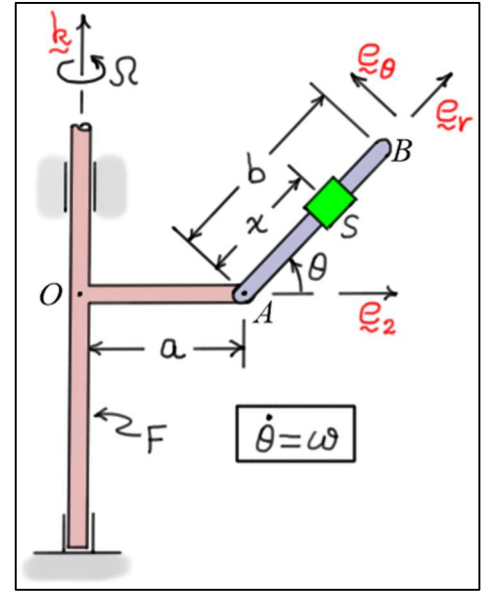
The figure shows a **three-body** system with **frame** F , **arm** AB , and **slider** S . Frame F rotates about the **fixed vertical direction** with rate $\dot{\phi} = \Omega$, and the arm AB rotates **relative** to F with rate $\dot{\theta} = \omega$. Slider S is connected to bar AB with a **prismatic joint**, meaning it is free slide along AB but not rotate relative to it. Its **position** relative to point A is given by the **variable distance** x .

Reference frames: (all frames align when $\phi = \theta = 0$)

$R : (\underline{i}, \underline{j}, \underline{k})$ (fixed frame)

$F : (\underline{e}_1, \underline{e}_2, \underline{k})$ (rotating with frame F)

$AB : (\underline{e}_1, \underline{e}_r, \underline{e}_\theta)$ (rotating with bar AB)



Let $\{\hat{\omega}_F\}$ and $\{\hat{\omega}_{AB}\}$ be the **base frame components** of the **relative angular velocities** ${}^R\omega_F$ and ${}^F\omega_{AB}$. Find the **fixed frame components** of the **time derivatives** of the **partial velocity matrices** and **acceleration** of slider S using **base frame** components or the relative angular velocity vectors. Use results from Example 3 of Unit 4 for the angular velocities, partial angular velocity matrices, velocities, and partial velocity matrices.

Solution:

Results from Unit 4:

The **base frame components** of the **relative angular velocities** can be written as follows.

$$\{\hat{\omega}_F\} = [0 \quad 0 \quad \Omega]^T \quad \{\hat{\omega}_{AB}\} = \{\hat{\omega}_S\} = [\omega \quad 0 \quad 0]^T$$

The **fixed frame components** of the **angular velocities** of the bodies are

$$\{\omega_F\} = \{\hat{\omega}_F\} \quad \{\omega_S\} = \{\omega_{AB}\} = \{\hat{\omega}_F\} + [R_F]^T \{\hat{\omega}_S\}$$

The **body frame components** of the **translation vector** $\{x'\}$ are

$$\{x'\} = [0 \quad x \quad 0]^T$$

Defining the column vector of **generalized speeds** to be $\{y\} \triangleq \begin{Bmatrix} \{\hat{\omega}_F\} \\ \{\hat{\omega}_S\} \\ \{\dot{x}'\} \end{Bmatrix}$, the **fixed frame components** of the

partial angular velocity matrices are observed to be

$$\begin{aligned} [{}^R\omega_{F,y}]_{3 \times 9} &= \begin{bmatrix} [{}^R\omega_{F,\hat{\omega}_F}] & [{}^R\omega_{F,\hat{\omega}_S}] & [{}^R\omega_{F,\dot{x}'}] \end{bmatrix} = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix} \\ [{}^R\omega_{S,y}]_{3 \times 9} &= \begin{bmatrix} [{}^R\omega_{S,\hat{\omega}_F}] & [{}^R\omega_{S,\hat{\omega}_S}] & [{}^R\omega_{S,\dot{x}'}] \end{bmatrix} = \begin{bmatrix} [I]_{3 \times 3} & [R_F]^T & [0]_{3 \times 3} \end{bmatrix} \end{aligned}$$

The **fixed frame components** of the **velocity** of S can be written as follows.

$$\{v_S\} = [{}^R v_{S,\hat{\omega}_F}] \{\hat{\omega}_F\} + [{}^R v_{S,\hat{\omega}_S}] \{\hat{\omega}_S\} + [{}^R v_{S,\dot{x}'}] \{\dot{x}'\} \quad (150)$$

The **fixed frame components** of the **partial velocity matrices** are as follows.

$$[{}^R v_{S,\hat{\omega}_F}] = -\left([{}^{\tilde{r}}_{A/O}] + [{}^{\tilde{r}}_{S/A}]\right) \quad (151)$$

$$[{}^R v_{S,\hat{\omega}_S}] = -[{}^{\tilde{r}}_{S/A}][R_F]^T \quad (152)$$

$$[{}^R v_{S,\dot{x}'}] = [R_{AB}]^T \quad (153)$$

Here,

$$[R_F]^T = \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \{r_{A/O}\} = [R_F]^T \begin{Bmatrix} 0 \\ a \\ 0 \end{Bmatrix} = \begin{Bmatrix} -aS_\phi \\ aC_\phi \\ 0 \end{Bmatrix} \quad [{}^{\tilde{r}}_{A/O}] = \begin{bmatrix} 0 & 0 & aC_\phi \\ 0 & 0 & aS_\phi \\ -aC_\phi & -aS_\phi & 0 \end{bmatrix}$$

$$[R_{AB}]^T = \begin{bmatrix} C_\phi & -S_\phi C_\theta & S_\phi S_\theta \\ S_\phi & C_\phi C_\theta & -C_\phi S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \quad \{r_{S/A}\} = [R_{AB}]^T \begin{Bmatrix} 0 \\ x \\ 0 \end{Bmatrix} = x \begin{Bmatrix} -S_\phi C_\theta \\ C_\phi C_\theta \\ S_\theta \end{Bmatrix}$$

$$[{}^{\tilde{r}}_{S/A}] = \begin{bmatrix} 0 & -xS_\theta & xC_\phi C_\theta \\ xS_\theta & 0 & xS_\phi C_\theta \\ -xC_\phi C_\theta & -xS_\phi C_\theta & 0 \end{bmatrix}$$

$$\{\omega_{AB}\} = \{\hat{\omega}_F\} + [R_F]^T \{\hat{\omega}_S\} = \begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix} + \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \omega \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \omega C_\phi \\ \omega S_\phi \\ \Omega \end{Bmatrix}$$

The explicit forms of the partial velocity matrices are as follows.

$$[{}^R v_{S,\hat{\omega}_F}] = \begin{bmatrix} 0 & xS_\theta & -(a + xC_\theta)C_\phi \\ -xS_\theta & 0 & -(a + xC_\theta)S_\phi \\ (a + xC_\theta)C_\phi & (a + xC_\theta)S_\phi & 0 \end{bmatrix} \quad (154)$$

$$[{}^R v_{S,\hat{\omega}_S}] = \begin{bmatrix} xS_\phi S_\theta & xC_\phi S_\theta & -xC_\phi C_\theta \\ -xC_\phi S_\theta & xS_\phi S_\theta & -xS_\phi C_\theta \\ xC_\theta & 0 & 0 \end{bmatrix} \quad \{v_S\} = \begin{Bmatrix} -(a + xC_\theta)\Omega C_\phi + (x\omega S_\theta - \dot{x}C_\theta)S_\phi \\ -(a + xC_\theta)\Omega S_\phi - (x\omega S_\theta - \dot{x}C_\theta)C_\phi \\ \dot{x}S_\theta + x\omega C_\theta \end{Bmatrix} \quad (155)$$

Acceleration:

The **fixed frame components** of ${}^R a_S$ the acceleration of S can be calculated by **differentiating** the fixed frame components of ${}^R v_P$ as follows.

$$\begin{aligned} \{a_s\} = \{\dot{v}_s\} = & \begin{bmatrix} R_{v_{S,\hat{\omega}_F}} \end{bmatrix} \{\dot{\hat{\omega}}_F\} + \begin{bmatrix} R_{v_{S,\hat{\omega}_S}} \end{bmatrix} \{\dot{\hat{\omega}}_S\} + \begin{bmatrix} R_{v_{S,\dot{x}'}} \end{bmatrix} \{\dot{x}'\} \\ & + \begin{bmatrix} R_{\dot{v}_{S,\hat{\omega}_F}} \end{bmatrix} \{\hat{\omega}_F\} + \begin{bmatrix} R_{\dot{v}_{S,\hat{\omega}_S}} \end{bmatrix} \{\hat{\omega}_S\} + \begin{bmatrix} R_{\dot{v}_{S,\dot{x}'}} \end{bmatrix} \{\dot{x}'\} \end{aligned} \quad (156)$$

Here,

$$\begin{bmatrix} R_{\dot{v}_{S,\hat{\omega}_F}} \end{bmatrix} = - \left(\begin{bmatrix} \dot{\tilde{r}}_{A/O} \end{bmatrix} + \begin{bmatrix} \dot{\tilde{r}}_{S/A} \end{bmatrix} \right) \quad (157)$$

$$\begin{bmatrix} R_{\dot{v}_{S,\hat{\omega}_S}} \end{bmatrix} = - \begin{bmatrix} \dot{\tilde{r}}_{S/A} \end{bmatrix} [R_F]^T - \begin{bmatrix} \tilde{r}_{S/A} \end{bmatrix} [\dot{R}_F]^T = - \begin{bmatrix} \dot{\tilde{r}}_{S/A} \end{bmatrix} [R_F]^T - \begin{bmatrix} \tilde{r}_{S/A} \end{bmatrix} [\tilde{\omega}_F] [R_F]^T \quad (158)$$

$$\begin{bmatrix} R_{\dot{v}_{S,\dot{x}'}} \end{bmatrix} = \begin{bmatrix} \dot{R}_{AB} \end{bmatrix}^T = [\tilde{\omega}_{AB}] [R_{AB}]^T \quad (159)$$

The **time derivatives** of the **two skew symmetric matrices** are built using the **time derivatives** of the **fixed frame components** of the **position vectors** $\tilde{r}_{A/O}$ and $\tilde{r}_{S/A}$. **Differentiating** these components gives

$$\left\{ \dot{r}_{A/O} \right\} = \begin{bmatrix} \dot{R}_F \end{bmatrix}^T \begin{Bmatrix} 0 \\ a \\ 0 \end{Bmatrix} = [\tilde{\omega}_F] [R_F]^T \begin{Bmatrix} 0 \\ a \\ 0 \end{Bmatrix} = \begin{bmatrix} 0 & -\Omega & 0 \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ a \\ 0 \end{Bmatrix} = \begin{bmatrix} -S_\phi \Omega & -C_\phi \Omega & 0 \\ C_\phi \Omega & -S_\phi \Omega & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ a \\ 0 \end{Bmatrix}$$

$$\left\{ \dot{r}_{A/O} \right\} = \begin{Bmatrix} -aC_\phi \Omega \\ -aS_\phi \Omega \\ 0 \end{Bmatrix} \quad \begin{bmatrix} \dot{\tilde{r}}_{A/O} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -aS_\phi \Omega \\ 0 & 0 & aC_\phi \Omega \\ aS_\phi \Omega & -aC_\phi \Omega & 0 \end{bmatrix}$$

$$\begin{aligned} \left\{ \dot{r}_{S/A} \right\} &= \begin{bmatrix} \dot{R}_{AB} \end{bmatrix}^T \begin{Bmatrix} 0 \\ x \\ 0 \end{Bmatrix} + [R_{AB}]^T \begin{Bmatrix} 0 \\ \dot{x} \\ 0 \end{Bmatrix} = [\tilde{\omega}_{AB}] [R_{AB}]^T \begin{Bmatrix} 0 \\ x \\ 0 \end{Bmatrix} + [R_{AB}]^T \begin{Bmatrix} 0 \\ \dot{x} \\ 0 \end{Bmatrix} \\ &= \begin{bmatrix} 0 & -\Omega & \omega S_\phi \\ \Omega & 0 & -\omega C_\phi \\ -\omega S_\phi & \omega C_\phi & 0 \end{bmatrix} \begin{bmatrix} C_\phi & -S_\phi C_\theta & S_\phi S_\theta \\ S_\phi & C_\phi C_\theta & -C_\phi S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \begin{Bmatrix} 0 \\ x \\ 0 \end{Bmatrix} + \begin{bmatrix} C_\phi & -S_\phi C_\theta & S_\phi S_\theta \\ S_\phi & C_\phi C_\theta & -C_\phi S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \begin{Bmatrix} 0 \\ \dot{x} \\ 0 \end{Bmatrix} \\ &= \begin{bmatrix} -S_\phi \Omega & -C_\phi C_\theta \Omega + S_\phi S_\theta \omega & C_\phi S_\theta \Omega + S_\phi C_\theta \omega \\ \Omega C_\phi & -S_\phi C_\theta \Omega - C_\phi S_\theta \omega & S_\phi S_\theta \Omega - C_\phi C_\theta \omega \\ -S_\phi C_\phi \omega + S_\phi C_\phi \omega & C_\theta \omega (S_\phi S_\phi + C_\phi C_\phi) & -\omega S_\theta (S_\phi S_\phi + C_\phi C_\phi) \end{bmatrix} \begin{Bmatrix} 0 \\ x \\ 0 \end{Bmatrix} + \begin{bmatrix} -S_\phi C_\theta \\ C_\phi C_\theta \\ S_\theta \end{bmatrix} \dot{x} \end{aligned}$$

$$\left\{ \dot{r}_{S/A} \right\} = \begin{Bmatrix} -xC_\phi C_\theta \Omega + xS_\phi S_\theta \omega - \dot{x}S_\phi C_\theta \\ -xS_\phi C_\theta \Omega - xC_\phi S_\theta \omega + \dot{x}C_\phi C_\theta \\ xC_\theta \omega + \dot{x}S_\theta \end{Bmatrix}$$

$$\begin{bmatrix} \dot{\tilde{r}}_{S/A} \end{bmatrix} = \begin{bmatrix} 0 & -xC_\theta \omega - \dot{x}S_\theta & -xS_\phi C_\theta \Omega - xC_\phi S_\theta \omega + \dot{x}C_\phi C_\theta \\ xC_\theta \omega + \dot{x}S_\theta & 0 & xC_\phi C_\theta \Omega - xS_\phi S_\theta \omega + \dot{x}S_\phi C_\theta \\ xS_\phi C_\theta \Omega + xC_\phi S_\theta \omega - \dot{x}C_\phi C_\theta & -xC_\phi C_\theta \Omega + xS_\phi S_\theta \omega - \dot{x}S_\phi C_\theta & 0 \end{bmatrix}$$

Substituting into Equations (157)-(159) gives explicit forms for the **fixed frame components** of the **time derivatives** of the **partial velocity matrices**.

$$\begin{aligned} \left[{}^R \dot{V}_{S, \hat{\omega}_F} \right] &= - \left(\left[\tilde{r}_{A/O} \right] + \left[\tilde{r}_{S/A} \right] \right) \\ &= \begin{bmatrix} 0 & 0 & aS_\phi \Omega \\ 0 & 0 & -aC_\phi \Omega \\ -aS_\phi \Omega & aC_\phi \Omega & 0 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & xC_\theta \omega + \dot{x}S_\theta & xS_\phi C_\theta \Omega + xC_\phi S_\theta \omega - \dot{x}C_\phi C_\theta \\ -xC_\theta \omega - \dot{x}S_\theta & 0 & -xC_\phi C_\theta \Omega + xS_\phi S_\theta \omega - \dot{x}S_\phi C_\theta \\ -xC_\phi C_\theta \Omega - xC_\phi S_\theta \omega + \dot{x}C_\phi C_\theta & xC_\phi C_\theta \Omega - xS_\phi S_\theta \omega + \dot{x}S_\phi C_\theta & 0 \end{bmatrix} \end{aligned} \quad (160)$$

$$\begin{aligned} \left[{}^R \dot{V}_{S, \hat{\omega}_S} \right] &= - \left[\tilde{r}_{S/A} \right] \left[R_F \right]^T - \left[\tilde{r}_{S/A} \right] \left[\tilde{\omega}_F \right] \left[R_F \right]^T \\ &= \begin{bmatrix} 0 & xC_\theta \omega + \dot{x}S_\theta & xS_\phi C_\theta \Omega + xC_\phi S_\theta \omega - \dot{x}C_\phi C_\theta \\ -xC_\theta \omega - \dot{x}S_\theta & 0 & -xC_\phi C_\theta \Omega + xS_\phi S_\theta \omega - \dot{x}S_\phi C_\theta \\ -xC_\phi C_\theta \Omega - xC_\phi S_\theta \omega + \dot{x}C_\phi C_\theta & xC_\phi C_\theta \Omega - xS_\phi S_\theta \omega + \dot{x}S_\phi C_\theta & 0 \end{bmatrix} \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & xS_\theta & -xC_\phi C_\theta \\ -xS_\theta & 0 & -xS_\phi C_\theta \\ xC_\phi C_\theta & xS_\phi C_\theta & 0 \end{bmatrix} \begin{bmatrix} 0 & -\Omega & 0 \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (xC_\theta \omega + \dot{x}S_\theta)S_\phi & (xC_\theta \omega + \dot{x}S_\theta)C_\phi & xS_\phi C_\theta \Omega + xC_\phi S_\theta \omega - \dot{x}C_\phi C_\theta \\ (-xC_\theta \omega - \dot{x}S_\theta)C_\phi & (xC_\theta \omega + \dot{x}S_\theta)S_\phi & -xC_\phi C_\theta \Omega + xS_\phi S_\theta \omega - \dot{x}S_\phi C_\theta \\ VS_{31} & VS_{32} & 0 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & xS_\theta & -xC_\phi C_\theta \\ -xS_\theta & 0 & -xS_\phi C_\theta \\ xC_\phi C_\theta & xS_\phi C_\theta & 0 \end{bmatrix} \begin{bmatrix} -\Omega S_\phi & -\Omega C_\phi & 0 \\ \Omega C_\phi & -\Omega S_\phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \left[{}^R \dot{V}_{S, \hat{\omega}_S} \right] &= \begin{bmatrix} (xC_\theta \omega + \dot{x}S_\theta)S_\phi & (xC_\theta \omega + \dot{x}S_\theta)C_\phi & xS_\phi C_\theta \Omega + xC_\phi S_\theta \omega - \dot{x}C_\phi C_\theta \\ (-xC_\theta \omega - \dot{x}S_\theta)C_\phi & (xC_\theta \omega + \dot{x}S_\theta)S_\phi & -xC_\phi C_\theta \Omega + xS_\phi S_\theta \omega - \dot{x}S_\phi C_\theta \\ VS_{31} & VS_{32} & 0 \end{bmatrix} \\ &+ \begin{bmatrix} xC_\phi S_\theta \Omega & -xS_\phi S_\theta \Omega & 0 \\ xS_\phi S_\theta \Omega & xC_\phi S_\theta \Omega & 0 \\ 0 & -xC_\theta \Omega & 0 \end{bmatrix} \end{aligned} \quad (161)$$

Simplifying the entries VS_{31} and VS_{32} gives

$$\begin{aligned} VS_{31} &= (-xC_\phi C_\theta \Omega - xC_\phi S_\theta \omega + \dot{x}C_\phi C_\theta)C_\phi + (xC_\phi C_\theta \Omega - xS_\phi S_\theta \omega + \dot{x}S_\phi C_\theta)S_\phi \\ &= (-S_\phi C_\phi C_\theta + S_\phi C_\phi C_\theta)x\Omega - (C_\phi C_\phi + S_\phi S_\phi)x\omega S_\theta + (C_\phi C_\phi + S_\phi S_\phi)\dot{x}C_\theta \end{aligned}$$

$$VS_{31} = -x\omega S_\theta + \dot{x}C_\theta$$

$$\begin{aligned} VS_{32} &= -S_\phi \left(-xS_\phi C_\theta \Omega - xC_\phi S_\theta \omega + \dot{x}C_\phi C_\theta \right) + C_\phi \left(xC_\phi C_\theta \Omega - xS_\phi S_\theta \omega + \dot{x}S_\phi C_\theta \right) \\ &= \left(S_\phi S_\phi + C_\phi C_\phi \right) x\Omega C_\theta + \left(S_\phi C_\phi S_\theta - S_\phi C_\phi S_\theta \right) x\omega + \left(-S_\phi C_\phi C_\theta + S_\phi C_\phi C_\theta \right) \dot{x} \end{aligned}$$

$$VS_{32} = x\Omega C_\theta$$

Finally,

$$\begin{aligned} \begin{bmatrix} {}^R\dot{V}_{S,\ddot{x}'} \end{bmatrix} &= [\tilde{\omega}_{AB}] [R_{AB}]^T = \begin{bmatrix} 0 & -\Omega & \omega S_\phi \\ \Omega & 0 & -\omega C_\phi \\ -\omega S_\phi & \omega C_\phi & 0 \end{bmatrix} \begin{bmatrix} C_\phi & -S_\phi C_\theta & S_\phi S_\theta \\ S_\phi & C_\phi C_\theta & -C_\phi S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \\ &= \begin{bmatrix} -\Omega S_\phi & -\Omega C_\phi C_\theta + \omega S_\phi S_\theta & \Omega C_\phi S_\theta + \omega S_\phi C_\theta \\ \Omega C_\phi & -\Omega S_\phi C_\theta - \omega C_\phi S_\theta & \Omega S_\phi S_\theta - \omega C_\phi C_\theta \\ -\omega S_\phi C_\phi + \omega S_\phi C_\phi & (S_\phi S_\phi + C_\phi C_\phi) \omega C_\theta & -(S_\phi S_\phi + C_\phi C_\phi) \omega S_\theta \end{bmatrix} \\ \begin{bmatrix} {}^R\dot{V}_{S,\ddot{x}'} \end{bmatrix} &= \begin{bmatrix} -\Omega S_\phi & -\Omega C_\phi C_\theta + \omega S_\phi S_\theta & \Omega C_\phi S_\theta + \omega S_\phi C_\theta \\ \Omega C_\phi & -\Omega S_\phi C_\theta - \omega C_\phi S_\theta & \Omega S_\phi S_\theta - \omega C_\phi C_\theta \\ 0 & \omega C_\theta & -\omega S_\theta \end{bmatrix} \end{aligned}$$

Substituting into Equation (156) and simplifying gives

$$\begin{aligned} \{a_S\} &= \begin{bmatrix} {}^R\mathbf{v}_{S,\hat{\omega}_F} \end{bmatrix} \{\dot{\hat{\omega}}_F\} + \begin{bmatrix} {}^R\mathbf{v}_{S,\hat{\omega}_S} \end{bmatrix} \{\dot{\hat{\omega}}_S\} + \begin{bmatrix} {}^R\mathbf{v}_{S,\ddot{x}'} \end{bmatrix} \{\ddot{x}'\} + \begin{bmatrix} {}^R\dot{\mathbf{v}}_{S,\hat{\omega}_F} \end{bmatrix} \{\hat{\omega}_F\} + \begin{bmatrix} {}^R\dot{\mathbf{v}}_{S,\hat{\omega}_S} \end{bmatrix} \{\hat{\omega}_S\} + \begin{bmatrix} {}^R\dot{\mathbf{v}}_{S,\ddot{x}'} \end{bmatrix} \{\ddot{x}'\} \\ &= \begin{bmatrix} 0 & xS_\theta & -(a + xC_\theta)C_\phi \\ -xS_\theta & 0 & -(a + xC_\theta)S_\phi \\ (a + xC_\theta)C_\phi & (a + xC_\theta)S_\phi & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\Omega} \end{Bmatrix} + \begin{bmatrix} xS_\phi S_\theta & xC_\phi S_\theta & -xC_\phi C_\theta \\ -xC_\phi S_\theta & xS_\phi S_\theta & -xS_\phi C_\theta \\ xC_\theta & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\omega} \\ 0 \\ 0 \end{Bmatrix} \\ &+ \begin{bmatrix} C_\phi & -S_\phi C_\theta & S_\phi S_\theta \\ S_\phi & C_\phi C_\theta & -C_\phi S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \begin{Bmatrix} 0 \\ \ddot{x} \\ 0 \end{Bmatrix} + \begin{bmatrix} 0 & 0 & aS_\phi \Omega \\ 0 & 0 & -aC_\phi \Omega \\ -aS_\phi \Omega & aC_\phi \Omega & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix} \\ &+ \begin{bmatrix} 0 & xC_\theta \omega + \dot{x}S_\theta & xS_\phi C_\theta \Omega + xC_\phi S_\theta \omega - \dot{x}C_\phi C_\theta \\ -xC_\theta \omega - \dot{x}S_\theta & 0 & -xC_\phi C_\theta \Omega + xS_\phi S_\theta \omega - \dot{x}S_\phi C_\theta \\ -xS_\phi C_\theta \Omega - xC_\phi S_\theta \omega + \dot{x}C_\phi C_\theta & xC_\phi C_\theta \Omega - xS_\phi S_\theta \omega + \dot{x}S_\phi C_\theta & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \Omega \end{Bmatrix} \\ &\vdots \end{aligned}$$

$$\begin{aligned}
& \vdots \\
& + \begin{bmatrix} (xC_\theta\omega + \dot{x}S_\theta)S_\phi & (xC_\theta\omega + \dot{x}S_\theta)C_\phi & xS_\phi C_\theta\Omega + xC_\phi S_\theta\omega - \dot{x}C_\phi C_\theta \\ (-xC_\theta\omega - \dot{x}S_\theta)C_\phi & (xC_\theta\omega + \dot{x}S_\theta)S_\phi & -xC_\phi C_\theta\Omega + xS_\phi S_\theta\omega - \dot{x}S_\phi C_\theta \\ -x\omega S_\theta + \dot{x}C_\theta & x\Omega C_\theta & 0 \end{bmatrix} \begin{Bmatrix} \omega \\ 0 \\ 0 \end{Bmatrix} \\
& + \begin{bmatrix} xC_\phi S_\theta\Omega & -xS_\phi S_\theta\Omega & 0 \\ xS_\phi S_\theta\Omega & xC_\phi S_\theta\Omega & 0 \\ 0 & -xC_\theta\Omega & 0 \end{bmatrix} \begin{Bmatrix} \omega \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} -\Omega S_\phi & -\Omega C_\phi C_\theta + \omega S_\phi S_\theta & \Omega C_\phi S_\theta + \omega S_\phi C_\theta \\ \Omega C_\phi & -\Omega S_\phi C_\theta - \omega C_\phi S_\theta & \Omega S_\phi S_\theta - \omega C_\phi C_\theta \\ 0 & \omega C_\theta & -\omega S_\theta \end{bmatrix} \begin{Bmatrix} 0 \\ \dot{x} \\ 0 \end{Bmatrix} \\
& = \begin{Bmatrix} -(a + xC_\theta)C_\phi \\ -(a + xC_\theta)S_\phi \\ 0 \end{Bmatrix} \dot{\Omega} + \begin{Bmatrix} xS_\phi S_\theta \\ -xC_\phi S_\theta \\ xC_\theta \end{Bmatrix} \dot{\omega} + \begin{Bmatrix} -S_\phi C_\theta \\ C_\phi C_\theta \\ S_\theta \end{Bmatrix} \ddot{x} + \begin{Bmatrix} aS_\phi \\ -aC_\phi \\ 0 \end{Bmatrix} \Omega^2 + \begin{Bmatrix} xS_\phi C_\theta\Omega + xC_\phi S_\theta\omega - \dot{x}C_\phi C_\theta \\ -xC_\phi C_\theta\Omega + xS_\phi S_\theta\omega - \dot{x}S_\phi C_\theta \\ 0 \end{Bmatrix} \Omega \\
& + \begin{Bmatrix} (xC_\theta\omega + \dot{x}S_\theta)S_\phi \\ (-xC_\theta\omega - \dot{x}S_\theta)C_\phi \\ -x\omega S_\theta + \dot{x}C_\theta \end{Bmatrix} \omega + \begin{Bmatrix} xC_\phi S_\theta\Omega \\ xS_\phi S_\theta\Omega \\ 0 \end{Bmatrix} \omega + \begin{Bmatrix} -\Omega C_\phi C_\theta + \omega S_\phi S_\theta \\ -\Omega S_\phi C_\theta - \omega C_\phi S_\theta \\ \omega C_\theta \end{Bmatrix} \dot{x} \\
& \boxed{\begin{aligned} \{a_S\} = & \begin{Bmatrix} -(a + xC_\theta)C_\phi \\ -(a + xC_\theta)S_\phi \\ 0 \end{Bmatrix} \dot{\Omega} + \begin{Bmatrix} S_\phi S_\theta \\ -C_\phi S_\theta \\ C_\theta \end{Bmatrix} x\dot{\omega} + \begin{Bmatrix} -S_\phi C_\theta \\ C_\phi C_\theta \\ S_\theta \end{Bmatrix} \ddot{x} + \begin{Bmatrix} (a + xC_\theta)S_\phi \\ -(a + xC_\theta)C_\phi \\ 0 \end{Bmatrix} \Omega^2 + \begin{Bmatrix} S_\phi C_\theta \\ -C_\phi C_\theta \\ -S_\theta \end{Bmatrix} x\omega^2 \\ & + \begin{Bmatrix} 2C_\phi S_\theta \\ 2S_\phi S_\theta \\ 0 \end{Bmatrix} x\Omega\omega + \begin{Bmatrix} -2C_\phi C_\theta \\ -2S_\phi C_\theta \\ 0 \end{Bmatrix} \dot{x}\Omega + \begin{Bmatrix} 2S_\phi S_\theta \\ -2C_\phi S_\theta \\ 2C_\theta \end{Bmatrix} \dot{x}\omega \end{aligned}} \quad (162)
\end{aligned}$$

Comparison with previous results:

In Volume I, Unit 4, the frame F components of the acceleration of S were calculated to be as follows.

$$\begin{aligned}
{}^R \underline{a}_S = & \left[-(a + xC_\theta) \dot{\Omega} + 2x\omega\Omega S_\theta - 2\dot{x}\Omega C_\theta \right] \underline{e}_1 + \left[\ddot{x}C_\theta - x\dot{\omega}S_\theta - (a + xC_\theta)\Omega^2 - xC_\theta\omega^2 - 2\dot{x}\omega S_\theta \right] \underline{e}_2 \\
& + \left[\ddot{x}S_\theta + x\dot{\omega}C_\theta - x\omega^2 S_\theta + 2\dot{x}\omega C_\theta \right] \underline{k}
\end{aligned}$$

These results can be **converted** to the fixed frame as follows.

$$\{a_S\} = [R_F]^T \{a_S\}_F = \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -(a + xC_\theta) \dot{\Omega} + 2x\omega\Omega S_\theta - 2\dot{x}\Omega C_\theta \\ \ddot{x}C_\theta - x\dot{\omega}S_\theta - (a + xC_\theta)\Omega^2 - xC_\theta\omega^2 - 2\dot{x}\omega S_\theta \\ \ddot{x}S_\theta + x\dot{\omega}C_\theta - x\omega^2 S_\theta + 2\dot{x}\omega C_\theta \end{Bmatrix}$$

Expanding the matrix product gives the following.

$$\Rightarrow \left\{ a_s \right\} = \begin{Bmatrix} -(a + xC_\theta)C_\phi \\ -(a + xC_\theta)S_\phi \\ 0 \end{Bmatrix} \dot{\Omega} + \begin{Bmatrix} S_\phi S_\theta \\ -C_\phi S_\theta \\ C_\theta \end{Bmatrix} x\dot{\omega} + \begin{Bmatrix} -S_\phi C_\theta \\ C_\phi C_\theta \\ S_\theta \end{Bmatrix} \ddot{x} + \begin{Bmatrix} (a + xC_\theta)S_\phi \\ -(a + xC_\theta)C_\phi \\ 0 \end{Bmatrix} \Omega^2 + \begin{Bmatrix} S_\phi C_\theta \\ -C_\phi C_\theta \\ -S_\theta \end{Bmatrix} x\omega^2$$

$$+ \begin{Bmatrix} 2C_\phi S_\theta \\ 2S_\phi S_\theta \\ 0 \end{Bmatrix} x\Omega\omega + \begin{Bmatrix} -2C_\phi C_\theta \\ -2S_\phi C_\theta \\ 0 \end{Bmatrix} \dot{x}\Omega + \begin{Bmatrix} 2S_\phi S_\theta \\ -2C_\phi S_\theta \\ 2C_\theta \end{Bmatrix} \dot{x}\omega$$

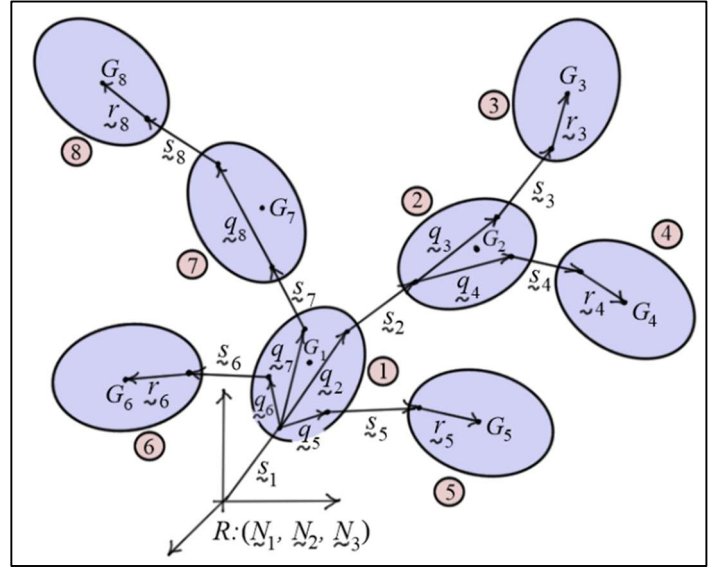
This result is **identical** to the result in Equation (162).

Example 5

The figure shows an eight-body system numbered using the numbering scheme presented in Unit 1. Body 1 is the system **reference body**, and the rest of the bodies are numbered in **ascending progression** outward along the branches. As structured, the lower numbered body array for the system is as follows.

$$\mathcal{L}(1, \dots, 8) = (0, 1, 2, 2, 1, 1, 7)$$

The orientation of body 1 is defined **relative** to the **fixed frame** $R: (\underline{N}_1, \underline{N}_2, \underline{N}_3)$, and the orientations of all the other bodies are defined **relative** to their **adjacent, lower-numbered** bodies. Using **base frame components** of the **relative angular velocities** of the bodies, the **system generalized speed matrix** is defined as follows.



$$\left\{ y \right\}_{48 \times 1} \triangleq \begin{Bmatrix} \left\{ y_1 \right\}_{24 \times 1} \\ \left\{ y_2 \right\}_{24 \times 1} \end{Bmatrix} \quad (163)$$

$$\left\{ y_1 \right\}_{24 \times 1} \triangleq [\hat{\omega}_{11} \quad \hat{\omega}_{12} \quad \hat{\omega}_{13} \quad \hat{\omega}_{21} \quad \hat{\omega}_{22} \quad \hat{\omega}_{23} \quad \cdots \quad \hat{\omega}_{71} \quad \hat{\omega}_{72} \quad \hat{\omega}_{73} \quad \hat{\omega}_{81} \quad \hat{\omega}_{82} \quad \hat{\omega}_{83}]^T \quad (164)$$

$$\left\{ y_2 \right\}_{24 \times 1} \triangleq [\dot{s}'_{11} \quad \dot{s}'_{12} \quad \dot{s}'_{13} \quad \dot{s}'_{21} \quad \dot{s}'_{22} \quad \dot{s}'_{23} \quad \cdots \quad \dot{s}'_{71} \quad \dot{s}'_{72} \quad \dot{s}'_{73} \quad \dot{s}'_{81} \quad \dot{s}'_{82} \quad \dot{s}'_{83}]^T \quad (165)$$

For each body B in the system, **find** the **time derivatives** of the **fixed frame components** of the **partial velocity matrices** of its **origin** and **mass center**. Then find the **fixed frame components** of the **acceleration** of G_3 the **mass center** of body 3. Use the results from Unit 2 for the **fixed frame components** of the **partial angular velocity matrices** of the bodies and use the results from Example 10 of Unit 4 for the **fixed frame components** of the **partial velocity matrices** of the **origins** and **mass centers** of the bodies. Follow the procedure presented in Equations (52)-(56), (81), and (82).

Results from Unit 2:

$$\begin{bmatrix} {}^R\omega_{1,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\omega_{2,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [R_1]^T_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\omega_{3,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [R_1]^T_{3 \times 3} & [R_2]^T_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\omega_{4,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [R_1]^T_{3 \times 3} & [0]_{3 \times 3} & [R_2]^T_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\omega_{5,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [R_1]^T_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\omega_{6,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [R_1]^T_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\omega_{7,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [R_1]^T_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\omega_{8,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [R_1]^T_{3 \times 3} & [R_7]^T_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\omega_{B,y_2} \end{bmatrix}_{3 \times 24} = [0]_{3 \times 24} \quad (B = 1, \dots, 8)$$

Results from Unit 4 for body *origins*:

Body 1:

$$\begin{bmatrix} {}^Rv_{O_1,y_1} \end{bmatrix}_{3 \times 24} = [0]_{3 \times 24} \quad \begin{bmatrix} {}^Rv_{O_1,y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 2:

$$\begin{bmatrix} {}^Rv_{O_2,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} {}^Rv_{O_2,\hat{a}_1} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^Rv_{O_2,\hat{a}_1} \end{bmatrix}_{3 \times 3} = -\begin{bmatrix} [\tilde{q}_2] + [\tilde{s}_2] \end{bmatrix}$$

$$\begin{bmatrix} {}^Rv_{O_2,y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [R_1]^T_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 3:

$$\begin{bmatrix} {}^Rv_{O_3,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} {}^Rv_{O_3,\hat{a}_1} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} {}^Rv_{O_3,\hat{a}_2} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^Rv_{O_3,\hat{a}_1} \end{bmatrix}_{3 \times 3} = -\begin{bmatrix} [\tilde{q}_2] + [\tilde{s}_2] + [\tilde{q}_3] + [\tilde{s}_3] \end{bmatrix} \quad \begin{bmatrix} {}^Rv_{O_3,\hat{a}_2} \end{bmatrix}_{3 \times 3} = -\begin{bmatrix} [\tilde{q}_3] + [\tilde{s}_3] \end{bmatrix} [R_1]^T$$

$$\begin{bmatrix} {}^Rv_{O_3,y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [R_1]^T_{3 \times 3} & [R_2]^T_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 4:

$$\begin{bmatrix} {}^Rv_{O_4,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} {}^Rv_{O_4,\hat{a}_1} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} {}^Rv_{O_4,\hat{a}_2} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^Rv_{O_4,\hat{a}_1} \end{bmatrix}_{3 \times 3} = -\begin{bmatrix} [\tilde{q}_2] + [\tilde{s}_2] + [\tilde{q}_4] + [\tilde{s}_4] \end{bmatrix} \quad \begin{bmatrix} {}^Rv_{O_4,\hat{a}_2} \end{bmatrix}_{3 \times 3} = -\begin{bmatrix} [\tilde{q}_4] + [\tilde{s}_4] \end{bmatrix} [R_1]^T$$

$$\begin{bmatrix} {}^R v_{O_4, y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [R_1]_{3 \times 3}^T & [0]_{3 \times 3} & [R_2]_{3 \times 3}^T & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 5:

$$\begin{bmatrix} {}^R v_{O_5, y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} {}^R v_{O_5, \hat{a}_1} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R v_{O_5, \hat{a}_1} \end{bmatrix}_{3 \times 3} = -\left[\begin{bmatrix} \tilde{q}_5 \end{bmatrix} + \begin{bmatrix} \tilde{s}_5 \end{bmatrix}\right]$$

$$\begin{bmatrix} {}^R v_{O_5, y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [R_1]_{3 \times 3}^T & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 6:

$$\begin{bmatrix} {}^R v_{O_6, y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} {}^R v_{O_6, \hat{a}_1} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R v_{O_6, \hat{a}_1} \end{bmatrix}_{3 \times 3} = -\left[\begin{bmatrix} \tilde{q}_6 \end{bmatrix} + \begin{bmatrix} \tilde{s}_6 \end{bmatrix}\right]$$

$$\begin{bmatrix} {}^R v_{O_6, y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [R_1]_{3 \times 3}^T & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 7:

$$\begin{bmatrix} {}^R v_{O_7, y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} {}^R v_{O_7, \hat{a}_1} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R v_{O_7, \hat{a}_1} \end{bmatrix}_{3 \times 3} = -\left[\begin{bmatrix} \tilde{q}_7 \end{bmatrix} + \begin{bmatrix} \tilde{s}_7 \end{bmatrix}\right]$$

$$\begin{bmatrix} {}^R v_{O_7, y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [R_1]_{3 \times 3}^T & [0]_{3 \times 3} \end{bmatrix}$$

Body 8:

$$\begin{bmatrix} {}^R v_{O_8, y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} {}^R v_{O_8, \hat{a}_1} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & \begin{bmatrix} {}^R v_{O_8, \hat{a}_7} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R v_{O_8, \hat{a}_1} \end{bmatrix}_{3 \times 3} = -\left[\begin{bmatrix} \tilde{q}_7 \end{bmatrix} + \begin{bmatrix} \tilde{s}_7 \end{bmatrix} + \begin{bmatrix} \tilde{q}_8 \end{bmatrix} + \begin{bmatrix} \tilde{s}_8 \end{bmatrix}\right] \quad \begin{bmatrix} {}^R v_{O_8, \hat{a}_7} \end{bmatrix}_{3 \times 3} = -\left(\begin{bmatrix} \tilde{q}_8 \end{bmatrix} + \begin{bmatrix} \tilde{s}_8 \end{bmatrix}\right)[R_1]^T$$

$$\begin{bmatrix} {}^R v_{O_8, y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [R_1]_{3 \times 3}^T & [R_7]_{3 \times 3}^T \end{bmatrix}$$

Results from Unit 4 for body *mass centers*:

Body 1:

$$\begin{bmatrix} {}^R v_{1, y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} {}^R v_{1, \hat{a}_1'} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R v_{1, \hat{a}_1'} \end{bmatrix}_{3 \times 3} = -\left[\begin{bmatrix} \tilde{r}_1 \end{bmatrix}\right]$$

$$\begin{bmatrix} {}^R v_{1, y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 2:

$$\begin{bmatrix} {}^R v_{2, y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} {}^R v_{2, \hat{a}_1} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} {}^R v_{2, \hat{a}_2} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R v_{2, \hat{a}_1} \end{bmatrix}_{3 \times 3} = -\left[\begin{bmatrix} \tilde{q}_2 \end{bmatrix} + \begin{bmatrix} \tilde{s}_2 \end{bmatrix} + \begin{bmatrix} \tilde{r}_2 \end{bmatrix}\right] \quad \begin{bmatrix} {}^R v_{2, \hat{a}_2} \end{bmatrix}_{3 \times 3} = -\left[\begin{bmatrix} \tilde{r}_2 \end{bmatrix}\right][R_1]^T$$

$$\begin{bmatrix} R_{v_2, y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [R_1]_{3 \times 3}^T & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 3:

$$\begin{bmatrix} R_{v_3, y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} R_{v_3, \hat{o}_1} \end{bmatrix}_{3 \times 3} \begin{bmatrix} R_{v_3, \hat{o}_2} \end{bmatrix}_{3 \times 3} \begin{bmatrix} R_{v_3, \hat{o}_3} \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3}$$

$$\begin{aligned} \boxed{\left[{}^R v_{3, \hat{\omega}_1} \right]_{3 \times 3}} &= -\left[[\tilde{q}_2] + [\tilde{s}_2] + [\tilde{q}_3] + [\tilde{s}_3] + [\tilde{r}_3] \right] \\ \boxed{\left[{}^R v_{3, \hat{\omega}_2} \right]_{3 \times 3}} &= -\left[[\tilde{q}_3] + [\tilde{s}_3] + [\tilde{r}_3] \right] [R_1]^T \end{aligned}$$

$$\begin{bmatrix} R_{v_3, \hat{w}_3} \end{bmatrix}_{3 \times 3} = -[\tilde{r}_3][R_2]^T$$

$$\begin{bmatrix} R_{v_3, y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [R_1]_{3 \times 3}^T & [R_2]_{3 \times 3}^T & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 4:

$$\begin{bmatrix} R_{v_4, y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} R_{v_4, \hat{w}_1} \end{bmatrix}_{3 \times 3} \quad \begin{bmatrix} R_{v_4, \hat{w}_2} \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 3} \quad \begin{bmatrix} R_{v_4, \hat{w}_4} \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 3} \quad [0]_{3 \times 3} \quad [0]_{3 \times 3} \quad [0]_{3 \times 3} \end{bmatrix}$$

$$\left[\begin{matrix} R_{\mathbf{v}_4, \hat{\omega}_1} \end{matrix} \right]_{3 \times 3} = - \left[[\tilde{q}_2] + [\tilde{s}_2] + [\tilde{q}_4] + [\tilde{s}_4] + [\tilde{r}_4] \right] \quad \left[\begin{matrix} R_{\mathbf{v}_4, \hat{\omega}_2} \end{matrix} \right]_{3 \times 3} = - \left[[\tilde{q}_4] + [\tilde{s}_4] + [\tilde{r}_4] \right] [R_1]^T$$

$$\begin{bmatrix} R_{v_4, \hat{w}_3} \end{bmatrix}_{3 \times 3} = -[\tilde{r}_4][R_2]^T$$

$$\begin{bmatrix} R_{v_4, y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [R_1]_{3 \times 3}^T & [0]_{3 \times 3} & [R_2]_{3 \times 3}^T & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 5:

$$\begin{bmatrix} R_{V_{5,y_1}} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} R_{V_{5,\hat{a}_1}} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & \begin{bmatrix} R_{V_{5,\hat{a}_5}} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\mathbf{v}_{5,\hat{w}_1} \end{bmatrix}_{3 \times 3} = -\begin{bmatrix} [\tilde{q}_5] + [\tilde{s}_5] + [\tilde{r}_5] \end{bmatrix} \quad \begin{bmatrix} {}^R\mathbf{v}_{5,\hat{w}_5} \end{bmatrix}_{3 \times 3} = -[\tilde{r}_5][R_1]^T$$

$$\begin{bmatrix} R_{\mathbf{v}_{\mathbf{y}_2}} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [R_1]^T_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 6:

$$\begin{bmatrix} R_{V_{6,y_1}} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} R_{V_{6,\hat{a}_1}} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & \begin{bmatrix} R_{V_{6,\hat{a}_6}} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\left[\begin{matrix} R_{V_{6,\hat{w}_1}} \end{matrix} \right]_{3 \times 3} = - \left[\begin{matrix} \tilde{q}_6 \end{matrix} \right] + \left[\begin{matrix} \tilde{s}_6 \end{matrix} \right] + \left[\begin{matrix} \tilde{r}_6 \end{matrix} \right] \quad \left[\begin{matrix} R_{V_{6,\hat{w}_6}} \end{matrix} \right]_{3 \times 3} = - \left[\begin{matrix} \tilde{r}_6 \end{matrix} \right] \left[\begin{matrix} R_1 \end{matrix} \right]^T$$

$$\begin{bmatrix} R_{\mathbf{v}_{6,y_2}} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [R_l]^T_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 7:

$$\begin{bmatrix} R_{v_{7,y_1}} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} R_{v_{7,\hat{w}_1}} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & \begin{bmatrix} R_{v_{7,\hat{w}_7}} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{aligned} \left[\begin{matrix} {}^R\mathbf{v}_{7,\hat{\omega}_1} \end{matrix} \right]_{3 \times 3} &= -\left[\begin{matrix} [\tilde{q}_7] + [\tilde{s}_7] + [\tilde{r}_7] \end{matrix} \right] & \left[\begin{matrix} {}^R\mathbf{v}_{7,\hat{\omega}_7} \end{matrix} \right]_{3 \times 3} &= -[\tilde{r}_7][R_1]^T \end{aligned}$$

$$\begin{bmatrix} R_{v_7, y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [R_1]^T_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 8:

$$\begin{bmatrix} R_{v_{8,y_1}} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} R_{v_{8,\hat{a}_1}} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} & \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} & \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} & \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} & \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} & \begin{bmatrix} R_{v_{8,\hat{a}_7}} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} R_{v_{8,\hat{a}_8}} \end{bmatrix}_{3 \times 3} \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} {}^R\mathbf{v}_{8,\hat{\omega}_1} \end{bmatrix}_{3 \times 3} &= -\left[\begin{bmatrix} \tilde{q}_7 \end{bmatrix} + \begin{bmatrix} \tilde{s}_7 \end{bmatrix} + \begin{bmatrix} \tilde{q}_8 \end{bmatrix} + \begin{bmatrix} \tilde{s}_8 \end{bmatrix} + \begin{bmatrix} \tilde{r}_8 \end{bmatrix}\right] & \begin{bmatrix} {}^R\mathbf{v}_{8,\hat{\omega}_7} \end{bmatrix}_{3 \times 3} &= -\left(\begin{bmatrix} \tilde{q}_8 \end{bmatrix} + \begin{bmatrix} \tilde{s}_8 \end{bmatrix} + \begin{bmatrix} \tilde{r}_8 \end{bmatrix}\right)\begin{bmatrix} R_1 \end{bmatrix}_{3 \times 3}^T \\ \begin{bmatrix} {}^R\mathbf{v}_{8,\hat{\omega}_8} \end{bmatrix}_{3 \times 3} &= -\begin{bmatrix} \tilde{r}_8 \end{bmatrix}\begin{bmatrix} R_7 \end{bmatrix}_{3 \times 3}^T \\ \begin{bmatrix} {}^R\mathbf{v}_{8,y_2} \end{bmatrix}_{3 \times 24} &= \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [R_1]_{3 \times 3}^T & [R_7]_{3 \times 3}^T \end{bmatrix} \end{aligned}$$

Solution:

Time Derivatives of **Partial Velocities** of **Body Origins**:

Body 1:

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_1,y_1} \end{bmatrix}_{3 \times 24} = [0]_{3 \times 24} \quad \begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_1,y_2} \end{bmatrix}_{3 \times 24} = [0]_{3 \times 24}$$

Body 2:

$$1. \left\{ \begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_2,y} \end{bmatrix}_{3 \times 48} = \underbrace{\begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_1,y} \end{bmatrix}_{3 \times 48}}_{\text{zero}} - \left(\begin{bmatrix} \tilde{q}_2 \end{bmatrix} + \begin{bmatrix} \tilde{s}_2 \end{bmatrix}\right) \underbrace{\begin{bmatrix} {}^R\dot{\omega}_{1,y} \end{bmatrix}_{3 \times 48}}_{\text{zero}} - \left(\begin{bmatrix} \dot{\tilde{q}}_2 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{s}}_2 \end{bmatrix}\right) \begin{bmatrix} {}^R\omega_{1,y} \end{bmatrix}_{3 \times 48}$$

$$2. \left\{ \begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_2,y} \end{bmatrix}_{ik} = \begin{bmatrix} \tilde{\omega}_1 \end{bmatrix} \begin{bmatrix} R_1 \end{bmatrix}_{ij}^T \quad (i=1,2,3; j=1,2,3; k=27+j)$$

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_2,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_2,\hat{\omega}_1} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_2,\hat{\omega}_1} \end{bmatrix}_{3 \times 3} = -\left[\begin{bmatrix} \dot{\tilde{q}}_2 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{s}}_2 \end{bmatrix}\right]$$

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_2,y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [0]_{3 \times 3} & \begin{bmatrix} \tilde{\omega}_1 \end{bmatrix} \begin{bmatrix} R_1 \end{bmatrix}^T & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 3:

$$1. \left\{ \begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_3,y} \end{bmatrix}_{3 \times 48} = \begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_2,y} \end{bmatrix}_{3 \times 48} - \left(\begin{bmatrix} \tilde{q}_3 \end{bmatrix} + \begin{bmatrix} \tilde{s}_3 \end{bmatrix}\right) \begin{bmatrix} {}^R\dot{\omega}_{2,y} \end{bmatrix}_{3 \times 48} - \left(\begin{bmatrix} \dot{\tilde{q}}_3 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{s}}_3 \end{bmatrix}\right) \begin{bmatrix} {}^R\omega_{2,y} \end{bmatrix}_{3 \times 48}$$

$$2. \left\{ \begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_3,y} \end{bmatrix}_{ik} = \begin{bmatrix} \tilde{\omega}_2 \end{bmatrix} \begin{bmatrix} R_2 \end{bmatrix}_{ij}^T \quad (i=1,2,3; j=1,2,3; k=30+j)$$

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_3,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_3,\hat{\omega}_1} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_3,\hat{\omega}_2} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_3,\hat{\omega}_1} \end{bmatrix}_{3 \times 3} = -\left[\begin{bmatrix} \dot{\tilde{q}}_2 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{s}}_2 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{q}}_3 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{s}}_3 \end{bmatrix}\right] \quad \begin{bmatrix} {}^R\mathbf{v}_{O_3,\hat{\omega}_2} \end{bmatrix}_{3 \times 3} = -\left[\begin{bmatrix} \dot{\tilde{q}}_3 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{s}}_3 \end{bmatrix}\right] \begin{bmatrix} R_1 \end{bmatrix}^T - \left[\begin{bmatrix} \tilde{q}_3 \end{bmatrix} + \begin{bmatrix} \tilde{s}_3 \end{bmatrix}\right] \begin{bmatrix} \tilde{\omega}_1 \end{bmatrix} \begin{bmatrix} R_1 \end{bmatrix}^T$$

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_3,y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [0]_{3 \times 3} & \begin{bmatrix} \tilde{\omega}_1 \end{bmatrix} \begin{bmatrix} R_1 \end{bmatrix}^T & \begin{bmatrix} \tilde{\omega}_2 \end{bmatrix} \begin{bmatrix} R_2 \end{bmatrix}^T & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 4:

$$1. \left\{ \begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_4,y} \end{bmatrix}_{3 \times 48} = \begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_3,y} \end{bmatrix}_{3 \times 48} - \left(\begin{bmatrix} \tilde{q}_4 \end{bmatrix} + \begin{bmatrix} \tilde{s}_4 \end{bmatrix}\right) \begin{bmatrix} {}^R\dot{\omega}_{3,y} \end{bmatrix}_{3 \times 48} - \left(\begin{bmatrix} \dot{\tilde{q}}_4 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{s}}_4 \end{bmatrix}\right) \begin{bmatrix} {}^R\omega_{3,y} \end{bmatrix}_{3 \times 48}$$

$$2. \left\{ \begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_4,y} \end{bmatrix}_{ik} = \begin{bmatrix} \tilde{\omega}_3 \end{bmatrix} \begin{bmatrix} R_3 \end{bmatrix}_{ij}^T \quad (i=1,2,3; j=1,2,3; k=33+j)$$

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_4,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_4,\hat{\omega}_1} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_4,\hat{\omega}_2} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{O_4,\hat{\omega}_1} \end{bmatrix}_{3 \times 3} = -\left[\begin{bmatrix} \dot{\tilde{q}}_2 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{s}}_2 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{q}}_4 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{s}}_4 \end{bmatrix}\right] \quad \begin{bmatrix} {}^R\mathbf{v}_{O_4,\hat{\omega}_2} \end{bmatrix}_{3 \times 3} = -\left[\begin{bmatrix} \dot{\tilde{q}}_4 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{s}}_4 \end{bmatrix}\right] \begin{bmatrix} R_1 \end{bmatrix}^T - \left[\begin{bmatrix} \tilde{q}_4 \end{bmatrix} + \begin{bmatrix} \tilde{s}_4 \end{bmatrix}\right] \begin{bmatrix} \tilde{\omega}_1 \end{bmatrix} \begin{bmatrix} R_1 \end{bmatrix}^T$$

$$\begin{bmatrix} {}^R\dot{v}_{O_4,y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [0]_{3 \times 3} & [\tilde{\omega}_1][R_1]^T & [0]_{3 \times 3} & [\tilde{\omega}_2][R_2]^T & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 5:

$$1. \left\{ \begin{bmatrix} {}^R\dot{v}_{O_5,y} \end{bmatrix}_{3 \times 48} = \underbrace{\begin{bmatrix} {}^R\dot{v}_{O_1,y} \end{bmatrix}_{3 \times 48}}_{\text{zero}} - ([\tilde{q}_5] + [\tilde{s}_5]) \underbrace{\begin{bmatrix} {}^R\dot{\omega}_{1,y} \end{bmatrix}_{3 \times 48}}_{\text{zero}} - ([\dot{\tilde{q}}_5] + [\dot{\tilde{s}}_5]) \begin{bmatrix} {}^R\omega_{1,y} \end{bmatrix}_{3 \times 48} \right.$$

$$2. \left\{ \begin{bmatrix} {}^R\dot{v}_{O_5,y} \end{bmatrix}_{ik} = \begin{bmatrix} [\tilde{\omega}_1][R_1]^T \end{bmatrix}_{ij} \quad (i = 1, 2, 3; j = 1, 2, 3; k = 36 + j) \right.$$

$$\begin{bmatrix} {}^R\dot{v}_{O_5,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} {}^R\dot{v}_{O_5,\hat{\omega}_1} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\dot{v}_{O_5,\hat{\omega}_1} \end{bmatrix}_{3 \times 3} = - \left([\dot{\tilde{q}}_5] + [\dot{\tilde{s}}_5] \right)$$

$$\begin{bmatrix} {}^R\dot{v}_{O_5,y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [\tilde{\omega}_1][R_1]^T & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 6:

$$1. \left\{ \begin{bmatrix} {}^R\dot{v}_{O_6,y} \end{bmatrix}_{3 \times 48} = \underbrace{\begin{bmatrix} {}^R\dot{v}_{O_1,y} \end{bmatrix}_{3 \times 48}}_{\text{zero}} - ([\tilde{q}_6] + [\tilde{s}_6]) \underbrace{\begin{bmatrix} {}^R\dot{\omega}_{1,y} \end{bmatrix}_{3 \times 48}}_{\text{zero}} - ([\dot{\tilde{q}}_6] + [\dot{\tilde{s}}_6]) \begin{bmatrix} {}^R\omega_{1,y} \end{bmatrix}_{3 \times 48} \right.$$

$$2. \left\{ \begin{bmatrix} {}^R\dot{v}_{O_6,y} \end{bmatrix}_{ik} = \begin{bmatrix} [\tilde{\omega}_1][R_1]^T \end{bmatrix}_{ij} \quad (i = 1, 2, 3; j = 1, 2, 3; k = 39 + j) \right.$$

$$\begin{bmatrix} {}^R\dot{v}_{O_6,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} {}^R\dot{v}_{O_6,\hat{\omega}_1} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\dot{v}_{O_6,\hat{\omega}_1} \end{bmatrix}_{3 \times 3} = - \left([\dot{\tilde{q}}_6] + [\dot{\tilde{s}}_6] \right)$$

$$\begin{bmatrix} {}^R\dot{v}_{O_6,y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [\tilde{\omega}_1][R_1]^T & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 7:

$$1. \left\{ \begin{bmatrix} {}^R\dot{v}_{O_7,y} \end{bmatrix}_{3 \times 48} = \underbrace{\begin{bmatrix} {}^R\dot{v}_{O_1,y} \end{bmatrix}_{3 \times 48}}_{\text{zero}} - ([\tilde{q}_7] + [\tilde{s}_7]) \underbrace{\begin{bmatrix} {}^R\dot{\omega}_{1,y} \end{bmatrix}_{3 \times 48}}_{\text{zero}} - ([\dot{\tilde{q}}_7] + [\dot{\tilde{s}}_7]) \begin{bmatrix} {}^R\omega_{1,y} \end{bmatrix}_{3 \times 48} \right.$$

$$2. \left\{ \begin{bmatrix} {}^R\dot{v}_{O_7,y} \end{bmatrix}_{ik} = \begin{bmatrix} [\tilde{\omega}_1][R_1]^T \end{bmatrix}_{ij} \quad (i = 1, 2, 3; j = 1, 2, 3; k = 42 + j) \right.$$

$$\begin{bmatrix} {}^R\dot{v}_{O_7,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} {}^R\dot{v}_{O_7,\hat{\omega}_1} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\dot{v}_{O_7,\hat{\omega}_1} \end{bmatrix}_{3 \times 3} = - \left([\dot{\tilde{q}}_7] + [\dot{\tilde{s}}_7] \right)$$

$$\begin{bmatrix} {}^R\dot{v}_{O_7,y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [\tilde{\omega}_1][R_1]^T & [0]_{3 \times 3} \end{bmatrix}$$

Body 8:

$$\begin{aligned}
 1. & \left\{ \begin{aligned} \left[\dot{R} \dot{v}_{O_8,y} \right]_{3 \times 48} &= \left[\dot{R} \dot{v}_{O_7,y} \right]_{3 \times 48} - \left([\ddot{q}_8] + [\ddot{s}_8] \right) \left[\dot{R} \dot{\omega}_{7,y} \right]_{3 \times 48} - \left([\dot{\ddot{q}}_8] + [\dot{\ddot{s}}_8] \right) \left[{}^R \omega_{7,y} \right]_{3 \times 48} \end{aligned} \right. \\
 2. & \left\{ \begin{aligned} \left[\dot{R} \dot{v}_{O_8,y} \right]_{ik} &= \left[\tilde{\omega}_7 \right] \left[R_7 \right]^T \bigg|_{ij} \quad (i=1,2,3; j=1,2,3; k=45+j) \\ \left[\dot{R} \dot{v}_{O_8,y_1} \right]_{3 \times 24} &= \left[\begin{array}{cccccc} \left[\dot{R} \dot{v}_{O_8,\hat{\omega}_1} \right]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & \left[\dot{R} \dot{v}_{O_8,\hat{\omega}_7} \right]_{3 \times 3} & [0]_{3 \times 3} \end{array} \right] \\ \left[\dot{R} \dot{v}_{O_8,\hat{\omega}_1} \right]_{3 \times 3} &= - \left[[\dot{\ddot{q}}_7] + [\dot{\ddot{s}}_7] + [\dot{\ddot{q}}_8] + [\dot{\ddot{s}}_8] \right] \quad \left[\dot{R} \dot{v}_{O_8,\hat{\omega}_7} \right]_{3 \times 3} = - \left[[\dot{\ddot{q}}_8] + [\dot{\ddot{s}}_8] \right] \left[R_1 \right]^T - \left[[\ddot{q}_8] + [\ddot{s}_8] \right] \left[\tilde{\omega}_1 \right] \left[R_1 \right]^T \\ \left[\dot{R} \dot{v}_{O_8,y_2} \right]_{3 \times 24} &= \left[\begin{array}{cccccc} [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & \left[\tilde{\omega}_1 \right] \left[R_1 \right]^T & \left[\tilde{\omega}_7 \right] \left[R_7 \right]^T \end{array} \right] \end{aligned} \right.
 \end{aligned}$$

Time Derivatives of **Partial Velocities** of the **Mass Centers**:

Body 1:

$$\begin{aligned}
 1. & \left\{ \begin{aligned} \left[\dot{R} \dot{v}_{1,y} \right]_{3 \times 48} &= \underbrace{\left[\dot{R} \dot{v}_{O_1,y} \right]_{3 \times 48}}_{\text{zero}} - \underbrace{\left[\tilde{r}_1 \right] \left[\dot{R} \dot{\omega}_{1,y} \right]_{3 \times 48}}_{\text{zero}} - \left[\dot{\tilde{r}}_1 \right] \left[{}^R \omega_{1,y} \right]_{3 \times 48} \\ \left[\dot{R} \dot{v}_{1,y_1} \right]_{3 \times 24} &= \left[\begin{array}{cccccc} \left[\dot{R} \dot{v}_{1,\hat{\omega}_1} \right]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{array} \right] \\ \left[\dot{R} \dot{v}_{1,\hat{\omega}_1} \right]_{3 \times 3} &= - \left[\dot{\tilde{r}}_1 \right]_{3 \times 3} \quad \left[\dot{R} \dot{v}_{1,y_2} \right]_{3 \times 24} = [0]_{3 \times 24} \end{aligned} \right.
 \end{aligned}$$

Body 2:

$$\begin{aligned}
 1. & \left\{ \begin{aligned} \left[\dot{R} \dot{v}_{2,y} \right]_{3 \times 48} &= \left[\dot{R} \dot{v}_{O_2,y} \right]_{3 \times 48} - \left[\tilde{r}_2 \right] \left[\dot{R} \dot{\omega}_{2,y} \right]_{3 \times 48} - \left[\dot{\tilde{r}}_2 \right] \left[{}^R \omega_{2,y} \right]_{3 \times 48} \\ \left[\dot{R} \dot{v}_{2,y_1} \right]_{3 \times 24} &= \left[\begin{array}{cccccc} \left[\dot{R} \dot{v}_{2,\hat{\omega}_1} \right]_{3 \times 3} & \left[\dot{R} \dot{v}_{2,\hat{\omega}_2} \right]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{array} \right] \\ \left[\dot{R} \dot{v}_{2,\hat{\omega}_1} \right]_{3 \times 3} &= - \left[[\dot{\ddot{q}}_2] + [\dot{\ddot{s}}_2] + [\dot{\ddot{r}}_2] \right] \quad \left[\dot{R} \dot{v}_{2,\hat{\omega}_2} \right]_{3 \times 3} = - \left[\dot{\tilde{r}}_2 \right] \left[R_1 \right]^T - \left[\tilde{r}_2 \right] \left[\tilde{\omega}_1 \right] \left[R_1 \right]^T \\ \left[\dot{R} \dot{v}_{2,y_2} \right]_{3 \times 24} &= \left[\begin{array}{cccccc} [0]_{3 \times 3} & \left[\tilde{\omega}_1 \right] \left[R_1 \right]^T & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{array} \right] \end{aligned} \right.
 \end{aligned}$$

Body 3:

$$\begin{aligned}
 1. & \left\{ \begin{aligned} \left[\dot{R} \dot{v}_{3,y} \right]_{3 \times 48} &= \left[\dot{R} \dot{v}_{O_3,y} \right]_{3 \times 48} - \left[\tilde{r}_3 \right] \left[\dot{R} \dot{\omega}_{3,y} \right]_{3 \times 48} - \left[\dot{\tilde{r}}_3 \right] \left[{}^R \omega_{3,y} \right]_{3 \times 48} \\ \left[\dot{R} \dot{v}_{3,y_1} \right]_{3 \times 24} &= \left[\begin{array}{cccccc} \left[\dot{R} \dot{v}_{3,\hat{\omega}_1} \right]_{3 \times 3} & \left[\dot{R} \dot{v}_{3,\hat{\omega}_2} \right]_{3 \times 3} & \left[\dot{R} \dot{v}_{3,\hat{\omega}_3} \right]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{array} \right] \\ \left[\dot{R} \dot{v}_{3,\hat{\omega}_1} \right]_{3 \times 3} &= - \left[[\dot{\ddot{q}}_2] + [\dot{\ddot{s}}_2] + [\dot{\ddot{q}}_3] + [\dot{\ddot{s}}_3] + [\dot{\ddot{r}}_3] \right] \\ \left[\dot{R} \dot{v}_{3,\hat{\omega}_2} \right]_{3 \times 3} &= - \left[[\dot{\ddot{q}}_3] + [\dot{\ddot{s}}_3] + [\dot{\ddot{r}}_3] \right] \left[R_1 \right]^T - \left[[\ddot{q}_3] + [\ddot{s}_3] + [\ddot{r}_3] \right] \left[\tilde{\omega}_1 \right] \left[R_1 \right]^T \\ \left[\dot{R} \dot{v}_{3,\hat{\omega}_3} \right]_{3 \times 3} &= - \left[\dot{\tilde{r}}_3 \right] \left[R_2 \right]^T - \left[\tilde{r}_3 \right] \left[\tilde{\omega}_2 \right] \left[R_2 \right]^T \\ \left[\dot{R} \dot{v}_{3,y_2} \right]_{3 \times 24} &= \left[\begin{array}{cccccc} [0]_{3 \times 3} & \left[\tilde{\omega}_1 \right] \left[R_1 \right]^T & \left[\tilde{\omega}_2 \right] \left[R_2 \right]^T & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{array} \right] \end{aligned} \right.
 \end{aligned}$$

Body 4:

$$1. \left\{ \begin{bmatrix} R \dot{v}_{4,y} \end{bmatrix} \right\}_{3 \times 48} = \begin{bmatrix} R \dot{v}_{O_4,y} \end{bmatrix}_{3 \times 48} - [\tilde{r}_4] \begin{bmatrix} R \dot{\omega}_{4,y} \end{bmatrix}_{3 \times 48} - [\dot{\tilde{r}}_4] \begin{bmatrix} R \omega_{4,y} \end{bmatrix}_{3 \times 48}$$

$$\begin{bmatrix} R\dot{v}_{4,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} R\dot{v}_{4,\hat{w}_1} \end{bmatrix}_{3 \times 3} \begin{bmatrix} R\dot{v}_{4,\hat{w}_2} \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \begin{bmatrix} R\dot{v}_{4,\hat{w}_4} \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3}$$

$$\left[\begin{matrix} R\dot{V}_{4,\hat{\omega}_1} \end{matrix} \right]_{3 \times 3} = - \left[\begin{matrix} \ddot{q}_2 \end{matrix} \right] + \left[\begin{matrix} \ddot{s}_2 \end{matrix} \right] + \left[\begin{matrix} \ddot{q}_4 \end{matrix} \right] + \left[\begin{matrix} \ddot{s}_4 \end{matrix} \right] + \left[\begin{matrix} \ddot{r}_4 \end{matrix} \right]$$

$$\begin{bmatrix} {}^R\mathbf{v}_{4,\hat{\omega}_2} \end{bmatrix}_{3 \times 3} = - \left[\begin{bmatrix} \dot{\tilde{q}}_4 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{s}}_4 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{r}}_4 \end{bmatrix} \right] [R_1]^T - \left[\begin{bmatrix} \tilde{q}_4 \end{bmatrix} + \begin{bmatrix} \tilde{s}_4 \end{bmatrix} + \begin{bmatrix} \tilde{r}_4 \end{bmatrix} \right] [\tilde{\omega}_1] [R_1]^T$$

$$\left[\begin{matrix} R \dot{v}_{4, \hat{w}_4} \end{matrix} \right]_{3 \times 3} = - \left[\dot{\tilde{r}}_4 \right] [R_2]^T - [\tilde{r}_4][\tilde{w}_2][R_2]^T$$

$$\begin{bmatrix} R\dot{v}_{4,y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [0]_{3 \times 3} & [\tilde{a}_1][R_1]^T & [0]_{3 \times 3} & [\tilde{a}_2][R_2]^T & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 5:

$$1. \left\{ \begin{bmatrix} {}^R \dot{v}_{5,y} \end{bmatrix} \right\}_{3 \times 48} = \begin{bmatrix} {}^R \dot{v}_{O_5,y} \end{bmatrix}_{3 \times 48} - [\tilde{r}_5] \begin{bmatrix} {}^R \dot{\omega}_{5,y} \end{bmatrix}_{3 \times 48} - [\dot{\tilde{r}}_5] \begin{bmatrix} {}^R \omega_{5,y} \end{bmatrix}_{3 \times 48}$$

$$\begin{bmatrix} R\dot{v}_{5,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} R\dot{v}_{5,\hat{w}_1} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & \begin{bmatrix} R\dot{v}_{5,\hat{w}_5} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\dot{v}_{5,\hat{\omega}_1} \end{bmatrix}_{3 \times 3} = - \left[\begin{bmatrix} \ddot{q}_5 \end{bmatrix} + \begin{bmatrix} \ddot{s}_5 \end{bmatrix} + \begin{bmatrix} \ddot{r}_5 \end{bmatrix} \right] \quad \begin{bmatrix} {}^R\dot{v}_{5,\hat{\omega}_5} \end{bmatrix}_{3 \times 3} = - \begin{bmatrix} \ddot{r}_5 \end{bmatrix} [R_1]^T - \begin{bmatrix} \ddot{r}_5 \end{bmatrix} [\tilde{\omega}_1] [R_1]^T$$

$$\begin{bmatrix} {}^R\dot{v}_{5,y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [\tilde{w}_1][R_1]^T & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 6:

$$1. \left\{ \begin{bmatrix} R \dot{v}_{6,y} \end{bmatrix} \right\}_{3 \times 48} = \begin{bmatrix} R \dot{v}_{O_6,y} \end{bmatrix}_{3 \times 48} - [\tilde{r}_6] \begin{bmatrix} R \dot{\omega}_{6,y} \end{bmatrix}_{3 \times 48} - [\dot{\tilde{r}}_6] \begin{bmatrix} R \omega_{6,y} \end{bmatrix}_{3 \times 48}$$

$$\begin{bmatrix} R\dot{v}_{6,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} \begin{bmatrix} R\dot{v}_{6,\hat{w}_1} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} & \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} & \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} & \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} & \begin{bmatrix} R\dot{v}_{6,\hat{w}_6} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} & \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{6,\hat{\omega}_1} \end{bmatrix}_{3 \times 3} = - \left[\begin{bmatrix} \dot{\tilde{\mathbf{q}}}_6 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{\mathbf{s}}}_6 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{\mathbf{r}}}_6 \end{bmatrix} \right] \quad \begin{bmatrix} {}^R\dot{\mathbf{v}}_{6,\hat{\omega}_6} \end{bmatrix}_{3 \times 3} = - \begin{bmatrix} \dot{\tilde{\mathbf{r}}}_6 \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 \end{bmatrix}^T - \begin{bmatrix} \dot{\tilde{\mathbf{r}}}_6 \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{\omega}}_1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 \end{bmatrix}^T$$

$$\begin{bmatrix} R\dot{v}_{6,y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [\tilde{a}_1][R_1]^T & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 7:

$$1. \left\{ \begin{bmatrix} R \dot{v}_{7,y} \end{bmatrix} \right\}_{3 \times 48} = \begin{bmatrix} R \dot{v}_{O_7,y} \end{bmatrix}_{3 \times 48} - [\tilde{r}_7] \begin{bmatrix} R \dot{\omega}_{7,y} \end{bmatrix}_{3 \times 48} - [\dot{\tilde{r}}_7] \begin{bmatrix} R \omega_{7,y} \end{bmatrix}_{3 \times 48}$$

$$\begin{bmatrix} R\dot{v}_{7,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} R\dot{v}_{7,\hat{w}_1} \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \begin{bmatrix} R\dot{v}_{7,\hat{w}_7} \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3}$$

$$\begin{aligned} \left[\begin{matrix} R \dot{v}_{7, \hat{\omega}_1} \end{matrix} \right]_{3 \times 3} &= - \left[\begin{matrix} \dot{\tilde{q}}_7 \end{matrix} \right] + \left[\begin{matrix} \dot{\tilde{S}}_7 \end{matrix} \right] + \left[\begin{matrix} \dot{\tilde{r}}_7 \end{matrix} \right] \\ \left[\begin{matrix} R \dot{v}_{7, \hat{\omega}_7} \end{matrix} \right]_{3 \times 3} &= - \left[\begin{matrix} \dot{\tilde{r}}_7 \end{matrix} \right] [R_1]^T - [\tilde{r}_7] [\tilde{\omega}_1] [R_1]^T \end{aligned}$$

$$\begin{bmatrix} R\dot{v}_{7,y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [\tilde{a}_1][R_1]^T & [0]_{3 \times 3} \end{bmatrix}$$

Body 8:

$$1. \left\{ \begin{bmatrix} {}^R \dot{v}_{8,y} \end{bmatrix} \right\}_{3 \times 48} = \begin{bmatrix} {}^R \dot{v}_{O_8,y} \end{bmatrix}_{3 \times 48} - [\tilde{r}_8] \begin{bmatrix} {}^R \dot{\omega}_{8,y} \end{bmatrix}_{3 \times 48} - [\dot{\tilde{r}}_8] \begin{bmatrix} {}^R \omega_{8,y} \end{bmatrix}_{3 \times 48}$$

$$\begin{bmatrix} R\dot{v}_{8,y_1} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} R\dot{v}_{8,\hat{\omega}_1} \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \begin{bmatrix} R\dot{v}_{8,\hat{\omega}_7} \end{bmatrix}_{3 \times 3} \begin{bmatrix} R\dot{v}_{8,\hat{\omega}_8} \end{bmatrix}_{3 \times 3}$$

$$\begin{bmatrix} {}^R\dot{v}_{8,\hat{\omega}_1} \end{bmatrix}_{3 \times 3} = - \left[\begin{bmatrix} \dot{\tilde{q}}_7 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{s}}_7 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{q}}_8 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{s}}_8 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{r}}_8 \end{bmatrix} \right]$$

$$\begin{bmatrix} {}^R\dot{v}_{8,\hat{\omega}_7} \end{bmatrix}_{3 \times 3} = - \left[\begin{bmatrix} \dot{\tilde{q}}_8 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{s}}_8 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{r}}_8 \end{bmatrix} \right] [R_1]^T - \left[\begin{bmatrix} \tilde{q}_8 \end{bmatrix} + \begin{bmatrix} \tilde{s}_8 \end{bmatrix} + \begin{bmatrix} \tilde{r}_8 \end{bmatrix} \right] [\tilde{\omega}_1] [R_1]^T$$

$$\begin{bmatrix} {}^R\dot{v}_{8,\hat{\omega}_8} \end{bmatrix}_{3 \times 3} = - \begin{bmatrix} \dot{\tilde{r}}_8 \end{bmatrix} [R_7]^T - \begin{bmatrix} \tilde{r}_8 \end{bmatrix} [\tilde{\omega}_7] [R_7]^T$$

$$\begin{bmatrix} {}^R\dot{v}_{8,y_2} \end{bmatrix}_{3 \times 24} = \begin{bmatrix} [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [\tilde{\omega}_1] [R_1]^T & [\tilde{\omega}_7] [R_7]^T \end{bmatrix}$$

Acceleration of G_3 :

The acceleration of G_3 can be calculated as follows.

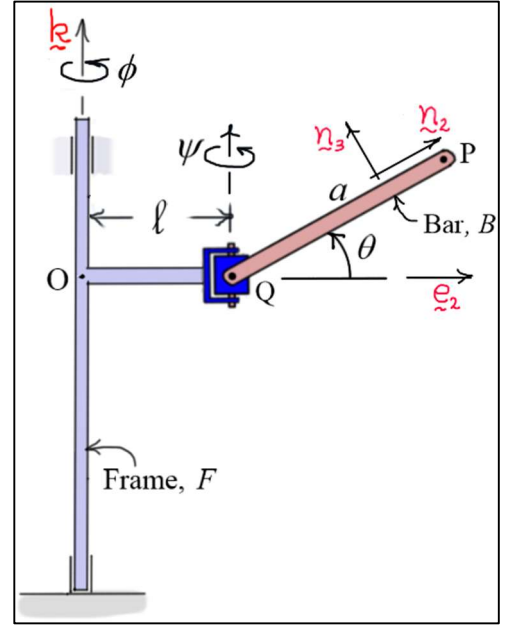
$$\begin{aligned} \{a_3\} &= \begin{bmatrix} {}^Rv_{3,y} \end{bmatrix} \{y\} + \begin{bmatrix} {}^R\dot{v}_{3,y} \end{bmatrix} \{y\} \\ &= \begin{bmatrix} {}^Rv_{3,y_1} \end{bmatrix} \{y_1\} + \begin{bmatrix} {}^Rv_{3,y_2} \end{bmatrix} \{y_2\} + \begin{bmatrix} {}^R\dot{v}_{3,y_1} \end{bmatrix} \{y_1\} + \begin{bmatrix} {}^R\dot{v}_{3,y_2} \end{bmatrix} \{y_2\} \\ &= \begin{bmatrix} \begin{bmatrix} {}^Rv_{3,\hat{\omega}_1} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} {}^Rv_{3,\hat{\omega}_2} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} {}^Rv_{3,\hat{\omega}_3} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix} \{y_1\} \\ &\quad + \begin{bmatrix} [I]_{3 \times 3} & [R_1]^T_{3 \times 3} & [R_2]^T_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix} \{y_2\} \\ &\quad + \begin{bmatrix} \begin{bmatrix} {}^R\dot{v}_{3,\hat{\omega}_1} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} {}^R\dot{v}_{3,\hat{\omega}_2} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} {}^R\dot{v}_{3,\hat{\omega}_3} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix} \{y_1\} \\ &\quad + \begin{bmatrix} [0]_{3 \times 3} & [\tilde{\omega}_1] [R_1]^T_{3 \times 3} & [\tilde{\omega}_2] [R_2]^T_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix} \{y_2\} \\ &= \begin{bmatrix} {}^Rv_{3,\hat{\omega}_1} \end{bmatrix} \{\dot{\hat{\omega}}_1\} + \begin{bmatrix} {}^Rv_{3,\hat{\omega}_2} \end{bmatrix} \{\dot{\hat{\omega}}_2\} + \begin{bmatrix} {}^Rv_{3,\hat{\omega}_3} \end{bmatrix} \{\dot{\hat{\omega}}_3\} + \{\dot{s}'_1\} + [R_1]^T \{\dot{s}'_2\} + [R_2]^T \{\dot{s}'_3\} \\ &\quad + \begin{bmatrix} {}^R\dot{v}_{3,\hat{\omega}_1} \end{bmatrix} \{\hat{\omega}_1\} + \begin{bmatrix} {}^R\dot{v}_{3,\hat{\omega}_2} \end{bmatrix} \{\hat{\omega}_2\} + \begin{bmatrix} {}^R\dot{v}_{3,\hat{\omega}_3} \end{bmatrix} \{\hat{\omega}_3\} + [\tilde{\omega}_1] [R_1]^T \{\dot{s}'_2\} + [\tilde{\omega}_2] [R_2]^T \{\dot{s}'_3\} \end{aligned}$$

$$\begin{aligned} \{a_3\} &= - \left[\begin{bmatrix} \tilde{q}_2 \end{bmatrix} + \begin{bmatrix} \tilde{s}_2 \end{bmatrix} + \begin{bmatrix} \tilde{q}_3 \end{bmatrix} + \begin{bmatrix} \tilde{s}_3 \end{bmatrix} + \begin{bmatrix} \tilde{r}_3 \end{bmatrix} \right] \{\dot{\hat{\omega}}_1\} - \left[\begin{bmatrix} \tilde{q}_3 \end{bmatrix} + \begin{bmatrix} \tilde{s}_3 \end{bmatrix} + \begin{bmatrix} \tilde{r}_3 \end{bmatrix} \right] [R_1]^T \{\dot{\hat{\omega}}_2\} - \begin{bmatrix} \tilde{r}_3 \end{bmatrix} [R_2]^T \{\dot{\hat{\omega}}_3\} \\ &\quad - \left[\begin{bmatrix} \dot{\tilde{q}}_2 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{s}}_2 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{q}}_3 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{s}}_3 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{r}}_3 \end{bmatrix} \right] \{\hat{\omega}_1\} - \left[\begin{bmatrix} \dot{\tilde{q}}_3 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{s}}_3 \end{bmatrix} + \begin{bmatrix} \dot{\tilde{r}}_3 \end{bmatrix} \right] [R_1]^T \{\hat{\omega}_2\} \\ &\quad - \left[\begin{bmatrix} \tilde{q}_3 \end{bmatrix} + \begin{bmatrix} \tilde{s}_3 \end{bmatrix} + \begin{bmatrix} \tilde{r}_3 \end{bmatrix} \right] [\tilde{\omega}_1] [R_1]^T \{\hat{\omega}_2\} - \begin{bmatrix} \dot{\tilde{r}}_3 \end{bmatrix} [R_2]^T \{\hat{\omega}_3\} - \begin{bmatrix} \tilde{r}_3 \end{bmatrix} [\tilde{\omega}_2] [R_2]^T \{\hat{\omega}_3\} \\ &\quad + \{\dot{s}'_1\} + [R_1]^T \{\dot{s}'_2\} + [R_2]^T \{\dot{s}'_3\} + [\tilde{\omega}_1] [R_1]^T \{\dot{s}'_2\} + [\tilde{\omega}_2] [R_2]^T \{\dot{s}'_3\} \end{aligned}$$

This result can *also* be *found* by other more *direct methods* such as *direct differentiation*.

Exercises

5.1 The diagram shows a **two-body** system with **frame** F and **bar** B . Frame F rotates ($\dot{\phi} = \Omega$) about a fixed vertical direction annotated by the unit vector $\hat{k}$, and bar B is attached to F with a joint allowing rotations about **two axes**. Rotation about the first axis is measured by the angle ψ , and rotation about the second axis is measured by the angle θ . The angles ϕ and ψ are both **zero** in the position shown. B is oriented with respect to F using a 3-1 rotation sequence. Three reference frames are defined for the system as follows.



Reference Frames:

$R: (\hat{i}, \hat{j}, \hat{k})$ fixed frame

$F: (\hat{e}_1, \hat{e}_2, \hat{k})$ rotating with Frame, F

$B: (\hat{n}_1, \hat{n}_2, \hat{n}_3)$ rotating with Bar, B

Let $\{\hat{\omega}'_F\}$ and $\{\hat{\omega}'_B\}$ be the **body frame components** of the **relative angular velocities** ${}^R\omega_F$ and ${}^F\omega_B$. Find the **fixed frame components** of the **time derivatives** of the **partial velocity matrices** and **acceleration** of point P using **body frame components** of the relative angular velocity vectors. Use results from Example 2 of Unit 4 for the angular velocities, partial angular velocity matrices, velocities, and partial velocity matrices.

Define the column vector of **generalized speeds** to be $\{y\} \triangleq \begin{Bmatrix} \{\hat{\omega}'_F\} \\ \{\hat{\omega}'_B\} \end{Bmatrix}$.

Answers:

$$\{\omega'_F\} = \{\hat{\omega}'_F\} = \{\hat{\omega}_F\} = \begin{bmatrix} 0 & 0 & \dot{\phi} \end{bmatrix}^T \quad \{\hat{\omega}'_B\} = \begin{bmatrix} \dot{\theta} & \dot{\psi} S_\theta & \dot{\psi} C_\theta \end{bmatrix}^T \quad \{\omega'_B\} = \begin{bmatrix} \dot{\theta} & (\dot{\phi} + \dot{\psi}) S_\theta & (\dot{\phi} + \dot{\psi}) C_\theta \end{bmatrix}^T$$

$$\begin{bmatrix} {}^R v_{P, \hat{\omega}'_F} \end{bmatrix} = -[R_F]^T [\tilde{r}'_{Q/O}] - [R_B]^T [\tilde{r}'_{P/Q}] [{}^F R_B] = \begin{bmatrix} a S_\theta S_\phi & a S_\theta C_\phi & -(\ell C_\phi + a C_\theta C_{\phi+\psi}) \\ -a S_\theta C_\phi & a S_\theta S_\phi & -(\ell S_\phi + a C_\theta S_{\phi+\psi}) \\ \ell + a C_\psi C_\theta & a S_\psi C_\theta & 0 \end{bmatrix}$$

$$\begin{bmatrix} {}^R v_{P, \hat{\omega}'_B} \end{bmatrix} = -[R_B]^T [\tilde{r}'_{P/Q}] = \begin{bmatrix} a S_{\phi+\psi} S_\theta & 0 & -a C_{\phi+\psi} \\ -a C_{\phi+\psi} S_\theta & 0 & -a S_{\phi+\psi} \\ a C_\theta & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} R_F \end{bmatrix}^T = \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} R_B \end{bmatrix}^T = \begin{bmatrix} C_{\phi+\psi} & -S_{\phi+\psi} C_\theta & S_{\phi+\psi} S_\theta \\ S_{\phi+\psi} & C_{\phi+\psi} C_\theta & -C_{\phi+\psi} S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \quad \begin{bmatrix} {}^F R_B \end{bmatrix} = \begin{bmatrix} C_\psi & S_\psi & 0 \\ -S_\psi C_\theta & C_\psi C_\theta & S_\theta \\ S_\psi S_\theta & -C_\psi S_\theta & C_\theta \end{bmatrix}$$

$$\begin{bmatrix} \tilde{\mathbf{r}}'_{Q/O} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \ell \\ 0 & 0 & 0 \\ -\ell & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \tilde{\mathbf{r}}'_{P/Q} \end{bmatrix} = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ -a & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} {}^R \dot{\mathbf{v}}_{P, \hat{\omega}'_F} \end{bmatrix} &= -[R_F]^T [\tilde{\omega}'_F] [\tilde{\mathbf{r}}'_{Q/O}] - [R_B]^T [\tilde{\omega}'_B] [\tilde{\mathbf{r}}'_{P/Q}] [{}^F R_B] - [R_B]^T [\tilde{\mathbf{r}}'_{P/Q}] [{}^F \tilde{\omega}'_B]^T [{}^F R_B] \\ &= \begin{bmatrix} aS_\theta C_\phi \dot{\phi} + aC_\theta S_\phi \dot{\theta} & -aS_\theta S_\phi \dot{\phi} + aC_\theta C_\phi \dot{\theta} & \ell S_\phi \dot{\phi} + aS_{\phi+\psi} C_\theta (\dot{\phi} + \dot{\psi}) + aC_{\phi+\psi} S_\theta \dot{\theta} \\ aS_\theta S_\phi \dot{\phi} - aC_\theta C_\phi \dot{\theta} & aS_\theta C_\phi \dot{\phi} + aC_\theta S_\phi \dot{\theta} & -\ell C_\phi \dot{\phi} - aC_{\phi+\psi} C_\theta (\dot{\phi} + \dot{\psi}) + aS_{\phi+\psi} S_\theta \dot{\theta} \\ -aC_\psi S_\theta \dot{\theta} - aS_\psi C_\theta \dot{\psi} & -aS_\psi S_\theta \dot{\theta} + aC_\psi C_\theta \dot{\psi} & 0 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} {}^R \dot{\mathbf{v}}_{P, \hat{\omega}'_B} \end{bmatrix} = -[R_B]^T [\tilde{\omega}'_B] [\tilde{\mathbf{r}}'_{P/Q}] = \begin{bmatrix} aC_{\phi+\psi} S_\theta (\dot{\phi} + \dot{\psi}) + aS_{\phi+\psi} C_\theta \dot{\theta} & 0 & aS_{\phi+\psi} (\dot{\phi} + \dot{\psi}) \\ aS_{\phi+\psi} S_\theta (\dot{\phi} + \dot{\psi}) - aC_{\phi+\psi} C_\theta \dot{\theta} & 0 & -aC_{\phi+\psi} (\dot{\phi} + \dot{\psi}) \\ -aS_\theta \dot{\theta} & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \{a_P\} &= [{}^R \mathbf{v}_{P, \hat{\omega}'_F}] \{\dot{\omega}'_F\} + [{}^R \mathbf{v}_{P, \hat{\omega}'_B}] \{\dot{\omega}'_B\} + [{}^R \dot{\mathbf{v}}_{P, \hat{\omega}'_F}] \{\omega'_F\} + [{}^R \dot{\mathbf{v}}_{P, \hat{\omega}'_B}] \{\omega'_B\} \\ &= \begin{bmatrix} -\ell C_\phi - aC_\theta C_{\phi+\psi} \\ -\ell S_\phi - aC_\theta S_{\phi+\psi} \\ 0 \end{bmatrix} \ddot{\phi} + \begin{bmatrix} -aC_{\phi+\psi} C_\theta \\ -aS_{\phi+\psi} C_\theta \\ 0 \end{bmatrix} \ddot{\psi} + \begin{bmatrix} aS_{\phi+\psi} S_\theta \\ -aC_{\phi+\psi} S_\theta \\ aC_\theta \end{bmatrix} \ddot{\theta} + \begin{bmatrix} \ell S_\phi + aS_{\phi+\psi} C_\theta \\ -\ell C_\phi - aC_{\phi+\psi} C_\theta \\ 0 \end{bmatrix} \dot{\phi}^2 + \begin{bmatrix} aS_{\phi+\psi} C_\theta \\ -aC_{\phi+\psi} C_\theta \\ 0 \end{bmatrix} \dot{\psi}^2 \\ &\quad + \begin{bmatrix} aS_{\phi+\psi} C_\theta \\ -aC_{\phi+\psi} C_\theta \\ -aS_\theta \end{bmatrix} \dot{\theta}^2 + \begin{bmatrix} 2aS_{\phi+\psi} C_\theta \\ -2aC_{\phi+\psi} C_\theta \\ 0 \end{bmatrix} \dot{\phi} \dot{\psi} + \begin{bmatrix} 2aC_{\phi+\psi} S_\theta \\ 2aS_{\phi+\psi} S_\theta \\ 0 \end{bmatrix} \dot{\phi} \dot{\theta} + \begin{bmatrix} 2aC_{\phi+\psi} S_\theta \\ 2aS_{\phi+\psi} S_\theta \\ 0 \end{bmatrix} \dot{\psi} \dot{\theta} \end{aligned}$$

This result for $\{a_P\}$ is **identical** to the result in Example 2.

5.2 Write a MATLAB script to **numerically** evaluate the **matrix equations** you derived in Exercise 5.1 using the data below. First calculate the **partial angular velocity matrices** and the **angular velocity components**. Then calculate the **partial velocity matrices** and the **velocity components** of point P . Use the results of Unit 4 as a basis for these calculations. Finally, calculate the **time derivatives** of the **partial velocity matrices** and the **acceleration components** of point P .

Data:

$$\begin{aligned} \ell &= 19 \text{ (in)} & a &= 11 \text{ (in)} \\ \phi &= 0 & \psi &= 20 \text{ (deg)} & \theta &= 30 \text{ (deg)} \\ \dot{\phi} &= 1.5 \text{ (rad/s)} & \dot{\psi} &= 2 \text{ (rad/s)} & \dot{\theta} &= -0.5 \text{ (rad/s)} \\ \ddot{\phi} &= 3 \text{ (rad/s}^2\text{)} & \ddot{\psi} &= 5 \text{ (rad/s}^2\text{)} & \ddot{\theta} &= -1.5 \text{ (rad/s}^2\text{)} \end{aligned}$$

Results:

Body frame components:

$$\begin{bmatrix} {}^R \omega'_{F,y} \end{bmatrix}_{3 \times 6} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

$$\left[{}^R \omega'_{B,y} \right]_{3 \times 6} = \begin{bmatrix} 0.939693 & 0.342020 & 0 \\ -0.296198 & 0.813798 & 0.500000 \\ 0.171010 & -0.469846 & 0.866025 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\{\omega'_F\} = [0 \ 0 \ 1.5]^T \text{ (r/s)}$$

$$\{\omega'_B\} = [-0.500000 \ 1.75000 \ 3.03109]^T \text{ (r/s)}$$

Fixed frame components:

$$\left[{}^R v_{P,y} \right]_{3 \times 6} = \begin{bmatrix} 0 & 0.458333 & -2.32931 \\ -0.458333 & 0 & -0.271515 \\ 2.32931 & 0.271515 & 0 \end{bmatrix} \begin{bmatrix} 0.156759 & 0 & -0.861385 \\ -0.430692 & 0 & -0.313518 \\ 0.793857 & 0 & 0 \end{bmatrix}$$

$$\{v_P\} = [-5.06431 \ -0.734956 \ -0.396928]^T \text{ (ft/s)}$$

$$\left[{}^R \dot{v}_{P,y} \right]_{3 \times 6} = \begin{bmatrix} 0.687500 & -0.396928 & 0.734956 \\ 0.396928 & 0.687500 & -5.06431 \\ -0.327684 & 1.57034 & 0 \end{bmatrix} \begin{bmatrix} 1.37167 & 0 & 1.09731 \\ 0.921648 & 0 & -3.01485 \\ 0.229167 & 0 & 0 \end{bmatrix}$$

$$\{a_P\} = [-9.06648 \ -14.9620 \ -1.30537]^T \text{ (ft/s}^2\text{)}$$

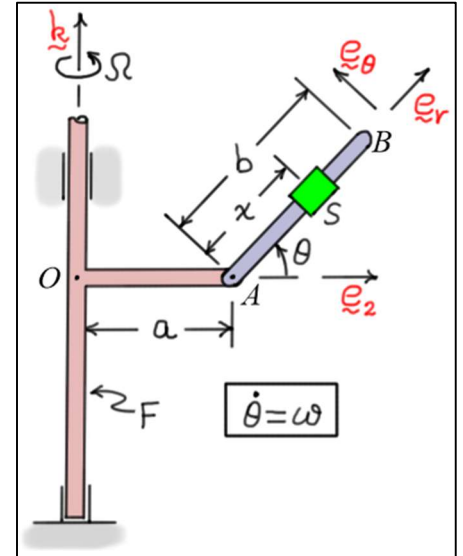
5.3 The figure shows a *three-body* system with *frame F*, *arm AB*, and *slider S*. Frame *F* rotates about the *fixed vertical direction* with rate $\dot{\phi} = \Omega$, and the arm *AB* rotates *relative* to *F* with rate $\dot{\theta} = \omega$. Slider *S* is connected to bar *AB* with a *prismatic joint*, meaning it is free slide along *AB* but not rotate relative to it. Its *position* relative to point *A* is given by the *variable distance* *x*.

Reference frames: (all frames align when $\phi = \theta = 0$)

$R : (\underline{i}, \underline{j}, \underline{k})$ (fixed frame)

$F : (\underline{e}_1, \underline{e}_2, \underline{k})$ (rotating with frame *F*)

$AB : (\underline{e}_1, \underline{e}_r, \underline{e}_\theta)$ (rotating with bar *AB*)



Let $\{\hat{\omega}'_F\}$ and $\{\hat{\omega}'_{AB}\}$ be the *body frame components* of the *relative angular velocities* ${}^R \omega_F$ and ${}^F \omega_{AB}$.

a) Find the *fixed frame components* of the *time derivatives* of the *partial velocity matrices* and *acceleration* of slider *S* associated with the *body frame components* of the relative angular velocity vectors. b) Find the *body frame components* of the *time derivatives* of the *partial velocity matrices* and *acceleration* of slider *S* associated with the *body frame components* of the relative angular velocity vectors. Use results from Example 3 of Unit 4 for the angular velocities, partial angular velocity matrices, velocities, and partial velocity matrices.

Answers:

$$\text{a) } \left\{ \hat{\omega}'_F \right\} = [0 \ 0 \ \Omega]^T \quad \left\{ \hat{\omega}'_{AB} \right\} = \left\{ \hat{\omega}'_S \right\} = [\omega \ 0 \ 0]^T \quad \left\{ \omega'_S \right\} = \left\{ \omega'_{AB} \right\} = \begin{bmatrix} \omega & \Omega S_\theta & \Omega C_\theta \end{bmatrix}^T$$

$$\begin{bmatrix} \tilde{r}'_{A/O} \end{bmatrix} = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ -a & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \tilde{r}'_{S/A} \end{bmatrix} = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ -x & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} R_F \end{bmatrix}^T = \begin{bmatrix} C_\phi & -S_\phi & 0 \\ S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} R_{AB} \end{bmatrix}^T = \begin{bmatrix} C_\phi & -S_\phi C_\theta & S_\phi S_\theta \\ S_\phi & C_\phi C_\theta & -C_\phi S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \quad \begin{bmatrix} {}^F R_{AB} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & S_\theta \\ 0 & -S_\theta & C_\theta \end{bmatrix}$$

$$\begin{bmatrix} {}^R v_{S, \tilde{\omega}'_F} \end{bmatrix} = - \left(\begin{bmatrix} R_F \end{bmatrix}^T \begin{bmatrix} \tilde{r}'_{A/O} \end{bmatrix} + \begin{bmatrix} R_{AB} \end{bmatrix}^T \begin{bmatrix} \tilde{r}'_{S/A} \end{bmatrix} \begin{bmatrix} {}^F R_{AB} \end{bmatrix} \right) = \begin{bmatrix} xS_\phi S_\theta & xC_\phi S_\theta & -(a + xC_\theta)C_\phi \\ -xC_\phi S_\theta & xS_\phi S_\theta & -(a + xC_\theta)S_\phi \\ a + xC_\theta & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} {}^R v_{S, \tilde{\omega}'_S} \end{bmatrix} = - \begin{bmatrix} R_{AB} \end{bmatrix}^T \begin{bmatrix} \tilde{r}'_{S/A} \end{bmatrix} = \begin{bmatrix} xS_\phi S_\theta & 0 & -xC_\phi \\ -xC_\phi S_\theta & 0 & -xS_\phi \\ xC_\theta & 0 & 0 \end{bmatrix} \quad \{ \dot{x}' \} = \begin{bmatrix} 0 & \dot{x} & 0 \end{bmatrix}^T \quad \begin{bmatrix} {}^R v_{S, \dot{x}'} \end{bmatrix} = \begin{bmatrix} R_{AB} \end{bmatrix}^T$$

$$\begin{aligned} \begin{bmatrix} {}^R \dot{v}_{S, \tilde{\omega}'_F} \end{bmatrix} &= - \begin{bmatrix} R_F \end{bmatrix}^T \begin{bmatrix} \tilde{\omega}'_F \end{bmatrix} \begin{bmatrix} \tilde{r}'_{A/O} \end{bmatrix} - \begin{bmatrix} R_{AB} \end{bmatrix}^T \begin{bmatrix} \tilde{\omega}'_{AB} \end{bmatrix} \begin{bmatrix} \tilde{r}'_{S/A} \end{bmatrix} \begin{bmatrix} {}^F R_{AB} \end{bmatrix} - \begin{bmatrix} R_{AB} \end{bmatrix}^T \begin{bmatrix} \dot{\tilde{r}}'_{S/A} \end{bmatrix} \begin{bmatrix} {}^F R_{AB} \end{bmatrix} \\ &\quad - \begin{bmatrix} R_{AB} \end{bmatrix}^T \begin{bmatrix} \tilde{r}'_{S/A} \end{bmatrix} \begin{bmatrix} {}^F \tilde{\omega}'_{AB} \end{bmatrix}^T \begin{bmatrix} {}^F R_{AB} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & a\Omega S_\phi \\ 0 & 0 & -a\Omega C_\phi \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} x\Omega C_\phi S_\theta + x\omega S_\phi C_\theta & -x\Omega S_\phi S_\theta & x\Omega S_\phi C_\theta \\ x\Omega S_\phi S_\theta - x\omega C_\phi C_\theta & x\Omega C_\phi S_\theta & -x\Omega C_\phi C_\theta \\ -x\omega S_\theta & 0 & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} \dot{x}S_\phi S_\theta & \dot{x}C_\phi S_\theta & -\dot{x}C_\phi C_\theta \\ -\dot{x}C_\phi S_\theta & \dot{x}S_\phi S_\theta & -\dot{x}S_\phi C_\theta \\ \dot{x}C_\theta & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & x\omega C_\phi C_\theta & x\omega C_\phi S_\theta \\ 0 & x\omega S_\phi C_\theta & x\omega S_\phi S_\theta \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} {}^R \dot{v}_{S, \tilde{\omega}'_S} \end{bmatrix} &= - \begin{bmatrix} \dot{R}_{AB} \end{bmatrix}^T \begin{bmatrix} \tilde{r}'_{S/A} \end{bmatrix} - \begin{bmatrix} R_{AB} \end{bmatrix}^T \begin{bmatrix} \dot{\tilde{r}}'_{S/A} \end{bmatrix} = - \begin{bmatrix} R_{AB} \end{bmatrix}^T \begin{bmatrix} \tilde{\omega}'_{AB} \end{bmatrix} \begin{bmatrix} \tilde{r}'_{S/A} \end{bmatrix} - \begin{bmatrix} R_{AB} \end{bmatrix}^T \begin{bmatrix} \dot{\tilde{r}}'_{S/A} \end{bmatrix} \\ &= \begin{bmatrix} x\Omega C_\phi S_\theta + x\omega S_\phi C_\theta & 0 & x\Omega S_\phi \\ x\Omega S_\phi S_\theta - x\omega C_\phi C_\theta & 0 & -x\Omega C_\phi \\ -x\omega S_\theta & 0 & 0 \end{bmatrix} + \begin{bmatrix} \dot{x}S_\phi S_\theta & 0 & -\dot{x}C_\phi \\ -\dot{x}C_\phi S_\theta & 0 & -\dot{x}S_\phi \\ \dot{x}C_\theta & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} {}^R \dot{v}_{S, \dot{x}'} \end{bmatrix} = \begin{bmatrix} R_{AB} \end{bmatrix}^T \begin{bmatrix} \tilde{\omega}'_{AB} \end{bmatrix} = \begin{bmatrix} -\Omega S_\phi & -\Omega C_\phi C_\theta + \omega S_\phi S_\theta & \Omega C_\phi S_\theta + \omega S_\phi C_\theta \\ \Omega C_\phi & -\Omega S_\phi C_\theta - \omega C_\phi S_\theta & \Omega S_\phi S_\theta - \omega C_\phi C_\theta \\ 0 & \omega C_\theta & -\omega S_\theta \end{bmatrix}$$

$$\begin{aligned} \{a_S\} = & \begin{Bmatrix} -(a + xC_\theta)C_\phi \\ -(a + xC_\theta)S_\phi \\ 0 \end{Bmatrix} \dot{\Omega} + \begin{Bmatrix} S_\phi S_\theta \\ -C_\phi S_\theta \\ C_\theta \end{Bmatrix} x\dot{\omega} + \begin{Bmatrix} -S_\phi C_\theta \\ C_\phi C_\theta \\ S_\theta \end{Bmatrix} \ddot{x} + \begin{Bmatrix} (a + xC_\theta)S_\phi \\ -(a + xC_\theta)C_\phi \\ 0 \end{Bmatrix} \Omega^2 + \begin{Bmatrix} S_\phi C_\theta \\ -C_\phi C_\theta \\ -S_\theta \end{Bmatrix} x\omega^2 \\ & + \begin{Bmatrix} 2C_\phi S_\theta \\ 2S_\phi S_\theta \\ 0 \end{Bmatrix} x\Omega\omega + \begin{Bmatrix} -2C_\phi C_\theta \\ -2S_\phi C_\theta \\ 0 \end{Bmatrix} \dot{x}\Omega + \begin{Bmatrix} 2S_\phi S_\theta \\ -2C_\phi S_\theta \\ 2C_\theta \end{Bmatrix} \dot{x}\omega \end{aligned}$$

This result is **identical** to the result from Example 4.

b) $\{\hat{\omega}'_F\} = [0 \ 0 \ \Omega]^T$ $\{\hat{\omega}'_{AB}\} = \{\hat{\omega}'_S\} = [\omega \ 0 \ 0]^T$ $\{\omega'_S\} = \{\omega'_{AB}\} = [\omega \ \Omega S_\theta \ \Omega C_\theta]^T$

$\{x'\} = [0 \ x \ 0]^T$ $[\tilde{r}'_{A/O}] = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ -a & 0 & 0 \end{bmatrix}$ $[\tilde{r}'_{S/A}] = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ -x & 0 & 0 \end{bmatrix}$ $[{}^F R_{AB}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & S_\theta \\ 0 & -S_\theta & C_\theta \end{bmatrix}$

$\{v'_S\} = -\left([{}^F R_{AB}][\tilde{r}'_{A/O}] + [\tilde{r}'_{S/A}][{}^F R_{AB}]\right)\{\hat{\omega}'_F\} - [\tilde{r}'_{S/A}]\{\hat{\omega}'_S\} + \{\dot{x}'\}$
 $= [{}^R v'_{S,\hat{\omega}'_F}]\{\hat{\omega}'_F\} + [{}^R v'_{S,\hat{\omega}'_S}]\{\hat{\omega}'_S\} + [{}^R v'_{S,\dot{x}'}]\{\dot{x}'\}$

$[v'_{S,\hat{\omega}'_F}] = \begin{bmatrix} 0 & xS_\theta & -(a + xC_\theta) \\ aS_\theta & 0 & 0 \\ aC_\theta + x & 0 & 0 \end{bmatrix}$ $[v'_{S,\hat{\omega}'_S}] = \begin{bmatrix} 0 & 0 & -x \\ 0 & 0 & 0 \\ x & 0 & 0 \end{bmatrix}$ $[v'_{S,\dot{x}'}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\{v'_S\} = [-(a + xC_\theta)\Omega \ \dot{x} \ x\omega]^T$

$\{a'_S\} = \{\dot{v}'_S\} + [\tilde{\omega}'_S]\{v'_S\}$
 $= [v'_{S,\hat{\omega}'_F}]\{\dot{\hat{\omega}}'_F\} + [v'_{S,\hat{\omega}'_S}]\{\dot{\hat{\omega}}'_S\} + [v'_{S,\dot{x}'}]\{\dot{x}'\} + [\dot{v}'_{S,\hat{\omega}'_F}]\{\hat{\omega}'_F\} + [\dot{v}'_{S,\hat{\omega}'_S}]\{\hat{\omega}'_S\} + [\tilde{\omega}'_S]\{v'_S\}$

$[\dot{v}'_{S,\hat{\omega}'_F}] = \begin{bmatrix} 0 & \dot{x}S_\theta + x\omega C_\theta & -\dot{x}C_\theta + x\omega S_\theta \\ a\omega C_\theta & 0 & 0 \\ -a\omega S_\theta + \dot{x} & 0 & 0 \end{bmatrix}$ $[\dot{v}'_{S,\hat{\omega}'_S}] = -[\tilde{r}'_{S/A}] = \begin{bmatrix} 0 & 0 & -\dot{x} \\ 0 & 0 & 0 \\ \dot{x} & 0 & 0 \end{bmatrix}$

$[\tilde{\omega}'_S] = \begin{bmatrix} 0 & -\Omega C_\theta & \Omega S_\theta \\ \Omega C_\theta & 0 & -\omega \\ -\Omega S_\theta & \omega & 0 \end{bmatrix}$ $\{a'_S\} = \begin{Bmatrix} -(a + xC_\theta)\dot{\Omega} - 2\dot{x}\Omega C_\theta + 2x\omega\Omega S_\theta \\ \ddot{x} - (a + xC_\theta)C_\theta\Omega^2 - x\omega^2 \\ x\dot{\omega} + (a + xC_\theta)S_\theta\Omega^2 + 2\dot{x}\omega \end{Bmatrix}$

5.4 Write MATLAB scripts to *numerically* evaluate the *matrix equations* you derived in Exercise 5.3 using the data below. First calculate the *partial angular velocity matrices* and the *angular velocity components*. Then calculate the *partial velocity matrices* and the *velocity components* of slider S . Use the results of Unit 4 as a basis for these calculations. Finally, calculate the *time derivatives* of the *partial velocity matrices* and the *acceleration* components of slider S .

Data:

$$\begin{aligned} a &= 0.75 \text{ (ft)} \\ \phi &= 0 & \theta &= 60 \text{ (deg)} & x &= 0.5 \text{ (ft)} \\ \dot{\phi} &= 3 \text{ (r/s)} & \dot{\theta} &= -5 \text{ (r/s)} & \dot{x} &= 1.5 \text{ (ft/s)} \\ \ddot{\phi} &= 2 \text{ (r/s}^2\text{)} & \ddot{\theta} &= -10 \text{ (r/s}^2\text{)} & \ddot{x} &= 5 \text{ (ft/s}^2\text{)} \end{aligned}$$

Results:

a) Body frame components of angular velocities

$$[{}^R\omega'_{F,y}]_{3 \times 9} = \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right]$$

$$[{}^R\omega'_{S,y}]_{3 \times 9} = \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5000 & 0.8660 \\ 0 & -0.8660 & 0.5000 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right]$$

$$\{\omega'_F\} = [0 \quad 0 \quad 3]^T \text{ (r/s)} \quad \{\omega'_S\} = [-5.0000 \quad 2.5981 \quad 1.5000]^T \text{ (r/s)}$$

Fixed frame components of velocity and acceleration of S :

$$[{}^Rv_{S,y}]_{3 \times 9} = \left[\begin{bmatrix} 0 & 0.4330 & -1.0 \\ -0.4330 & 0 & 0 \\ 1.0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -0.5000 \\ -0.4330 & 0 & 0 \\ 0.2500 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5000 & -0.8660 \\ 0 & 0.8660 & 0.5000 \end{bmatrix} \right]$$

$$\{v_S\} = [-3.0000 \quad 2.9151 \quad 0.0490]^T \text{ (ft/s)}$$

$$[{}^R\dot{v}_{S,y}]_{3 \times 9} = \left[\begin{bmatrix} 1.2990 & 0.0490 & -2.9151 \\ -0.0490 & 1.2990 & -3.0000 \\ 2.9151 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1.2990 & 0 & -1.5000 \\ -0.0490 & 0 & -1.5000 \\ 2.9151 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1.5000 & 2.5981 \\ 3.0000 & 4.3301 & 2.5000 \\ 0 & -2.5000 & 4.3301 \end{bmatrix} \right]$$

$$\{a_S\} = [-19.4904 \quad 4.5705 \quad -16.4952]^T \text{ (ft/s}^2\text{)}$$

b) Body frame components of angular velocities – same as part (a)

Body frame components of velocity and acceleration of S :

$$[{}^Rv'_{S,y}]_{3 \times 9} = \left[\begin{bmatrix} 0 & 0.4330 & -1.0000 \\ 0.6495 & 0 & 0 \\ 0.8750 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -0.5000 \\ 0 & 0 & 0 \\ 0.5000 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right]$$

$$\{v'_S\} = [-3.0000 \quad 1.5000 \quad -2.5000]^T \text{ (ft/s)}$$

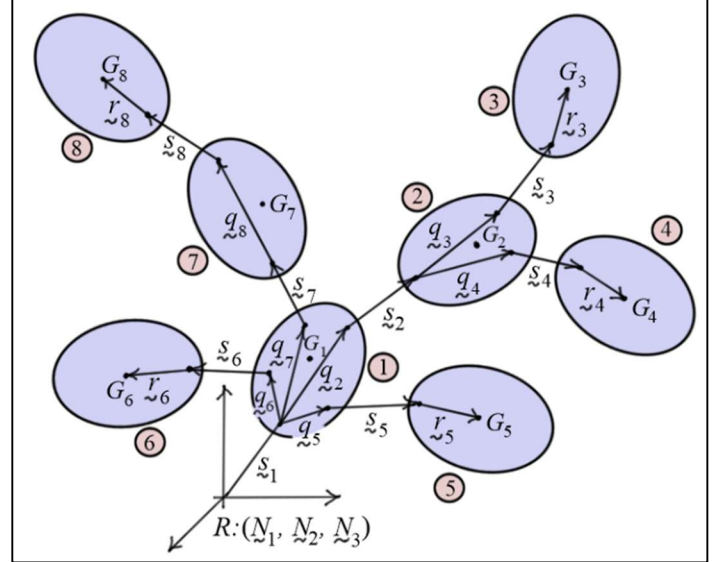
$$\left[{}^R\hat{\mathbf{v}}'_{S,y} \right]_{3 \times 9} = \left[\begin{array}{ccc|ccc|ccc} 0 & 0.0490 & -2.9151 & 0 & 0 & -1.5000 & 0 & 0 & 0 \\ -1.8750 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4.7476 & 0 & 0 & 1.5000 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\{a'_S\} = [-19.4904 \quad -12.0000 \quad -12.2058]^T \text{ (ft/s}^2\text{)}$$

5.5 The figure shows an eight-body system numbered using the numbering scheme presented in Unit 1. Body 1 is the system *reference body*, and the rest of the bodies are numbered in *ascending progression* outward along the branches. As structured, the lower numbered body array for the system is as follows.

$$\mathcal{L}(1, \dots, 8) = (0, 1, 2, 2, 1, 1, 1, 7)$$

The orientation of body 1 is defined *relative* to the fixed frame $R: (\tilde{N}_1, \tilde{N}_2, \tilde{N}_3)$, and the orientations



of all the other bodies are defined *relative* to their adjacent, lower numbered bodies. Using *body frame components* of the *relative angular velocities* of the bodies, the *system generalized speed matrix* is defined as follows.

$$\{y\}_{48 \times 1} \triangleq \left\{ \begin{array}{l} \{y_1\}_{24 \times 1} \\ \{y_2\}_{24 \times 1} \end{array} \right\}$$

$$\{y_1\}_{24 \times 1} \triangleq [\hat{\omega}'_{11} \quad \hat{\omega}'_{12} \quad \hat{\omega}'_{13} \quad \hat{\omega}'_{21} \quad \hat{\omega}'_{22} \quad \hat{\omega}'_{23} \quad \cdots \quad \hat{\omega}'_{71} \quad \hat{\omega}'_{72} \quad \hat{\omega}'_{73} \quad \hat{\omega}'_{81} \quad \hat{\omega}'_{82} \quad \hat{\omega}'_{83}]^T$$

$$\{y_2\}_{24 \times 1} \triangleq [\dot{s}'_{11} \quad \dot{s}'_{12} \quad \dot{s}'_{13} \quad \dot{s}'_{21} \quad \dot{s}'_{22} \quad \dot{s}'_{23} \quad \cdots \quad \dot{s}'_{71} \quad \dot{s}'_{72} \quad \dot{s}'_{73} \quad \dot{s}'_{81} \quad \dot{s}'_{82} \quad \dot{s}'_{83}]^T$$

a) For each body B in the system, find the *time derivatives* of the *fixed frame components* of the *partial velocity matrices* of its *origin* and *mass center*. Then find the *fixed frame components* of the *acceleration* of G_3 the *mass center* of body 3. Use the results from Unit 2 for the *body frame components* of the *partial angular velocity matrices* of the bodies and the results from Unit 4 for the *fixed frame components* of the *partial velocity matrices* of the *origins* and *mass centers* of the bodies. Follow the procedure presented in Equations (53)-(58), (73), and (75).

b) For each body B in the system, find $\left[{}^R\hat{\mathbf{v}}_{O_B,y} \right]_{\mathcal{L}(B)}$ the *body* $\mathcal{L}(K)$ *frame components* of the *partial velocity matrix* associated with ${}^R\mathbf{v}_{O_B/O_{\mathcal{L}(B)}}$ the *velocity* of O_B the origin of body B *relative* to $O_{\mathcal{L}(B)}$ the origin

of body $\mathcal{L}(B)$ and $\left[{}^R\dot{\mathbf{v}}_{O_B,y} \right]_{\mathcal{L}(B)}$ it's **time derivative**. The matrix $\left[{}^R\hat{\mathbf{v}}_{O_B,y} \right]_{\mathcal{L}(B)}$ is defined in Equation (48).

Then find the **body frame components** of the accelerations of the origins of the bodies 1-3 using the procedure presented in Equation (66). Then find the **body frame components** of the **acceleration** of G_3 the **mass center** of body 3 using Equation (86). Use the results from Unit 2 for the **body frame components** of the **partial angular velocity matrices** of the bodies.

Answers – Part (a)

Body 1:

$$\left[{}^R\dot{\mathbf{v}}_{O_1,y_1} \right] = [0]_{3 \times 24} \quad \left[{}^R\dot{\mathbf{v}}_{O_1,y_2} \right] = [0]_{3 \times 24}$$

Body 2:

$$\left[{}^R\dot{\mathbf{v}}_{O_2,y_1} \right]_{3 \times 24} = \left[\left[{}^R\dot{\mathbf{v}}_{O_2,\hat{\omega}_1'} \right]_{3 \times 3} \quad [0]_{3 \times 21} \right] \quad \left[{}^R\dot{\mathbf{v}}_{O_2,\hat{\omega}_1'} \right] = -[R_1]^T [\tilde{\omega}_1'] ([\tilde{q}_2'] + [\tilde{s}_2']) - [R_1]^T [\dot{\tilde{s}}_2']$$

$$\left[{}^R\dot{\mathbf{v}}_{O_2,y_2} \right]_{3 \times 24} = \left[[0]_{3 \times 3} \quad [R_1]^T [\tilde{\omega}_1']_{3 \times 3} \quad [0]_{3 \times 18} \right]$$

Body 3:

$$\left[{}^R\dot{\mathbf{v}}_{O_3,y_1} \right] = \left[\left[{}^R\dot{\mathbf{v}}_{O_3,\hat{\omega}_1'} \right]_{3 \times 3} \quad \left[{}^R\dot{\mathbf{v}}_{O_3,\hat{\omega}_2'} \right]_{3 \times 3} \quad [0]_{3 \times 18} \right]$$

$$\left[{}^R\dot{\mathbf{v}}_{O_3,\hat{\omega}_1'} \right] = -[R_1]^T [\tilde{\omega}_1'] ([\tilde{q}_2'] + [\tilde{s}_2']) - [R_1]^T [\dot{\tilde{s}}_2'] - [R_2]^T ([\tilde{q}_3'] + [\tilde{s}_3']) [\tilde{\omega}_2']^T [{}^1R_2]$$

$$- [R_2]^T [\tilde{\omega}_2'] ([\tilde{q}_3'] + [\tilde{s}_3']) [{}^1R_2] - [R_2]^T [\dot{\tilde{s}}_3'] [{}^1R_2]$$

$$\left[{}^R\dot{\mathbf{v}}_{O_3,\hat{\omega}_2'} \right] = -[R_2]^T [\tilde{\omega}_2'] ([\tilde{q}_3'] + [\tilde{s}_3']) - [R_2]^T [\dot{\tilde{s}}_3']$$

$$\left[{}^R\dot{\mathbf{v}}_{O_3,y_3} \right] = \left[[0]_{3 \times 3} \quad [R_1]^T [\tilde{\omega}_1']_{3 \times 3} \quad [R_2]^T [\tilde{\omega}_2']_{3 \times 3} \quad [0]_{3 \times 15} \right]$$

Body 4:

$$\left[{}^R\dot{\mathbf{v}}_{O_4,y_1} \right] = \left[\left[{}^R\dot{\mathbf{v}}_{O_4,\hat{\omega}_1'} \right]_{3 \times 3} \quad \left[{}^R\dot{\mathbf{v}}_{O_4,\hat{\omega}_2'} \right]_{3 \times 3} \quad [0]_{3 \times 18} \right]$$

$$\left[{}^R\dot{\mathbf{v}}_{O_4,\hat{\omega}_1'} \right] = -[R_1]^T [\tilde{\omega}_1'] ([\tilde{q}_2'] + [\tilde{s}_2']) - [R_1]^T [\dot{\tilde{s}}_2'] - [R_2]^T ([\tilde{q}_4'] + [\tilde{s}_4']) [\tilde{\omega}_2']^T [{}^1R_2]$$

$$- [R_2]^T [\tilde{\omega}_2'] ([\tilde{q}_4'] + [\tilde{s}_4']) [{}^1R_2] - [R_2]^T [\dot{\tilde{s}}_4'] [{}^1R_2]$$

$$\left[{}^R\dot{\mathbf{v}}_{O_4,\hat{\omega}_2'} \right] = -[R_2]^T [\tilde{\omega}_2'] ([\tilde{q}_4'] + [\tilde{s}_4']) - [R_2]^T [\dot{\tilde{s}}_4']$$

$$\left[{}^R\dot{\mathbf{v}}_{O_4,y_2} \right]_{3 \times 24} = \left[[0]_{3 \times 3} \quad [R_1]^T [\tilde{\omega}_1']_{3 \times 3} \quad [0]_{3 \times 3} \quad [R_2]^T [\tilde{\omega}_2']_{3 \times 3} \quad [0]_{3 \times 12} \right]$$

Body 5:

$$\left[{}^R\dot{\mathbf{v}}_{O_5,y_1} \right] = \left[\left[{}^R\dot{\mathbf{v}}_{O_5,\hat{\omega}_1'} \right]_{3 \times 3} \quad [0]_{3 \times 21} \right] \quad \left[{}^R\dot{\mathbf{v}}_{O_5,\hat{\omega}_1'} \right] = -[R_1]^T [\tilde{\omega}_1'] ([\tilde{q}_5'] + [\tilde{s}_5']) - [R_1]^T [\dot{\tilde{s}}_5']$$

$$\left[{}^R\dot{\mathbf{v}}_{O_5,y_2} \right] = \left[[0]_{3 \times 12} \quad [R_1]^T [\tilde{\omega}_1']_{3 \times 3} \quad [0]_{3 \times 9} \right]$$

Body 6:

$$\begin{bmatrix} {}^R\dot{v}_{O_6,y_1} \end{bmatrix} = \begin{bmatrix} {}^R\dot{v}_{O_6,\dot{\omega}_1} \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 21} \quad \begin{bmatrix} {}^R\dot{v}_{O_6,\dot{\omega}_1} \end{bmatrix} = -[R_1]^T [\tilde{\omega}_1'] ([\tilde{q}_6'] + [\tilde{s}_6']) - [R_1]^T [\dot{s}_6']$$

$$\begin{bmatrix} {}^R\dot{v}_{O_6,y_2} \end{bmatrix} = \begin{bmatrix} [0]_{3 \times 15} \quad [R_1]^T [\tilde{\omega}_1'] \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 6}$$

Body 7:

$$\begin{bmatrix} {}^R\dot{v}_{O_7,y_1} \end{bmatrix} = \begin{bmatrix} {}^R\dot{v}_{O_7,\dot{\omega}_1} \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 21} \quad \begin{bmatrix} {}^R\dot{v}_{O_7,\dot{\omega}_1} \end{bmatrix} = -[R_1]^T [\tilde{\omega}_1'] ([\tilde{q}_7'] + [\tilde{s}_7']) - [R_1]^T [\dot{s}_7']$$

$$\begin{bmatrix} {}^R\dot{v}_{O_7,y_2} \end{bmatrix} = \begin{bmatrix} [0]_{3 \times 18} \quad [R_1]^T [\tilde{\omega}_1'] \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 3}$$

Body 8:

$$\begin{bmatrix} {}^R\dot{v}_{O_8,y_1} \end{bmatrix} = \begin{bmatrix} {}^R\dot{v}_{O_8,\dot{\omega}_1} \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 15} \quad \begin{bmatrix} {}^R\dot{v}_{O_8,\dot{\omega}_7} \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 3}$$

$$\begin{bmatrix} {}^R\dot{v}_{O_8,\dot{\omega}_1} \end{bmatrix} = -[R_1]^T [\tilde{\omega}_1'] ([\tilde{q}_7'] + [\tilde{s}_7']) - [R_1]^T [\dot{s}_7'] - [R_7]^T ([\tilde{q}_8'] + [\tilde{s}_8']) [{}^1\tilde{\omega}_7']^T [{}^1R_7] \\ - [R_7]^T [\tilde{\omega}_7'] ([\tilde{q}_8'] + [\tilde{s}_8']) [{}^1R_7] - [R_7]^T [\dot{s}_8'] [{}^1R_7]$$

$$\begin{bmatrix} {}^R\dot{v}_{O_8,\dot{\omega}_7} \end{bmatrix} = -[R_7]^T [\tilde{\omega}_7'] ([\tilde{q}_8'] + [\tilde{s}_8']) - [R_7]^T [\dot{s}_8']$$

$$\begin{bmatrix} {}^R\dot{v}_{O_8,y_2} \end{bmatrix} = \begin{bmatrix} [0]_{3 \times 18} \quad [R_1]^T [\tilde{\omega}_1'] \end{bmatrix}_{3 \times 3} \quad \begin{bmatrix} [R_7]^T [\tilde{\omega}_7'] \end{bmatrix}_{3 \times 3}$$

Time Derivatives of **Partial Velocities** of **Body Mass Centers**:

Body 1:

$$\begin{bmatrix} {}^R\dot{v}_{1,y_1} \end{bmatrix} = \begin{bmatrix} {}^R\dot{v}_{1,\dot{\omega}_1} \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 21} \quad \begin{bmatrix} {}^R\dot{v}_{1,\dot{\omega}_1} \end{bmatrix} = -[R_1]^T [\tilde{\omega}_1'] [\tilde{r}_1'] \quad \begin{bmatrix} {}^R\dot{v}_{1,y_2} \end{bmatrix} = [0]_{3 \times 24}$$

Body 2:

$$\begin{bmatrix} {}^R\dot{v}_{2,y_1} \end{bmatrix} = \begin{bmatrix} {}^R\dot{v}_{2,\dot{\omega}_1} \end{bmatrix}_{3 \times 3} \quad \begin{bmatrix} {}^R\dot{v}_{2,\dot{\omega}_2} \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 18}$$

$$\begin{bmatrix} {}^R\dot{v}_{2,\dot{\omega}_1} \end{bmatrix} = -[R_1]^T [\tilde{\omega}_1'] ([\tilde{q}_2'] + [\tilde{s}_2']) - [R_1]^T [\dot{s}_2'] - [R_2]^T [\tilde{r}_2'] [{}^1\tilde{\omega}_2']^T [{}^1R_2] - [R_2]^T [\tilde{\omega}_2'] [\tilde{r}_2'] [{}^1R_2]$$

$$\begin{bmatrix} {}^R\dot{v}_{2,\dot{\omega}_2} \end{bmatrix} = -[R_2]^T [\tilde{\omega}_2'] [\tilde{r}_2'] \quad \begin{bmatrix} {}^R\dot{v}_{2,y_2} \end{bmatrix} = \begin{bmatrix} [0]_{3 \times 3} \quad [R_1]^T [\tilde{\omega}_1'] \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 18}$$

Body 3:

$$\begin{bmatrix} {}^R\dot{v}_{3,y_1} \end{bmatrix} = \begin{bmatrix} {}^R\dot{v}_{3,\dot{\omega}_1} \end{bmatrix}_{3 \times 3} \quad \begin{bmatrix} {}^R\dot{v}_{3,\dot{\omega}_2} \end{bmatrix}_{3 \times 3} \quad \begin{bmatrix} {}^R\dot{v}_{3,\dot{\omega}_3} \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 15}$$

$$\begin{bmatrix} {}^R\dot{v}_{3,\dot{\omega}_1} \end{bmatrix} = -[R_1]^T [\tilde{\omega}_1'] ([\tilde{q}_2'] + [\tilde{s}_2']) - [R_1]^T [\dot{s}_2'] - [R_2]^T ([\tilde{q}_3'] + [\tilde{s}_3']) [{}^1\tilde{\omega}_2']^T [{}^1R_2] \\ - [R_2]^T [\tilde{\omega}_2'] ([\tilde{q}_3'] + [\tilde{s}_3']) [{}^1R_2] - [R_2]^T [\dot{s}_3'] [{}^1R_2] \\ - [R_3]^T [\tilde{r}_3'] [{}^1\tilde{\omega}_3']^T [{}^1R_3] - [R_3]^T [\tilde{\omega}_3'] [\tilde{r}_3'] [{}^1R_3]$$

$$\begin{bmatrix} {}^R\dot{v}_{3,\dot{\omega}_2} \end{bmatrix} = -[R_2]^T [\tilde{\omega}_2'] ([\tilde{q}_3'] + [\tilde{s}_3']) - [R_2]^T [\dot{s}_3'] - [R_3]^T [\tilde{r}_3'] [{}^2\tilde{\omega}_3']^T [{}^2R_3] - [R_3]^T [\tilde{\omega}_3'] [\tilde{r}_3'] [{}^2R_3]$$

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{3,\hat{\omega}'_3} \end{bmatrix} = -[R_3]^T [\tilde{\omega}'_3] [\tilde{r}'_3] \quad \begin{bmatrix} {}^R\dot{\mathbf{v}}_{3,y_2} \end{bmatrix} = \begin{bmatrix} [0]_{3 \times 3} & [R_1]^T [\tilde{\omega}'_1]_{3 \times 3} & [R_2]^T [\tilde{\omega}'_2]_{3 \times 3} & [0]_{3 \times 15} \end{bmatrix}$$

Body 4:

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{4,y_1} \end{bmatrix} = \begin{bmatrix} {}^R\dot{\mathbf{v}}_{4,\hat{\omega}'_1} \end{bmatrix}_{3 \times 3} \quad {}^R\dot{\mathbf{v}}_{4,\hat{\omega}'_2} \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 3} \quad {}^R\dot{\mathbf{v}}_{4,\hat{\omega}'_4} \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 12}$$

$$\begin{aligned} {}^R\dot{\mathbf{v}}_{4,\hat{\omega}'_1} \end{bmatrix} = & -[R_1]^T [\tilde{\omega}'_1] ([\tilde{q}'_2] + [\tilde{s}'_2]) - [R_1]^T [\dot{\tilde{s}}'_2] - [R_2]^T ([\tilde{q}'_4] + [\tilde{s}'_4]) [{}^1\tilde{\omega}'_2]^T [{}^1R_2] \\ & - [R_2]^T [\tilde{\omega}'_2] ([\tilde{q}'_4] + [\tilde{s}'_4]) [{}^1R_2] - [R_2]^T [\dot{\tilde{s}}'_4] [{}^1R_2] \\ & - [R_4]^T [\tilde{r}'_4] [{}^1\tilde{\omega}'_4]^T [{}^1R_4] - [R_4]^T [\tilde{\omega}'_4] [\tilde{r}'_4] [{}^1R_4] \end{aligned}$$

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{4,\hat{\omega}'_2} \end{bmatrix} = -[R_2]^T [\tilde{\omega}'_2] ([\tilde{q}'_4] + [\tilde{s}'_4]) - [R_2]^T [\dot{\tilde{s}}'_4] - [R_4]^T [\tilde{r}'_4] [{}^2\tilde{\omega}'_4]^T [{}^2R_4] - [R_4]^T [\tilde{\omega}'_4] [\tilde{r}'_4] [{}^2R_4]$$

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{4,\hat{\omega}'_4} \end{bmatrix} = -[R_4]^T [\tilde{\omega}'_4] [\tilde{r}'_4] \quad \begin{bmatrix} {}^R\dot{\mathbf{v}}_{4,y_2} \end{bmatrix} = \begin{bmatrix} [0]_{3 \times 3} & [R_1]^T [\tilde{\omega}'_1]_{3 \times 3} & [0]_{3 \times 3} & [R_2]^T [\tilde{\omega}'_2]_{3 \times 3} & [0]_{3 \times 12} \end{bmatrix}$$

Body 5:

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{5,y_1} \end{bmatrix} = \begin{bmatrix} {}^R\dot{\mathbf{v}}_{5,\hat{\omega}'_1} \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 9} \quad {}^R\dot{\mathbf{v}}_{5,\hat{\omega}'_5} \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 9}$$

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{5,\hat{\omega}'_1} \end{bmatrix} = -[R_1]^T [\tilde{\omega}'_1] ([\tilde{q}'_5] + [\tilde{s}'_5]) - [R_1]^T [\dot{\tilde{s}}'_5] - [R_5]^T [\tilde{r}'_5] [{}^1\tilde{\omega}'_5]^T [{}^1R_5] - [R_5]^T [\tilde{\omega}'_5] [\tilde{r}'_5] [{}^1R_5]$$

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{5,\hat{\omega}'_5} \end{bmatrix} = -[R_5]^T [\tilde{\omega}'_5] [\tilde{r}'_5] \quad \begin{bmatrix} {}^R\dot{\mathbf{v}}_{5,y_2} \end{bmatrix} = \begin{bmatrix} [0]_{3 \times 12} & [R_1]^T [\tilde{\omega}'_1]_{3 \times 3} & [0]_{3 \times 9} \end{bmatrix}$$

Body 6:

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{6,y_1} \end{bmatrix} = \begin{bmatrix} {}^R\dot{\mathbf{v}}_{6,\hat{\omega}'_1} \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 12} \quad {}^R\dot{\mathbf{v}}_{6,\hat{\omega}'_6} \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 6}$$

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{6,\hat{\omega}'_1} \end{bmatrix} = -[R_1]^T [\tilde{\omega}'_1] ([\tilde{q}'_6] + [\tilde{s}'_6]) - [R_1]^T [\dot{\tilde{s}}'_6] - [R_6]^T [\tilde{r}'_6] [{}^1\tilde{\omega}'_6]^T [{}^1R_6] - [R_6]^T [\tilde{\omega}'_6] [\tilde{r}'_6] [{}^1R_6]$$

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{6,\hat{\omega}'_6} \end{bmatrix} = -[R_6]^T [\tilde{\omega}'_6] [\tilde{r}'_6] \quad \begin{bmatrix} {}^R\dot{\mathbf{v}}_{6,y_2} \end{bmatrix} = \begin{bmatrix} [0]_{3 \times 15} & [R_1]^T [\tilde{\omega}'_1]_{3 \times 3} & [0]_{3 \times 6} \end{bmatrix}$$

Body 7:

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{7,y_1} \end{bmatrix} = \begin{bmatrix} {}^R\dot{\mathbf{v}}_{7,\hat{\omega}'_1} \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 15} \quad {}^R\dot{\mathbf{v}}_{7,\hat{\omega}'_7} \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 3}$$

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{7,\hat{\omega}'_1} \end{bmatrix} = -[R_1]^T [\tilde{\omega}'_1] ([\tilde{q}'_7] + [\tilde{s}'_7]) - [R_1]^T [\dot{\tilde{s}}'_7] - [R_7]^T [\tilde{r}'_7] [{}^1\tilde{\omega}'_7]^T [{}^1R_7] - [R_7]^T [\tilde{\omega}'_7] [\tilde{r}'_7] [{}^1R_7]$$

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{7,\hat{\omega}'_7} \end{bmatrix} = -[R_7]^T [\tilde{\omega}'_7] [\tilde{r}'_7] \quad \begin{bmatrix} {}^R\dot{\mathbf{v}}_{7,y_2} \end{bmatrix} = \begin{bmatrix} [0]_{3 \times 18} & [R_1]^T [\tilde{\omega}'_1]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix}$$

Body 8:

$$\begin{bmatrix} {}^R\dot{\mathbf{v}}_{8,y_1} \end{bmatrix} = \begin{bmatrix} {}^R\dot{\mathbf{v}}_{8,\hat{\omega}'_1} \end{bmatrix}_{3 \times 3} \quad [0]_{3 \times 15} \quad {}^R\dot{\mathbf{v}}_{8,\hat{\omega}'_7} \end{bmatrix}_{3 \times 3} \quad {}^R\dot{\mathbf{v}}_{8,\hat{\omega}'_8} \end{bmatrix}_{3 \times 3}$$

$$\begin{aligned} {}^R\dot{\mathbf{v}}_{8,\hat{\omega}'_1} \end{bmatrix} = & -[R_1]^T [\tilde{\omega}'_1] ([\tilde{q}'_7] + [\tilde{s}'_7]) - [R_1]^T [\dot{\tilde{s}}'_7] - [R_7]^T ([\tilde{q}'_8] + [\tilde{s}'_8]) [{}^1\tilde{\omega}'_7]^T [{}^1R_7] \\ & - [R_7]^T [\tilde{\omega}'_7] ([\tilde{q}'_8] + [\tilde{s}'_8]) [{}^1R_7] - [R_7]^T [\dot{\tilde{s}}'_8] [{}^1R_7] \\ & - [R_8]^T [\tilde{r}'_8] [{}^1\tilde{\omega}'_8]^T [{}^1R_8] - [R_8]^T [\tilde{\omega}'_8] [\tilde{r}'_8] [{}^1R_8] \end{aligned}$$

$$\begin{bmatrix} {}^R\dot{v}_{8,\dot{\omega}_7'} \end{bmatrix} = -[R_7]^T [\tilde{\omega}_7'] ([\tilde{q}_8'] + [\tilde{s}_8']) - [R_7]^T [\dot{s}_8'] - [R_8]^T [\tilde{r}_8'] [\tilde{\omega}_8']^T [{}^7R_8] - [R_8]^T [\tilde{\omega}_8'] [\tilde{r}_8'] [{}^7R_8]$$

$$\begin{bmatrix} {}^R\dot{v}_{8,\dot{\omega}_8'} \end{bmatrix} = -[R_8]^T [\tilde{\omega}_8'] [\tilde{r}_8'] \quad \begin{bmatrix} {}^R\dot{v}_{8,y_2} \end{bmatrix} = \begin{bmatrix} [0]_{3 \times 18} & [R_1]^T [\tilde{\omega}_1']_{3 \times 3} & [R_7]^T [\tilde{\omega}_7']_{3 \times 3} \end{bmatrix}$$

Acceleration of G_3 :

$$\begin{aligned} \{a_3\} = & -\left[[R_1]^T ([\tilde{q}_2'] + [\tilde{s}_2']) + [R_2]^T ([\tilde{q}_3'] + [\tilde{s}_3']) [{}^1R_2] + [R_3]^T [\tilde{r}_3'] [{}^1R_3] \right] \{\dot{\omega}_1'\} \\ & - [R_2]^T ([\tilde{q}_3'] + [\tilde{s}_3']) + [R_3]^T [\tilde{r}_3'] [{}^2R_3] \{\dot{\omega}_2'\} - [R_3]^T [\tilde{r}_3'] \{\dot{\omega}_3'\} \\ & - [R_1]^T ([\tilde{\omega}_1'] ([\tilde{q}_2'] + [\tilde{s}_2']) + [\dot{s}_2']) \{\dot{\omega}_1'\} \\ & - [R_2]^T ([\tilde{\omega}_2'] ([\tilde{q}_3'] + [\tilde{s}_3']) + [\dot{s}_3'] + ([\tilde{q}_3'] + [\tilde{s}_3']) [{}^1\tilde{\omega}_2']^T) [{}^1R_2] \{\dot{\omega}_1'\} \\ & - [R_3]^T ([\tilde{\omega}_3'] [\tilde{r}_3'] + [\tilde{r}_3'] [{}^1\tilde{\omega}_3']^T) [{}^1R_3] \{\dot{\omega}_1'\} - [R_2]^T ([\tilde{\omega}_2'] ([\tilde{q}_3'] + [\tilde{s}_3']) + [\dot{s}_3']) \{\dot{\omega}_2'\} \\ & - [R_3]^T ([\tilde{\omega}_3'] [\tilde{r}_3'] + [\tilde{r}_3'] [{}^2\tilde{\omega}_3']^T) [{}^2R_3] \{\dot{\omega}_2'\} - [R_3]^T [\tilde{\omega}_3'] [\tilde{r}_3'] \{\dot{\omega}_3'\} \\ & + \{\ddot{s}_1'\} + [R_1]^T \{\ddot{s}_2'\} + [R_2]^T \{\ddot{s}_3'\} + [R_1]^T [\tilde{\omega}_1'] \{\dot{s}_2'\} + [R_2]^T [\tilde{\omega}_2'] \{\dot{s}_3'\} \end{aligned}$$

Answers – Part (b)

Body 1:

$$\begin{bmatrix} {}^R\hat{v}_{O_1,y_1} \end{bmatrix}_R = [0]_{3 \times 24} \quad \begin{bmatrix} {}^R\hat{v}_{O_1,y_2} \end{bmatrix}_R = \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 21} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\dot{\hat{v}}_{O_1,y_1} \end{bmatrix}_{\mathcal{E}(1)=R} = [0]_{3 \times 24} \quad \begin{bmatrix} {}^R\dot{\hat{v}}_{O_1,y_2} \end{bmatrix}_{\mathcal{E}(1)=R} = [0]_{3 \times 24}$$

Body 2:

$$\begin{bmatrix} {}^R\hat{v}_{O_2,y_1} \end{bmatrix}_1 = -([\tilde{q}_2'] + [\tilde{s}_2']) \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 21} \end{bmatrix} \quad \begin{bmatrix} {}^R\hat{v}_{O_2,y_2} \end{bmatrix}_1 = \begin{bmatrix} [0]_{3 \times 3} & [I]_{3 \times 3} & [0]_{3 \times 18} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\dot{\hat{v}}_{O_2,y_1} \end{bmatrix}_1 = -[\dot{s}_2'] \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 21} \end{bmatrix} \quad \begin{bmatrix} {}^R\dot{\hat{v}}_{O_2,y_2} \end{bmatrix}_1 = [0]_{3 \times 24}$$

Body 3:

$$\begin{bmatrix} {}^R\hat{v}_{O_3,y_1} \end{bmatrix}_2 = -([\tilde{q}_3'] + [\tilde{s}_3']) \begin{bmatrix} [{}^1R_2] & [I]_{3 \times 3} & [0]_{3 \times 18} \end{bmatrix} \quad \begin{bmatrix} {}^R\hat{v}_{O_3,y_2} \end{bmatrix}_2 = [S_{s_3'}]_{y_2} = \begin{bmatrix} [0]_{3 \times 6} & [I]_{3 \times 3} & [0]_{3 \times 15} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\dot{\hat{v}}_{O_3,y_1} \end{bmatrix}_2 = -([\tilde{q}_3'] + [\tilde{s}_3']) [{}^1\tilde{\omega}_2']^T \begin{bmatrix} [{}^1R_2] & [0]_{3 \times 21} \end{bmatrix} - [\dot{s}_3'] \begin{bmatrix} [{}^1R_2] & [I]_{3 \times 3} & [0]_{3 \times 18} \end{bmatrix} \quad \begin{bmatrix} {}^R\dot{\hat{v}}_{O_3,y_2} \end{bmatrix}_2 = [0]_{3 \times 24}$$

Body 4:

$$\begin{bmatrix} {}^R\hat{v}_{O_4,y_1} \end{bmatrix}_2 = -([\tilde{q}_4'] + [\tilde{s}_4']) \begin{bmatrix} [{}^1R_2] & [I]_{3 \times 3} & [0]_{3 \times 18} \end{bmatrix} \quad \begin{bmatrix} {}^R\hat{v}_{O_4,y_2} \end{bmatrix}_2 = \begin{bmatrix} [0]_{3 \times 9} & [I]_{3 \times 3} & [0]_{3 \times 12} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\dot{\hat{v}}_{O_4,y_1} \end{bmatrix}_2 = -([\tilde{q}_4'] + [\tilde{s}_4']) [{}^1\tilde{\omega}_2']^T \begin{bmatrix} [{}^1R_2] & [I]_{3 \times 3} & [0]_{3 \times 18} \end{bmatrix} - [\dot{s}_4'] \begin{bmatrix} [{}^1R_2] & [I]_{3 \times 3} & [0]_{3 \times 18} \end{bmatrix}$$

$$\begin{bmatrix} {}^R\dot{\hat{v}}_{O_4,y_2} \end{bmatrix}_2 = [0]_{3 \times 24}$$

Body 5:

$$\begin{aligned} \begin{bmatrix} {}^R\hat{v}_{O_5,y_1} \end{bmatrix}_1 &= -([\tilde{q}'_5] + [\tilde{s}'_5]) \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 21} \end{bmatrix} & \begin{bmatrix} {}^R\hat{v}_{O_5,y_2} \end{bmatrix}_1 &= \begin{bmatrix} [0]_{3 \times 12} & [I]_{3 \times 3} & [0]_{3 \times 9} \end{bmatrix} \\ \begin{bmatrix} {}^R\dot{\hat{v}}_{O_5,y_1} \end{bmatrix}_1 &= -[\dot{\tilde{s}}'_5] \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 21} \end{bmatrix} & \begin{bmatrix} {}^R\dot{\hat{v}}_{O_5,y_2} \end{bmatrix}_1 &= [0]_{3 \times 24} \end{aligned}$$

Body 6:

$$\begin{aligned} \begin{bmatrix} {}^R\hat{v}_{O_6,y_1} \end{bmatrix}_1 &= -([\tilde{q}'_6] + [\tilde{s}'_6]) \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 21} \end{bmatrix} & \begin{bmatrix} {}^R\hat{v}_{O_6,y_2} \end{bmatrix}_1 &= \begin{bmatrix} [0]_{3 \times 15} & [I]_{3 \times 3} & [0]_{3 \times 6} \end{bmatrix} \\ \begin{bmatrix} {}^R\dot{\hat{v}}_{O_6,y_1} \end{bmatrix}_1 &= -[\dot{\tilde{s}}'_6] \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 21} \end{bmatrix} & \begin{bmatrix} {}^R\dot{\hat{v}}_{O_6,y_2} \end{bmatrix}_1 &= [0]_{3 \times 24} \end{aligned}$$

Body 7:

$$\begin{aligned} \begin{bmatrix} {}^R\hat{v}_{O_7,y_1} \end{bmatrix}_1 &= -([\tilde{q}'_7] + [\tilde{s}'_7]) \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 21} \end{bmatrix} & \begin{bmatrix} {}^R\hat{v}_{O_7,y_2} \end{bmatrix}_1 &= \begin{bmatrix} [0]_{3 \times 18} & [I]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix} \\ \begin{bmatrix} {}^R\dot{\hat{v}}_{O_7,y_1} \end{bmatrix}_1 &= -[\dot{\tilde{s}}'_7] \begin{bmatrix} [I]_{3 \times 3} & [0]_{3 \times 21} \end{bmatrix} & \begin{bmatrix} {}^R\dot{\hat{v}}_{O_7,y_2} \end{bmatrix}_1 &= [0]_{3 \times 24} \end{aligned}$$

Body 8:

$$\begin{aligned} \begin{bmatrix} {}^R\hat{v}_{O_8,y_1} \end{bmatrix}_2 &= -([\tilde{q}'_8] + [\tilde{s}'_8]) \begin{bmatrix} [{}^1R_7] & [0]_{3 \times 15} & [I]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix} & \begin{bmatrix} {}^R\hat{v}_{O_8,y_2} \end{bmatrix}_2 &= [S'_{s'_8}]_{y_2} = \begin{bmatrix} [0]_{3 \times 21} & [I]_{3 \times 3} \end{bmatrix} \\ \begin{bmatrix} {}^R\dot{\hat{v}}_{O_8,y_1} \end{bmatrix}_7 &= -([\tilde{q}'_8] + [\tilde{s}'_8]) [{}^1\tilde{\omega}'_7]^T \begin{bmatrix} [{}^1R_7] & [0]_{3 \times 21} \end{bmatrix} - [\dot{\tilde{s}}'_8] \begin{bmatrix} [{}^1R_7] & [0]_{3 \times 15} & [I]_{3 \times 3} & [0]_{3 \times 3} \end{bmatrix} \\ \begin{bmatrix} {}^R\dot{\hat{v}}_{O_8,y_2} \end{bmatrix}_7 &= [0]_{3 \times 24} \end{aligned}$$

Accelerations of Body Origins:

$$\begin{aligned} \{a'_{O_1}\} &= [R_1] \{\dot{s}'_1\} \\ \{a'_{O_2}\} &= [R_2] \{\dot{s}'_1\} + [{}^1R_2] [\tilde{\omega}'_1] (\{q'_2\} + \{s'_2\}) + [{}^1R_2] [\tilde{\omega}'_1] [\tilde{\omega}'_1] (\{q'_2\} + \{s'_2\}) + [{}^1R_2] \{\dot{s}'_2\} + 2[{}^1R_2] [\tilde{\omega}'_1] \{\dot{s}'_2\} \\ \{a'_{O_3}\} &= ([R_3] \{\dot{s}'_1\} + [{}^1R_3] [\tilde{\omega}'_1] (\{q'_2\} + \{s'_2\}) + [{}^1R_3] [\tilde{\omega}'_1] [\tilde{\omega}'_1] (\{q'_2\} + \{s'_2\}) + [{}^1R_3] \{\dot{s}'_2\} + 2[{}^1R_3] [\tilde{\omega}'_1] \{\dot{s}'_2\}) \\ &\quad + [{}^2R_3] [\tilde{\omega}'_2] (\{q'_3\} + \{s'_3\}) + [{}^2R_3] [\tilde{\omega}'_2] [\tilde{\omega}'_2] (\{q'_3\} + \{s'_3\}) + [{}^2R_3] \{\dot{s}'_3\} + 2[{}^2R_3] [\tilde{\omega}'_2] \{\dot{s}'_3\} \end{aligned}$$

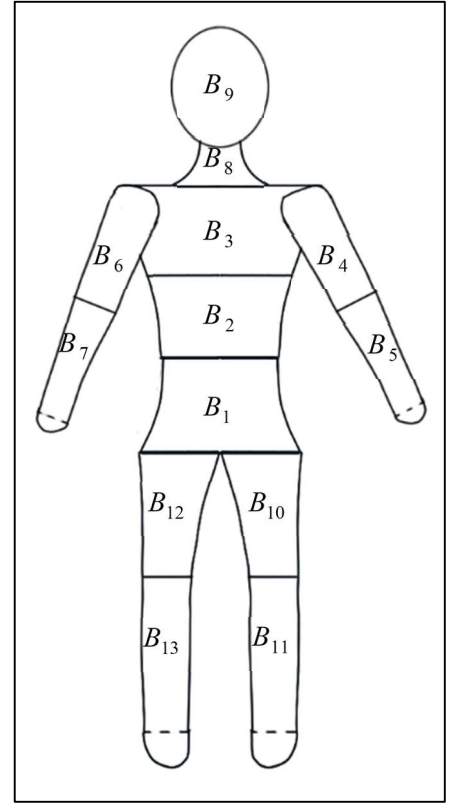
Acceleration of G_3 :

$$\begin{aligned} \{a'_3\} &= [R_3] \{\dot{s}'_1\} + [{}^1R_3] [\tilde{\omega}'_1] (\{q'_2\} + \{s'_2\}) + [{}^1R_3] [\tilde{\omega}'_1] [\tilde{\omega}'_1] (\{q'_2\} + \{s'_2\}) + [{}^1R_3] \{\dot{s}'_2\} + 2[{}^1R_3] [\tilde{\omega}'_1] \{\dot{s}'_2\} \\ &\quad + [{}^2R_3] [\tilde{\omega}'_2] (\{q'_3\} + \{s'_3\}) + [{}^2R_3] [\tilde{\omega}'_2] [\tilde{\omega}'_2] (\{q'_3\} + \{s'_3\}) + [{}^2R_3] \{\dot{s}'_3\} + 2[{}^2R_3] [\tilde{\omega}'_2] \{\dot{s}'_3\} \\ &\quad + [\tilde{\omega}'_3] \{r'_3\} + [\tilde{\omega}'_3] [\tilde{\omega}'_3] \{r'_3\} \end{aligned}$$

5.6 The figure shows a thirteen-body model of the human body numbered using the numbering scheme presented in Unit 1. Body 1 is the **lower torso**, and it is the **system reference body**. The rest of the bodies are numbered in ascending progression outward along the branches. As structured, the lower numbered body array for the system is as follows.

$$\mathcal{L}(1, \dots, 13) = (0, 1, 2, 3, 4, 3, 6, 3, 8, 1, 10, 1, 12)$$

The position and orientation of body 1 is defined **relative** to the **fixed frame** $R: (\underline{N}_1, \underline{N}_2, \underline{N}_3)$, and the orientations of all the other bodies are defined **relative** to their **adjacent, lower numbered bodies**. Body 1 can **translate** and **rotate** relative to the fixed frame, but **all other bodies** can **only rotate** relative to their lower numbered bodies, not translate. Using **base frame components** of the **relative angular velocities** of the bodies, the **system generalized speed matrix** is defined as follows.



$$\{\mathcal{Y}\}_{42 \times 1} \triangleq \begin{Bmatrix} \{\mathcal{Y}_1\}_{39 \times 1} \\ \{\mathcal{Y}_2\}_{3 \times 1} \end{Bmatrix}$$

$$\{\mathcal{Y}_1\}_{39 \times 1} \triangleq [\hat{\omega}_{11} \quad \hat{\omega}_{12} \quad \hat{\omega}_{13} \quad \hat{\omega}_{21} \quad \hat{\omega}_{22} \quad \hat{\omega}_{23} \quad \cdots \quad \hat{\omega}_{11,1} \quad \hat{\omega}_{11,2} \quad \hat{\omega}_{11,3} \quad \hat{\omega}_{12,1} \quad \hat{\omega}_{12,2} \quad \hat{\omega}_{12,3}]^T$$

$$\{\mathcal{Y}_2\}_{3 \times 1} \triangleq [\dot{s}'_{11} \quad \dot{s}'_{12} \quad \dot{s}'_{13}]^T$$

a) For body 7 (lower right arm), find the **time derivatives** of the **fixed frame components** of the **partial velocity matrix** of its **origin** O_7 and its **mass center** G_7 . Then find the **fixed frame components** of the **acceleration** of G_7 , the **mass center** of body 7. Use the results presented in Unit 2 for the **fixed frame components** of the **partial angular velocity matrices** of the bodies and the results from Unit 4 for the **fixed frame components** of the **partial velocity matrices** of the **origins** and **mass centers** of the bodies. Follow the procedures outlined in Equations (54)-(56) and Equation (82). b) Use **direct differentiation** to find the **fixed frame components** of the **acceleration** of G_7 in terms of the **base frame components** of the **relative angular velocity** vectors and compare the results with those obtained in part (a). Assume that $s'_{Bi} (B = 2, \dots, 13; i = 1, 2, 3)$ and $\dot{s}'_{Bi} (B = 2, \dots, 13; i = 1, 2, 3)$ are all **zero**.

Answers:

Body Origins:

Body 1:

$$\begin{bmatrix} {}^R \dot{\mathcal{V}}_{O_1, y_1} \end{bmatrix}_{3 \times 39} = [0]_{3 \times 39} \quad \begin{bmatrix} {}^R \dot{\mathcal{V}}_{O_1, y_2} \end{bmatrix}_{3 \times 3} = [0]_{3 \times 3}$$

Body 2:

$$\begin{bmatrix} {}^R \dot{\mathcal{V}}_{O_2, y_1} \end{bmatrix}_{3 \times 39} = \begin{bmatrix} \begin{bmatrix} {}^R \dot{\mathcal{V}}_{O_2, \hat{a}_1} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & \cdots & [0]_{3 \times 3} \end{bmatrix} \quad \begin{bmatrix} {}^R \dot{\mathcal{V}}_{O_2, \hat{a}_1} \end{bmatrix}_{3 \times 3} = -[\dot{\hat{q}}_2] \quad \begin{bmatrix} {}^R \dot{\mathcal{V}}_{O_2, y_2} \end{bmatrix}_{3 \times 3} = [0]_{3 \times 3}$$

Body 3:

$$\begin{aligned} \left[\begin{matrix} R\dot{v}_{O_3, y_1} \end{matrix} \right]_{3 \times 39} &= \left[\begin{matrix} \left[\begin{matrix} R\dot{v}_{O_3, \hat{w}_1} \end{matrix} \right]_{3 \times 3} & \left[\begin{matrix} R\dot{v}_{O_3, \hat{w}_2} \end{matrix} \right]_{3 \times 3} & [0]_{3 \times 3} & \cdots & [0]_{3 \times 3} \end{matrix} \right] \\ \left[\begin{matrix} R\dot{v}_{O_3, \hat{w}_1} \end{matrix} \right]_{3 \times 3} &= -\left[\begin{matrix} \ddot{q}_2 \end{matrix} \right] - \left[\begin{matrix} \ddot{q}_3 \end{matrix} \right] & \left[\begin{matrix} R\dot{v}_{O_3, \hat{w}_2} \end{matrix} \right]_{3 \times 3} &= -\left(\left[\begin{matrix} \tilde{q}_3 \end{matrix} \right] \left[\begin{matrix} \tilde{w}_1 \end{matrix} \right] + \left[\begin{matrix} \dot{\tilde{q}}_3 \end{matrix} \right] \right) \left[\begin{matrix} R_1 \end{matrix} \right]^T & \left[\begin{matrix} R\dot{v}_{O_3, y_2} \end{matrix} \right]_{3 \times 3} &= [0]_{3 \times 3} \end{aligned}$$

Body 6:

$$\begin{aligned} \left[\begin{matrix} R\dot{v}_{O_6, y_1} \end{matrix} \right]_{3 \times 39} &= \left[\begin{matrix} \left[R\dot{v}_{O_6, \hat{w}_1} \right]_{3 \times 3} & \left[R\dot{v}_{O_6, \hat{w}_2} \right]_{3 \times 3} & \left[R\dot{v}_{O_6, \hat{w}_3} \right]_{3 \times 3} & [0]_{3 \times 3} & \cdots & [0]_{3 \times 3} \end{matrix} \right] \\ \left[\begin{matrix} R\dot{v}_{O_6, \hat{w}_1} \end{matrix} \right]_{3 \times 3} &= -\left[\begin{matrix} \ddot{q}_2 \end{matrix} \right] - \left[\begin{matrix} \ddot{q}_3 \end{matrix} \right] - \left[\begin{matrix} \ddot{q}_6 \end{matrix} \right] & \left[\begin{matrix} R\dot{v}_{O_6, \hat{w}_2} \end{matrix} \right]_{3 \times 3} &= -\left(\left[\begin{matrix} \tilde{q}_3 \end{matrix} \right] \left[\begin{matrix} \tilde{w}_1 \end{matrix} \right] + \left[\begin{matrix} \tilde{q}_3 \end{matrix} \right] + \left[\begin{matrix} \tilde{q}_6 \end{matrix} \right] \left[\begin{matrix} \tilde{w}_1 \end{matrix} \right] + \left[\begin{matrix} \tilde{q}_6 \end{matrix} \right] \right) \left[R_1 \right]^T \\ \left[\begin{matrix} R\dot{v}_{O_6, \hat{w}_3} \end{matrix} \right]_{3 \times 3} &= -\left(\left[\begin{matrix} \tilde{q}_6 \end{matrix} \right] \left[\begin{matrix} \tilde{w}_2 \end{matrix} \right] + \left[\begin{matrix} \ddot{q}_6 \end{matrix} \right] \right) \left[R_2 \right]^T & \left[\begin{matrix} R\dot{v}_{O_6, y_2} \end{matrix} \right]_{3 \times 3} &= \left[0 \right]_{3 \times 3} \end{aligned}$$

Body 7:

$$\begin{aligned}
\begin{bmatrix} \dot{R}\dot{v}_{O_7, y_1} \end{bmatrix}_{3 \times 39} &= \begin{bmatrix} \dot{R}\dot{v}_{O_7, \hat{w}_1} \end{bmatrix}_{3 \times 3} \quad \begin{bmatrix} \dot{R}\dot{v}_{O_7, \hat{w}_2} \end{bmatrix}_{3 \times 3} \quad \begin{bmatrix} \dot{R}\dot{v}_{O_7, \hat{w}_3} \end{bmatrix}_{3 \times 3} \quad \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \quad \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \quad \begin{bmatrix} \dot{R}\dot{v}_{O_7, \hat{w}_6} \end{bmatrix}_{3 \times 3} \quad \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \quad \cdots \quad \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \\
\begin{bmatrix} \dot{R}\dot{v}_{O_7, \hat{w}_1} \end{bmatrix}_{3 \times 3} &= -\begin{bmatrix} \ddot{q}_2 \end{bmatrix} - \begin{bmatrix} \ddot{q}_3 \end{bmatrix} - \begin{bmatrix} \ddot{q}_6 \end{bmatrix} - \begin{bmatrix} \ddot{q}_7 \end{bmatrix} \\
\begin{bmatrix} \dot{R}\dot{v}_{O_7, \hat{w}_2} \end{bmatrix}_{3 \times 3} &= -\left(\begin{bmatrix} \ddot{q}_3 \end{bmatrix} + \begin{bmatrix} \ddot{q}_6 \end{bmatrix} + \begin{bmatrix} \ddot{q}_7 \end{bmatrix}\right)[R_1]^T - \left(\begin{bmatrix} \ddot{q}_3 \end{bmatrix} + \begin{bmatrix} \ddot{q}_6 \end{bmatrix} + \begin{bmatrix} \ddot{q}_7 \end{bmatrix}\right)[\tilde{w}_1][R_1]^T \\
\begin{bmatrix} \dot{R}\dot{v}_{O_7, \hat{w}_3} \end{bmatrix}_{3 \times 3} &= -\left(\begin{bmatrix} \ddot{q}_6 \end{bmatrix} + \begin{bmatrix} \ddot{q}_7 \end{bmatrix}\right)[R_2]^T - \left(\begin{bmatrix} \ddot{q}_6 \end{bmatrix} + \begin{bmatrix} \ddot{q}_7 \end{bmatrix}\right)[\tilde{w}_2][R_2]^T \quad \begin{bmatrix} \dot{R}\dot{v}_{O_7, \hat{w}_6} \end{bmatrix}_{3 \times 3} = -\left(\begin{bmatrix} \ddot{q}_7 \end{bmatrix}[\tilde{w}_3] + \begin{bmatrix} \ddot{q}_7 \end{bmatrix}\right)[R_3]^T \\
\begin{bmatrix} \dot{R}\dot{v}_{O_7, y_2} \end{bmatrix}_{3 \times 3} &= \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3}
\end{aligned}$$

Mass Center of Body 7:

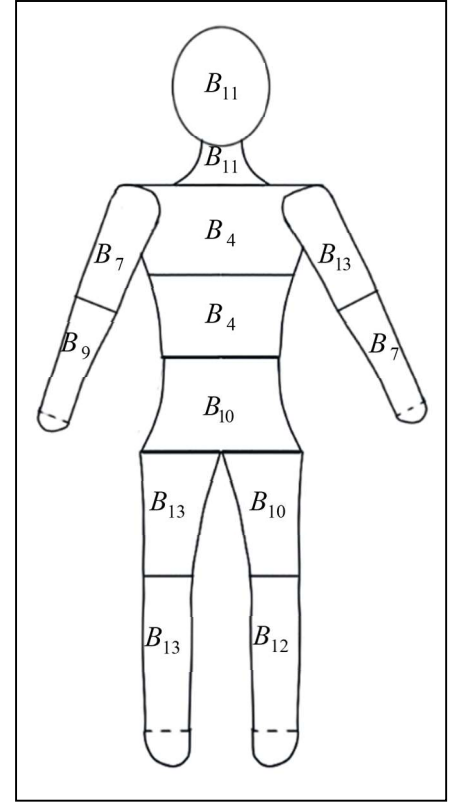
$$\begin{aligned} \left[\begin{smallmatrix} R\dot{V}_{7,y_1} \end{smallmatrix} \right]_{3 \times 39} &= \left[\left[\begin{smallmatrix} R\dot{V}_{7,\hat{\omega}_1} \end{smallmatrix} \right]_{3 \times 3} \quad \left[\begin{smallmatrix} R\dot{V}_{7,\hat{\omega}_2} \end{smallmatrix} \right]_{3 \times 3} \quad \left[\begin{smallmatrix} R\dot{V}_{7,\hat{\omega}_3} \end{smallmatrix} \right]_{3 \times 3} \quad [0]_{3 \times 3} \quad [0]_{3 \times 3} \quad \left[\begin{smallmatrix} R\dot{V}_{7,\hat{\omega}_6} \end{smallmatrix} \right]_{3 \times 3} \quad \left[\begin{smallmatrix} R\dot{V}_{7,\hat{\omega}_7} \end{smallmatrix} \right]_{3 \times 3} \quad [0]_{3 \times 3} \quad \cdots \quad [0]_{3 \times 3} \right] \\ \left[\begin{smallmatrix} R\dot{V}_{7,\hat{\omega}_1} \end{smallmatrix} \right]_{3 \times 3} &= -\left[\begin{smallmatrix} \dot{q}_2 \end{smallmatrix} \right] - \left[\begin{smallmatrix} \dot{q}_3 \end{smallmatrix} \right] - \left[\begin{smallmatrix} \dot{q}_6 \end{smallmatrix} \right] - \left[\begin{smallmatrix} \dot{q}_7 \end{smallmatrix} \right] - \left[\begin{smallmatrix} \dot{r}_7 \end{smallmatrix} \right] \\ \left[\begin{smallmatrix} R\dot{V}_{7,\hat{\omega}_2} \end{smallmatrix} \right]_{3 \times 3} &= -\left(\left[\begin{smallmatrix} \dot{q}_3 \end{smallmatrix} \right] + \left[\begin{smallmatrix} \dot{q}_6 \end{smallmatrix} \right] + \left[\begin{smallmatrix} \dot{q}_7 \end{smallmatrix} \right] + \left[\begin{smallmatrix} \dot{r}_7 \end{smallmatrix} \right] \right) [R_1]^T - \left(\left[\begin{smallmatrix} \tilde{q}_3 \end{smallmatrix} \right] + \left[\begin{smallmatrix} \tilde{q}_6 \end{smallmatrix} \right] + \left[\begin{smallmatrix} \tilde{q}_7 \end{smallmatrix} \right] + \left[\begin{smallmatrix} \tilde{r}_7 \end{smallmatrix} \right] \right) [\tilde{\omega}_1] [R_1]^T \\ \left[\begin{smallmatrix} R\dot{V}_{7,\hat{\omega}_3} \end{smallmatrix} \right]_{3 \times 3} &= -\left(\left[\begin{smallmatrix} \dot{q}_6 \end{smallmatrix} \right] + \left[\begin{smallmatrix} \dot{q}_7 \end{smallmatrix} \right] + \left[\begin{smallmatrix} \dot{r}_7 \end{smallmatrix} \right] \right) [R_2]^T - \left(\left[\begin{smallmatrix} \tilde{q}_6 \end{smallmatrix} \right] + \left[\begin{smallmatrix} \tilde{q}_7 \end{smallmatrix} \right] + \left[\begin{smallmatrix} \tilde{r}_7 \end{smallmatrix} \right] \right) [\tilde{\omega}_2] [R_2]^T \\ \left[\begin{smallmatrix} R\dot{V}_{7,\hat{\omega}_6} \end{smallmatrix} \right]_{3 \times 3} &= -\left(\left[\begin{smallmatrix} \dot{q}_7 \end{smallmatrix} \right] + \left[\begin{smallmatrix} \dot{r}_7 \end{smallmatrix} \right] \right) [R_3]^T - \left(\left[\begin{smallmatrix} \tilde{q}_7 \end{smallmatrix} \right] + \left[\begin{smallmatrix} \tilde{r}_7 \end{smallmatrix} \right] \right) [\tilde{\omega}_3] [R_3]^T \quad \left[\begin{smallmatrix} R\dot{V}_{7,\hat{\omega}_7} \end{smallmatrix} \right]_{3 \times 3} = -\left(\left[\begin{smallmatrix} \dot{r}_7 \end{smallmatrix} \right] + \left[\begin{smallmatrix} \tilde{r}_7 \end{smallmatrix} \right] [\tilde{\omega}_6] \right) [R_6]^T \\ \left[\begin{smallmatrix} R\dot{V}_{7,y_2} \end{smallmatrix} \right]_{3 \times 3} &= [0]_{3 \times 3} \end{aligned}$$

$$\begin{aligned} \{a_7\} = & \{\dot{s}'_1\} - [\tilde{q}_2] + [\tilde{q}_3] + [\tilde{q}_6] + [\tilde{q}_7] + [\tilde{r}_7] \left\{ \dot{\omega}_1 \right\} - [\tilde{q}_3] + [\tilde{q}_6] + [\tilde{q}_7] + [\tilde{r}_7] [R_1]^T \left\{ \dot{\omega}_2 \right\} \\ & - [\tilde{q}_6] + [\tilde{q}_7] + [\tilde{r}_7] [R_2]^T \left\{ \dot{\omega}_3 \right\} - [\tilde{q}_7] + [\tilde{r}_7] [R_3]^T \left\{ \dot{\omega}_6 \right\} - [\tilde{r}_7] [R_6]^T \left\{ \dot{\omega}_7 \right\} \\ & - [\dot{\tilde{q}}_2] + [\dot{\tilde{q}}_3] + [\dot{\tilde{q}}_6] + [\dot{\tilde{q}}_7] + [\dot{\tilde{r}}_7] \left\{ \dot{\omega}_1 \right\} \\ & - [\dot{\tilde{q}}_3] + [\dot{\tilde{q}}_6] + [\dot{\tilde{q}}_7] + [\dot{\tilde{r}}_7] [R_1]^T \left\{ \dot{\omega}_2 \right\} - [\tilde{q}_3] + [\tilde{q}_6] + [\tilde{q}_7] + [\tilde{r}_7] [\tilde{\omega}_1] [R_1]^T \left\{ \dot{\omega}_2 \right\} \\ & - [\dot{\tilde{q}}_6] + [\dot{\tilde{q}}_7] + [\dot{\tilde{r}}_7] [R_2]^T \left\{ \dot{\omega}_3 \right\} - [\tilde{q}_6] + [\tilde{q}_7] + [\tilde{r}_7] [\tilde{\omega}_2] [R_2]^T \left\{ \dot{\omega}_3 \right\} \\ & - [\dot{\tilde{q}}_7] + [\dot{\tilde{r}}_7] [R_3]^T \left\{ \dot{\omega}_6 \right\} - [\tilde{q}_7] + [\tilde{r}_7] [\tilde{\omega}_3] [R_3]^T \left\{ \dot{\omega}_6 \right\} - [\dot{\tilde{r}}_7] + [\tilde{r}_7] [\tilde{\omega}_6] [R_6]^T \left\{ \dot{\omega}_7 \right\} \end{aligned}$$

5.7 The figure shows a thirteen-body model of the human body numbered using the numbering scheme presented in Unit 1. Body 1 is the **lower torso**, and it is the **system reference body**. The rest of the bodies are numbered in ascending progression outward along the branches. As structured, the lower numbered body array for the system is as follows.

$$\mathcal{L}(1, \dots, 13) = (0, 1, 2, 3, 4, 3, 6, 3, 8, 1, 10, 1, 12)$$

The position and orientation of body 1 is defined **relative** to the **fixed frame** $R: (\underline{N}_1, \underline{N}_2, \underline{N}_3)$, and the orientations of all the other bodies are defined **relative** to their **adjacent, lower numbered bodies**. Body 1 can **translate** and **rotate** relative to the fixed frame, but **all other bodies** can **only rotate** relative to their lower numbered bodies, not translate. Using **body frame components** of the **relative angular velocities** of the bodies, the **system generalized speed matrix** is defined as follows.



$$\{\mathcal{Y}\}_{42 \times 1} \triangleq \begin{Bmatrix} \{\mathcal{Y}_1\}_{39 \times 1} \\ \{\mathcal{Y}_2\}_{3 \times 1} \end{Bmatrix}$$

$$\{\mathcal{Y}_1\}_{39 \times 1} \triangleq [\hat{\omega}'_{11} \quad \hat{\omega}'_{12} \quad \hat{\omega}'_{13} \quad \hat{\omega}'_{21} \quad \hat{\omega}'_{22} \quad \hat{\omega}'_{23} \quad \cdots \quad \hat{\omega}'_{11,1} \quad \hat{\omega}'_{11,2} \quad \hat{\omega}'_{11,3} \quad \hat{\omega}'_{12,1} \quad \hat{\omega}'_{12,2} \quad \hat{\omega}'_{12,3}]^T$$

$$\{\mathcal{Y}_2\}_{3 \times 1} \triangleq [\dot{s}'_{11} \quad \dot{s}'_{12} \quad \dot{s}'_{13}]^T$$

a) For body 7 (lower right arm), find the **time derivatives** of the **fixed frame components** of the **partial velocity matrix** of its **origin** O_7 and its **mass center** G_7 . Then find the **fixed frame components** of the **acceleration** of G_7 , the **mass center** of body 7. Use the results presented in Unit 2 for the **fixed frame components** of the **partial angular velocity matrices** of the bodies and the results from Unit 4 for the **fixed frame components** of the **partial velocity matrices** of the **origins** and **mass centers** of the bodies. Follow the procedures outlined in Equations (58)-(60) and Equation (83). b) Use **direct differentiation** to find the **fixed frame components** of the **acceleration** of G_7 in terms of the **body frame components** of the **relative angular velocity** vectors and compare the results with those obtained in part (a). Assume that $s'_{Bi} (B = 2, \dots, 13; i = 1, 2, 3)$ and $\dot{s}'_{Bi} (B = 2, \dots, 13; i = 1, 2, 3)$ are all **zero**.

Answers:

Body Origins:

Body 1:

$$\begin{bmatrix} {}^R v_{O_1, y_1} \end{bmatrix}_{3 \times 39} = [0]_{3 \times 39} \quad \begin{bmatrix} {}^R v_{O_1, y_2} \end{bmatrix}_{3 \times 3} = [0]_{3 \times 3}$$

Body 2:

$$\begin{bmatrix} \dot{R}_{O_2, y_1} \end{bmatrix}_{3 \times 39} = \begin{bmatrix} \begin{bmatrix} \dot{R}_{O_2, \hat{a}_1'} \end{bmatrix}_{3 \times 3} & [0]_{3 \times 3} & \cdots & [0]_{3 \times 3} \end{bmatrix} \begin{bmatrix} \dot{R}_{O_2, \hat{a}_1'} \end{bmatrix}_{3 \times 3} = -[R_1]^T [\tilde{a}_1'] [\tilde{q}_2'] \quad \begin{bmatrix} \dot{v}_{O_2, y_2} \end{bmatrix}_{3 \times 3} = [0]_{3 \times 3}$$

Body 3:

$$\begin{aligned} \left[\begin{matrix} R\dot{v}_{O_3, y_1} \end{matrix} \right]_{3 \times 39} &= \left[\begin{matrix} \left[R\dot{v}_{O_3, \hat{w}'_1} \right]_{3 \times 3} & \left[R\dot{v}_{O_3, \hat{w}'_2} \right]_{3 \times 3} & [0]_{3 \times 3} & \cdots & [0]_{3 \times 3} \end{matrix} \right] \\ \left[R\dot{v}_{O_3, \hat{w}'_1} \right]_{3 \times 3} &= -[R_1]^T [\tilde{w}'_1][\tilde{q}'_2] - [R_2]^T [\tilde{q}'_3][{}^1\tilde{w}'_2]^T [{}^1R_2] - [R_2]^T [\tilde{w}'_2][\tilde{q}'_3][{}^1R_2] \\ \left[R\dot{v}_{O_3, \hat{w}'_2} \right]_{3 \times 3} &= -[R_2]^T [\tilde{w}'_2][\tilde{q}'_3] & \left[Rv_{O_3, y_2} \right]_{3 \times 3} &= [0]_{3 \times 3} \end{aligned}$$

Body 6:

$$\begin{aligned} \left[\begin{smallmatrix} R\dot{v}_{O_6, y_1} \end{smallmatrix} \right]_{3 \times 39} &= \left[\begin{smallmatrix} R\dot{v}_{O_6, \tilde{w}_1'} \end{smallmatrix} \right]_{3 \times 3} \quad \left[\begin{smallmatrix} R\dot{v}_{O_6, \tilde{w}_2'} \end{smallmatrix} \right]_{3 \times 3} \quad \left[\begin{smallmatrix} R\dot{v}_{O_6, \tilde{w}_3'} \end{smallmatrix} \right]_{3 \times 3} \quad [0]_{3 \times 3} \quad \cdots \quad [0]_{3 \times 3} \\ \left[\begin{smallmatrix} R\dot{v}_{O_6, \tilde{w}_1'} \end{smallmatrix} \right]_{3 \times 3} &= -[R_1]^T [\tilde{w}_1'] [\tilde{q}_2'] - [R_2]^T [\tilde{q}_3'] [\tilde{w}_2']^T [{}^1R_2] - [R_2]^T [\tilde{w}_2'] [\tilde{q}_3'] [{}^1R_2] \\ &\quad - [R_3]^T [\tilde{q}_6'] [\tilde{w}_3']^T [{}^1R_3] - [R_3]^T [\tilde{w}_3'] [\tilde{q}_6'] [{}^1R_3] \\ \left[\begin{smallmatrix} R\dot{v}_{O_6, \tilde{w}_2'} \end{smallmatrix} \right]_{3 \times 3} &= -[R_2]^T [\tilde{w}_2'] [\tilde{q}_3'] - [R_3]^T [\tilde{q}_6'] [\tilde{w}_3']^T [{}^2R_3] - [R_3]^T [\tilde{w}_3'] [\tilde{q}_6'] [{}^2R_3] \\ \left[\begin{smallmatrix} R\dot{v}_{O_6, \tilde{w}_3'} \end{smallmatrix} \right]_{3 \times 3} &= -[R_3]^T [\tilde{w}_3'] [\tilde{q}_6'] \quad \left[\begin{smallmatrix} R\dot{v}_{O_6, y_2} \end{smallmatrix} \right]_{3 \times 3} = [0]_{3 \times 3} \end{aligned}$$

Body 7:

$$\begin{aligned}
\begin{bmatrix} {}^R\dot{v}_{O_7, y_1} \end{bmatrix}_{3 \times 39} &= \begin{bmatrix} \begin{bmatrix} {}^R\dot{v}_{O_7, \hat{\omega}'_1} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} {}^R\dot{v}_{O_7, \hat{\omega}'_2} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} {}^R\dot{v}_{O_7, \hat{\omega}'_3} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} & \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} & \begin{bmatrix} {}^R\dot{v}_{O_7, \hat{\omega}'_6} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} & \cdots & \begin{bmatrix} 0 \end{bmatrix}_{3 \times 3} \end{bmatrix} \\
\begin{bmatrix} {}^R\dot{v}_{O_7, \hat{\omega}'_1} \end{bmatrix}_{3 \times 3} &= -[R_1]^T [\tilde{\omega}'_1] [\tilde{q}'_2] - [R_2]^T [\tilde{q}'_3] [{}^1\tilde{\omega}'_2]^T [{}^1R_2] - [R_2]^T [\tilde{\omega}'_2] [\tilde{q}'_3] [{}^1R_2] \\
&\quad - [R_3]^T [\tilde{q}'_6] [{}^1\tilde{\omega}'_3]^T [{}^1R_3] - [R_3]^T [\tilde{\omega}'_3] [\tilde{q}'_6] [{}^1R_3] - [R_6]^T [\tilde{q}'_7] [{}^1\tilde{\omega}'_6]^T [{}^1R_6] \\
&\quad - [R_6]^T [\tilde{\omega}'_6] [\tilde{q}'_7] [{}^1R_6] \\
\begin{bmatrix} {}^R\dot{v}_{O_7, \hat{\omega}'_2} \end{bmatrix}_{3 \times 3} &= -[R_2]^T [\tilde{\omega}'_2] [\tilde{q}'_3] - [R_3]^T [\tilde{q}'_6] [{}^2\tilde{\omega}'_3]^T [{}^2R_3] - [R_3]^T [\tilde{\omega}'_3] [\tilde{q}'_6] [{}^2R_3] \\
&\quad - [R_6]^T [\tilde{q}'_7] [{}^2\tilde{\omega}'_6]^T [{}^2R_6] - [R_6]^T [\tilde{\omega}'_6] [\tilde{q}'_7] [{}^2R_6] \\
\begin{bmatrix} {}^R\dot{v}_{O_7, \hat{\omega}'_3} \end{bmatrix}_{3 \times 3} &= -[R_3]^T [\tilde{\omega}'_3] [\tilde{q}'_6] - [R_6]^T [\tilde{q}'_7] [{}^3\tilde{\omega}'_6]^T [{}^3R_6] - [R_6]^T [\tilde{\omega}'_6] [\tilde{q}'_7] [{}^3R_6] \\
\begin{bmatrix} {}^R\dot{v}_{O_7, \hat{\omega}'_6} \end{bmatrix}_{3 \times 3} - [R_6]^T [\tilde{\omega}'_6] [\tilde{q}'_7] &\quad \begin{bmatrix} {}^Rv_{O_7, y_2} \end{bmatrix}_{3 \times 3} = [0]_{3 \times 3}
\end{aligned}$$

Body 7 Mass Center G_7 :

$$\begin{bmatrix} R\dot{\mathbf{v}}_{7,y_1} \end{bmatrix}_{3 \times 39} = \begin{bmatrix} \begin{bmatrix} R\dot{\mathbf{v}}_{7,\hat{\omega}'_1} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} R\dot{\mathbf{v}}_{7,\hat{\omega}'_2} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} R\dot{\mathbf{v}}_{7,\hat{\omega}'_3} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} \mathbf{0} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} \mathbf{0} \end{bmatrix}_{3 \times 3} \\ & \begin{bmatrix} R\dot{\mathbf{v}}_{7,\hat{\omega}'_6} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} R\dot{\mathbf{v}}_{7,\hat{\omega}'_7} \end{bmatrix}_{3 \times 3} & \begin{bmatrix} \mathbf{0} \end{bmatrix}_{3 \times 3} & \cdots & \begin{bmatrix} \mathbf{0} \end{bmatrix}_{3 \times 3} \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} {}^R\dot{v}_{7,\hat{\omega}'_1} \end{bmatrix}_{3 \times 3} = & -[R_1]^T [\tilde{\omega}'_1][\tilde{q}'_2] - [R_2]^T [\tilde{q}'_3] \begin{bmatrix} {}^1\tilde{\omega}'_2 \end{bmatrix}^T \begin{bmatrix} {}^1R_2 \end{bmatrix} - [R_2]^T [\tilde{\omega}'_2][\tilde{q}'_3] \begin{bmatrix} {}^1R_2 \end{bmatrix} \\ & - [R_3]^T [\tilde{q}'_6] \begin{bmatrix} {}^1\tilde{\omega}'_3 \end{bmatrix}^T \begin{bmatrix} {}^1R_3 \end{bmatrix} - [R_3]^T [\tilde{\omega}'_3][\tilde{q}'_6] \begin{bmatrix} {}^1R_3 \end{bmatrix} - [R_6]^T [\tilde{q}'_7] \begin{bmatrix} {}^1\tilde{\omega}'_6 \end{bmatrix}^T \begin{bmatrix} {}^1R_6 \end{bmatrix} \\ & - [R_6]^T [\tilde{\omega}'_6][\tilde{q}'_7] \begin{bmatrix} {}^1R_6 \end{bmatrix} - [R_7]^T [\tilde{r}'_7] \begin{bmatrix} {}^1\tilde{\omega}'_7 \end{bmatrix}^T \begin{bmatrix} {}^1R_7 \end{bmatrix} - [R_7]^T [\tilde{\omega}'_7][\tilde{r}'_7] \begin{bmatrix} {}^1R_7 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} {}^R\dot{v}_{7,\hat{\omega}'_2} \end{bmatrix}_{3 \times 3} = & -[R_2]^T [\tilde{\omega}'_2][\tilde{q}'_3] - [R_3]^T [\tilde{q}'_6] \begin{bmatrix} {}^2\tilde{\omega}'_3 \end{bmatrix}^T \begin{bmatrix} {}^2R_3 \end{bmatrix} - [R_3]^T [\tilde{\omega}'_3][\tilde{q}'_6] \begin{bmatrix} {}^2R_3 \end{bmatrix} \\ & - [R_6]^T [\tilde{q}'_7] \begin{bmatrix} {}^2\tilde{\omega}'_6 \end{bmatrix}^T \begin{bmatrix} {}^2R_6 \end{bmatrix} - [R_6]^T [\tilde{\omega}'_6][\tilde{q}'_7] \begin{bmatrix} {}^2R_6 \end{bmatrix} - [R_7]^T [\tilde{r}'_7] \begin{bmatrix} {}^2\tilde{\omega}'_7 \end{bmatrix}^T \begin{bmatrix} {}^2R_7 \end{bmatrix} \\ & - [R_7]^T [\tilde{\omega}'_7][\tilde{r}'_7] \begin{bmatrix} {}^2R_7 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} {}^R\dot{v}_{7,\hat{\omega}'_3} \end{bmatrix}_{3 \times 3} = & -[R_3]^T [\tilde{\omega}'_3][\tilde{q}'_6] - [R_6]^T [\tilde{q}'_7] \begin{bmatrix} {}^3\tilde{\omega}'_6 \end{bmatrix}^T \begin{bmatrix} {}^3R_6 \end{bmatrix} - [R_6]^T [\tilde{\omega}'_6][\tilde{q}'_7] \begin{bmatrix} {}^3R_6 \end{bmatrix} \\ & - [R_7]^T [\tilde{r}'_7] \begin{bmatrix} {}^3\tilde{\omega}'_7 \end{bmatrix}^T \begin{bmatrix} {}^3R_7 \end{bmatrix} - [R_7]^T [\tilde{\omega}'_7][\tilde{r}'_7] \begin{bmatrix} {}^3R_7 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} {}^R\dot{v}_{7,\hat{\omega}'_6} \end{bmatrix}_{3 \times 3} = -[R_6]^T [\tilde{\omega}'_6][\tilde{q}'_7] - [R_7]^T [\tilde{r}'_7] \begin{bmatrix} {}^6\tilde{\omega}'_7 \end{bmatrix}^T \begin{bmatrix} {}^6R_7 \end{bmatrix} - [R_7]^T [\tilde{\omega}'_7][\tilde{r}'_7] \begin{bmatrix} {}^6R_7 \end{bmatrix}$$

$$\begin{bmatrix} {}^R\dot{v}_{7,\hat{\omega}'_7} \end{bmatrix}_{3 \times 3} = -[R_7]^T [\tilde{\omega}'_7][\tilde{r}'_7] \quad \begin{bmatrix} {}^Rv_{7,y_2} \end{bmatrix}_{3 \times 3} = [0]_{3 \times 3}$$

Acceleration of G_7 :

$$\begin{aligned} \{a_7\} = & -\left([R_1]^T [\tilde{q}'_2] + [R_2]^T [\tilde{q}'_3] \begin{bmatrix} {}^1R_2 \end{bmatrix} + [R_3]^T [\tilde{q}'_6] \begin{bmatrix} {}^1R_3 \end{bmatrix} + [R_6]^T [\tilde{q}'_7] \begin{bmatrix} {}^1R_6 \end{bmatrix} + [R_7]^T [\tilde{r}'_7] \begin{bmatrix} {}^1R_7 \end{bmatrix}\right) \{\dot{\omega}'_1\} \\ & -\left([R_2]^T [\tilde{q}'_3] + [R_3]^T [\tilde{q}'_6] \begin{bmatrix} {}^2R_3 \end{bmatrix} + [R_6]^T [\tilde{q}'_7] \begin{bmatrix} {}^2R_6 \end{bmatrix} + [R_7]^T [\tilde{r}'_7] \begin{bmatrix} {}^2R_7 \end{bmatrix}\right) \{\dot{\omega}'_2\} \\ & -\left([R_3]^T [\tilde{q}'_6] + [R_6]^T [\tilde{q}'_7] \begin{bmatrix} {}^3R_6 \end{bmatrix} + [R_7]^T [\tilde{r}'_7] \begin{bmatrix} {}^3R_7 \end{bmatrix}\right) \{\dot{\omega}'_3\} - \left([R_6]^T [\tilde{q}'_7] + [R_7]^T [\tilde{r}'_7] \begin{bmatrix} {}^6R_7 \end{bmatrix}\right) \{\dot{\omega}'_6\} \\ & -\left([R_7]^T [\tilde{r}'_7]\right) \{\dot{\omega}'_7\} - \left([R_1]^T [\tilde{\omega}'_1][\tilde{q}'_2] + [R_2]^T [\tilde{\omega}'_2][\tilde{q}'_3] \begin{bmatrix} {}^1R_2 \end{bmatrix} + [R_2]^T [\tilde{q}'_3] \begin{bmatrix} {}^1\tilde{\omega}'_2 \end{bmatrix}^T \begin{bmatrix} {}^1R_2 \end{bmatrix}\right) \{\dot{\omega}'_1\} \\ & -\left([R_3]^T [\tilde{\omega}'_3][\tilde{q}'_6] \begin{bmatrix} {}^1R_3 \end{bmatrix} + [R_3]^T [\tilde{q}'_6] \begin{bmatrix} {}^1\tilde{\omega}'_3 \end{bmatrix}^T \begin{bmatrix} {}^1R_3 \end{bmatrix} + [R_6]^T [\tilde{\omega}'_6][\tilde{q}'_7] \begin{bmatrix} {}^1R_6 \end{bmatrix} + [R_6]^T [\tilde{q}'_7] \begin{bmatrix} {}^1\tilde{\omega}'_6 \end{bmatrix}^T \begin{bmatrix} {}^1R_6 \end{bmatrix}\right) \{\dot{\omega}'_1\} \\ & -\left([R_7]^T [\tilde{\omega}'_7][\tilde{r}'_7] \begin{bmatrix} {}^1R_7 \end{bmatrix} + [R_7]^T [\tilde{r}'_7] \begin{bmatrix} {}^1\tilde{\omega}'_7 \end{bmatrix}^T \begin{bmatrix} {}^1R_7 \end{bmatrix}\right) \{\dot{\omega}'_1\} \\ & -\left([R_2]^T [\tilde{\omega}'_2][\tilde{q}'_3] + [R_3]^T [\tilde{\omega}'_3][\tilde{q}'_6] \begin{bmatrix} {}^2R_3 \end{bmatrix} + [R_3]^T [\tilde{q}'_6] \begin{bmatrix} {}^2\tilde{\omega}'_3 \end{bmatrix}^T \begin{bmatrix} {}^2R_3 \end{bmatrix}\right) \{\dot{\omega}'_2\} \\ & -\left([R_6]^T [\tilde{\omega}'_6][\tilde{q}'_7] \begin{bmatrix} {}^2R_6 \end{bmatrix} + [R_6]^T [\tilde{q}'_7] \begin{bmatrix} {}^2\tilde{\omega}'_6 \end{bmatrix}^T \begin{bmatrix} {}^2R_6 \end{bmatrix} + [R_7]^T [\tilde{\omega}'_7][\tilde{r}'_7] \begin{bmatrix} {}^2R_7 \end{bmatrix} + [R_7]^T [\tilde{r}'_7] \begin{bmatrix} {}^2\tilde{\omega}'_7 \end{bmatrix}^T \begin{bmatrix} {}^2R_7 \end{bmatrix}\right) \{\dot{\omega}'_2\} \\ & -\left([R_3]^T [\tilde{\omega}'_3][\tilde{q}'_6] + [R_6]^T [\tilde{\omega}'_6][\tilde{q}'_7] \begin{bmatrix} {}^3R_6 \end{bmatrix} + [R_6]^T [\tilde{q}'_7] \begin{bmatrix} {}^3\tilde{\omega}'_6 \end{bmatrix}^T \begin{bmatrix} {}^3R_6 \end{bmatrix}\right) \{\dot{\omega}'_3\} \\ & -\left([R_7]^T [\tilde{\omega}'_7][\tilde{r}'_7] \begin{bmatrix} {}^3R_7 \end{bmatrix} + [R_7]^T [\tilde{r}'_7] \begin{bmatrix} {}^3\tilde{\omega}'_7 \end{bmatrix}^T \begin{bmatrix} {}^3R_7 \end{bmatrix}\right) \{\dot{\omega}'_3\} \\ & -\left([R_6]^T [\tilde{\omega}'_6][\tilde{q}'_7] + [R_7]^T [\tilde{\omega}'_7][\tilde{r}'_7] \begin{bmatrix} {}^6R_7 \end{bmatrix} + [R_7]^T [\tilde{r}'_7] \begin{bmatrix} {}^6\tilde{\omega}'_7 \end{bmatrix}^T \begin{bmatrix} {}^6R_7 \end{bmatrix}\right) \{\dot{\omega}'_6\} - \left([R_7]^T [\tilde{\omega}'_7][\tilde{r}'_7]\right) \{\dot{\omega}'_7\} + \{\ddot{s}_1\} \end{aligned}$$

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Addendum A: Symbolic Calculation of Equations of Example 2

Volume III, Unit 5, Example 2, Part (a) – Symbolic Solution

```
% This script calculates symbolically the fixed frame components of the
% velocity and acceleration of point P of Example 2 (part (a)) of both
% Unit 4 and Unit 5 of Volume III.
% It uses the fixed frame components of the angular velocities and base
% frame components of the relative angular velocities.
%
clearvars

% Define variables
phi = sym("phi");    phiDot = sym("phiDot");    phiDDot = sym("phiDDot");
psi = sym("psi");    psiDot = sym("psiDot");    psiDDot = sym("psiDDot");
theta = sym("theta"); thetaDot = sym("thetaDot"); thetaDDot = sym("thetaDDot");
len_l = sym("len_l");
len_a = sym("len_a");

sinPhi = sym(sin(phi)); cosPhi = sym(cos(phi));
sinPsi = sym(sin(psi)); cosPsi = sym(cos(psi));
sinTheta = sym(sin(theta)); cosTheta = sym(cos(theta));

% Transformation matrix of F relative to ground (G)
RFrelG(1,1) = cosPhi; RFrelG(1,2) = sinPhi; RFrelG(1,3) = sym(0);
RFrelG(2,1) = -sinPhi; RFrelG(2,2) = cosPhi; RFrelG(2,3) = sym(0);
RFrelG(3,1) = sym(0); RFrelG(3,2) = sym(0); RFrelG(3,3) = sym(1);

% Transformation matrix of B relative to F
RBrelF = calculateTransformationMatrixFromAngleX(theta)* ...
        calculateTransformationMatrixFromAngleZ(psi);

% Transformation matrix of D relative to ground (G)
RBrelG = RBrelF*RFrelG;
RBrelG = simplify(RBrelG);

% Position vector of Q wrt O - fixed frame components
% Position vector of P wrt Q - fixed frame components
len_l_prime = [sym(0); len_l; sym(0)];
len_a_prime = [sym(0); len_a; sym(0)];
rQwrtO = transpose(RFrelG)*len_l_prime;
rQwrtO = collect(rQwrtO, len_l);
rPwrtQ = transpose(RBrelG)*len_a_prime;
rPwrtQ = collect(rPwrtQ, len_a);

rQwrtOTilde = buildTildeMatrix(rQwrtO);
rPwrtQTilde = buildTildeMatrix(rPwrtQ);

% Define system state vector, y
omegaFHat = [sym(0); sym(0); phiDot];
omegaBHat = [thetaDot*cosPsi; thetaDot*sinPsi; psiDot];
y = [omegaFHat; omegaBHat];

omegaFHatDot = [sym(0); sym(0); phiDDot];
omegaBHatDot = [(thetaDDot*cosPsi) - (thetaDot*psiDot*sinPsi); ...
                (thetaDDot*sinPsi) + (thetaDot*psiDot*cosPsi); ...
                psiDDot];
yDot = [omegaFHatDot; omegaBHatDot];
```

```

% Angular velocity of F in R
% Angular velocity of D in R
% fixed frame components
omegaF      = omegaFHat;
omegaB      = omegaF + (transpose(RFrelG)*omegaBHat);
omegaB      = simplify(omegaB);
omegaFTilde = buildTildeMatrix(omegaF);
omegaBTilde = buildTildeMatrix(omegaB);

% Partial angular velocity matrices
% fixed frame components
partialOmegaFwrtY = sym(zeros(3,6));
partialOmegaBwrtY = sym(zeros(3,6));

partialOmegaFwrtY(:,1:3) = sym(eye(3,3));
partialOmegaBwrtY(:,1:3) = sym(eye(3,3));
partialOmegaBwrtY(:,4:6) = transpose(RFrelG);

% Partial velocity matrices of point P
% fixed frame components
partialVelocityPwrtY = sym(zeros(3,6));

partialVelocityPwrtY(:,1:3) = -(rQwrtOTilde + rPwrtQTilde);
partialVelocityPwrtY(:,4:6) = -rPwrtQTilde*transpose(RFrelG);

partialVelocityPwrtY = simplify(partialVelocityPwrtY);

% Velocity of P - fixed frame components
v_PPart1 = partialVelocityPwrtY(:,1:3)*y(1:3);

v_PPart2 = partialVelocityPwrtY(:,4:6)*y(4:6);
v_PPart2 = collect(v_PPart2,thetaDot);
v_PPart2 = collect(v_PPart2,len_a);
v_PPart2 = collect(v_PPart2,sinTheta);
expr1     = sym(cos(phi)*sin(psi) + cos(psi)*sin(phi));
expr2     = sym(sin(psi + phi));
v_PPart2 = subs(v_PPart2,expr1,expr2);
expr1     = sym(cos(phi)*cos(psi) - sin(phi)*sin(psi));
expr2     = sym(cos(psi + phi));
v_PPart2  = subs(v_PPart2,expr1,expr2);

v_P       = v_PPart1 + v_PPart2;
expr1     = sym(cos(psi)^2*cos(theta) + cos(theta)*sin(psi)^2);
expr2     = cosTheta;
v_P       = subs(v_P,expr1,expr2);

disp(' ');
disp(' Body Frame Components of v_P')
disp(' ');
disp(' Component 1');
disp(v_P(1));
disp(' ')
disp(' Component 2');
disp(v_P(2));
disp(' ')
disp(' Component 3');
disp(v_P(3));

```

```

% Time derivative of position vector of Q wrt O - fixed frame components
% Time derivative of position vector of P wrt Q - fixed frame components
rQwrtoDot = omegaFTilde*transpose(RFrelG)*len_l_prime;
rQwrtoDot = collect(rQwrtoDot,len_l);
rPwrtoDot = omegaBTilde*transpose(RBrelG)*len_a_prime;
rPwrtoDot = simplify(rPwrtoDot);
rPwrtoDot = collect(rPwrtoDot,len_a);

rQwrtoTildeDot = buildTildeMatrix(rQwrtoDot);
rPwrtoTildeDot = buildTildeMatrix(rPwrtoDot);

% Time derivatives of partial velocity matrices of point P
% fixed frame components
partialVelocityPwrtoDot = sym(zeros(3,6));

partialVelocityPwrtoDot(:,1:3) = -(rQwrtoTildeDot + rPwrtoTildeDot);
partialVelocityPwrtoDot(:,4:6) = ...
    -((rPwrtoTildeDot) + (rPwrtoTilde*omegaFTilde))*transpose(RFrelG);

partialVelocityPwrtoDot = simplify(partialVelocityPwrtoDot);
partialVelocityPwrtoDot = collect(partialVelocityPwrtoDot,len_l);
partialVelocityPwrtoDot = collect(partialVelocityPwrtoDot,len_a);

% Acceleration of P - fixed frame components
a_PPart1 = partialVelocityPwrtoDot*yDot;
a_PPart1 = collect(a_PPart1,phiDDot);
a_PPart1 = collect(a_PPart1,psiDDot);
a_PPart1 = collect(a_PPart1,thetaDDot);
a_PPart1 = collect(a_PPart1,sinTheta);
a_PPart1 = collect(a_PPart1,len_a);
expr1 = sym(cos(phi)*sin(psi) + cos(psi)*sin(phi));
expr2 = sym(sin(psi + phi));
a_PPart1 = subs(a_PPart1,expr1,expr2);
a_PPart1 = collect(a_PPart1,psiDot);
a_PPart1 = collect(a_PPart1,thetaDot);
expr1 = sym(cos(phi)*cos(psi) - sin(phi)*sin(psi));
expr2 = sym(cos(psi + phi));
a_PPart1 = subs(a_PPart1,expr1,expr2);
expr1 = sym(cos(phi)*sin(psi) + cos(psi)*sin(phi));
expr2 = sym(sin(psi + phi));
a_PPart1 = subs(a_PPart1,expr1,expr2);
expr1 = sym(cos(psi)^2*cos(theta) + cos(theta)*sin(psi)^2);
expr2 = cosTheta;
a_PPart1 = subs(a_PPart1,expr1,expr2);

a_PPart2 = partialVelocityPwrtoDot*y;
a_PPart2 = expand(a_PPart2);
expr = sym(phiDot^2);
a_PPart2 = collect(a_PPart2,expr);
expr = sym(psiDot^2);
a_PPart2 = collect(a_PPart2,expr);
expr = sym(thetaDot^2);
a_PPart2 = collect(a_PPart2,expr);

expr1 = sym(phiDot*psiDot);
phiDotPsiDot = sym("phiDotPsiDot");
a_PPart2 = subs(a_PPart2,expr1,phiDotPsiDot);

```

```

a_PPart2      = collect(a_PPart2,phiDotPsiDot);

expr1          = sym(phiDot*thetaDot);
phiDotThetaDot = sym("phiDotThetaDot");
a_PPart2       = subs(a_PPart2,expr1,phiDotThetaDot);
a_PPart2       = collect(a_PPart2,phiDotThetaDot);

expr1          = sym(psiDot*thetaDot);
psiDotThetaDot = sym("psiDotThetaDot");
a_PPart2       = subs(a_PPart2,expr1,psiDotThetaDot);
a_PPart2       = collect(a_PPart2,psiDotThetaDot);

expr1  = sym(len_a*cos(phi)*cos(theta)*sin(psi) + ...
            len_a*cos(psi)*cos(theta)*sin(phi));
expr2  = sym(len_a*cos(theta)*sin(phi + psi));
a_PPart2 = subs(a_PPart2,expr1,expr2);

expr1  = sym(2*len_a*cos(phi)*cos(psi)*sin(theta) - ...
            2*len_a*sin(phi)*sin(psi)*sin(theta));
expr2  = sym(2*len_a*sin(theta)*cos(phi + psi));
a_PPart2 = subs(a_PPart2,expr1,expr2);

expr1  = sym(2*len_a*cos(phi)*cos(theta)*sin(psi) + ...
            2*len_a*cos(psi)*cos(theta)*sin(phi));
expr2  = sym(2*len_a*cos(theta)*sin(phi + psi));
a_PPart2 = subs(a_PPart2,expr1,expr2);

expr1  = sym(len_a*cos(phi)*cos(psi)*sin(theta) - ...
            len_a*sin(phi)*sin(psi)*sin(theta));
expr2  = sym(len_a*sin(theta)*cos(phi + psi));
a_PPart2 = subs(a_PPart2,expr1,expr2);

expr1  = sym(-len_a*cos(phi)*cos(psi)*cos(theta) + ...
            len_a*cos(theta)*sin(phi)*sin(psi));
expr2  = sym(-len_a*cos(theta)*cos(phi + psi));
a_PPart2 = subs(a_PPart2,expr1,expr2);

expr1  = sym(2*len_a*cos(phi)*sin(psi)*sin(theta) + ...
            2*len_a*cos(psi)*sin(phi)*sin(theta));
expr2  = sym(2*len_a*sin(theta)*sin(phi + psi));
a_PPart2 = subs(a_PPart2,expr1,expr2);

expr1  = sym(len_a*cos(phi)*sin(psi)*sin(theta) + ...
            len_a*cos(psi)*sin(phi)*sin(theta));
expr2  = sym(len_a*sin(theta)*sin(phi + psi));
a_PPart2 = subs(a_PPart2,expr1,expr2);

expr1  = sym(len_a*sin(theta)*cos(psi)^2 + ...
            len_a*sin(theta)*sin(psi)^2);
expr2  = sym(len_a*sin(theta));
a_PPart2 = subs(a_PPart2,expr1,expr2);

a_P      = a_PPart1 + a_PPart2;

expr1 = sym(psiDot*thetaDot);
a_P    = subs(a_P,expr1,psiDotThetaDot);
a_P    = collect(a_P,psiDotThetaDot);

```

```

disp(' ');
disp(' Body Frame Components of a_P')
disp(' ');
disp(' Component 1');
disp(a_P(1));
disp(' ')
disp(' Component 2');
disp(a_P(2));
disp(' ')
disp(' Component 3');
disp(a_P(3));

```

Body Frame Components of v_P

Component 1

$$\text{len}_a \cdot \theta \dot{} \sin(\phi + \psi) \sin(\theta) - \text{len}_a \cdot \psi \dot{} \cos(\phi + \psi) \cos(\theta) - \phi \dot{} (\text{len}_l \cos(\phi) + \text{len}_a \cos(\phi + \psi) \cos(\theta))$$

Component 2

$$- \phi \dot{} (\text{len}_l \sin(\phi) + \text{len}_a \sin(\phi + \psi) \cos(\theta)) - \text{len}_a \cdot \psi \dot{} \sin(\phi + \psi) \cos(\theta) - \text{len}_a \cdot \theta \dot{} \cos(\phi + \psi) \sin(\theta)$$

Component 3

$$\text{len}_a \cdot \theta \dot{} \cos(\theta)$$

Body Frame Components of a_P

Component 1

$$\begin{aligned} & (\text{len}_l \sin(\phi) + \text{len}_a \sin(\phi + \psi) \cos(\theta)) \phi \ddot{}^2 + \text{len}_a \sin(\phi + \psi) \cos(\theta) \psi \ddot{}^2 + \\ & \text{len}_a \sin(\phi + \psi) \cos(\theta) \theta \ddot{}^2 - \\ & \text{len}_a (\phi \ddot{} \cos(\phi + \psi) \cos(\theta) + \psi \ddot{} \cos(\phi + \psi) \cos(\theta) - \\ & \theta \ddot{} \sin(\phi + \psi) \sin(\theta)) - \text{len}_l \phi \ddot{} \cos(\phi) + \\ & 2 \text{len}_a \phi \dot{} \psi \dot{} \sin(\phi + \psi) \cos(\theta) + 2 \text{len}_a \phi \dot{} \theta \dot{} \cos(\phi + \psi) \sin(\theta) + \\ & 2 \text{len}_a \psi \dot{} \theta \dot{} \cos(\phi + \psi) \sin(\theta) \end{aligned}$$

Component 2

$$\begin{aligned} & (- \text{len}_l \cos(\phi) - \text{len}_a \cos(\phi + \psi) \cos(\theta)) \phi \dot{}^2 - \text{len}_a \cos(\phi + \psi) \cos(\theta) \psi \dot{}^2 - \\ & \text{len}_a \cos(\phi + \psi) \cos(\theta) \theta \dot{}^2 - \\ & \text{len}_a (\phi \ddot{} \sin(\phi + \psi) \cos(\theta) + \psi \ddot{} \sin(\phi + \psi) \cos(\theta) + \\ & \theta \ddot{} \cos(\phi + \psi) \sin(\theta)) - \\ & \text{len}_l \phi \ddot{} \sin(\phi) - 2 \text{len}_a \phi \dot{} \psi \dot{} \cos(\phi + \psi) \cos(\theta) + \\ & 2 \text{len}_a \phi \dot{} \theta \dot{} \sin(\phi + \psi) \sin(\theta) + 2 \text{len}_a \psi \dot{} \theta \dot{} \sin(\phi + \psi) \sin(\theta) \end{aligned}$$

Component 3

$$- \text{len}_a \sin(\theta) \theta \dot{}^2 + \text{len}_a \theta \ddot{} \cos(\theta)$$

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Volume III, Unit 5, Example 2, Part (b) – Symbolic Solution

```
% This script calculates symbolically the body frame components of the
% velocity and acceleration of point P of Example 2 (part (b)) of both
% Unit 4 and Unit 5 of Volume III.
% It uses the body frame components of the angular velocities and body
% frame components of the relative angular velocities.
%
clearvars

% Define variables
phi = sym("phi");    phiDot = sym("phiDot");    phiDDot = sym("phiDDot");
psi = sym("psi");    psiDot = sym("psiDot");    psiDDot = sym("psiDDot");
theta = sym("theta"); thetaDot = sym("thetaDot"); thetaDDot = sym("thetaDDot");
len_l = sym("len_l");
len_a = sym("len_a");

sinPhi = sym(sin(phi)); cosPhi = sym(cos(phi));
sinPsi = sym(sin(psi)); cosPsi = sym(cos(psi));
sinTheta = sym(sin(theta)); cosTheta = sym(cos(theta));

% Transformation matrix of F relative to ground (G)
RFrelG(1,1) = cosPhi; RFrelG(1,2) = sinPhi; RFrelG(1,3) = sym(0);
RFrelG(2,1) = -sinPhi; RFrelG(2,2) = cosPhi; RFrelG(2,3) = sym(0);
RFrelG(3,1) = sym(0); RFrelG(3,2) = sym(0); RFrelG(3,3) = sym(1);

% Transformation matrix of B relative to F
RBrelF = calculateTransformationMatrixFromAngleX(theta)* ...
        calculateTransformationMatrixFromAngleZ(psi);

% Transformation matrix of D relative to ground (G)
RBrelG = RBrelF*RFrelG;
RBrelG = simplify(RBrelG);

% Position vector of Q wrt O - body frame components
% Position vector of P wrt Q - body frame components
len_l_prime = [sym(0); len_l; sym(0)];
len_a_prime = [sym(0); len_a; sym(0)];
rQwrtOPrime = len_l_prime;
rPwrtQPrime = len_a_prime;

rQwrtOPrimeTilde = buildTildeMatrix(rQwrtOPrime);
rPwrtQPrimeTilde = buildTildeMatrix(rPwrtQPrime);

% Define system state vector, y
omegaFPrimeHat = [sym(0); sym(0); phiDot];
omegaBPrimeHat = [thetaDot; psiDot*sinTheta; psiDot*cosTheta];

y = [omegaFPrimeHat; omegaBPrimeHat];

omegaFPrimeHatDot = [sym(0); sym(0); phiDDot];
omegaBPrimeHatDot = [thetaDDot; ...
                    (psiDDot*sinTheta) + (psiDot*thetaDot*cosTheta); ...
                    (psiDDot*cosTheta) - (psiDot*thetaDot*sinTheta)];

yDot = [omegaFPrimeHatDot; omegaBPrimeHatDot];
```

```

% Angular velocity of F in R
% Angular velocity of D in R
% body frame components
omegaFPrime = omegaFPrimeHat;
omegaBPrime = (RBrelF*omegaFPrime) + (omegaBPrimeHat);
omegaBPrime = simplify(omegaBPrime);

omegaBPrimeHatTilde = buildTildeMatrix(omegaBPrimeHat);
omegaBPrimeTilde = buildTildeMatrix(omegaBPrime);

% Partial angular velocity matrices
% body frame components
partialOmegaFwrtY = sym(zeros(3,6));
partialOmegaBwrtY = sym(zeros(3,6));

partialOmegaFwrtY(:,1:3) = sym(eye(3,3));
partialOmegaBwrtY(:,1:3) = RBrelF;
partialOmegaBwrtY(:,4:6) = sym(eye(3,3));

% Partial velocity matrices of point P
% body frame components
partialVelocityPwrtY = sym(zeros(3,6));

partialVelocityPwrtY(:,1:3) = -(RBrelF*rQwrtOPrimeTilde) - ...
    (rPwrtQPrimeTilde*RBrelF);
partialVelocityPwrtY(:,4:6) = -rPwrtQPrimeTilde;

partialVelocityPwrtY = simplify(partialVelocityPwrtY);

% Velocity of P - body frame components
v_P = partialVelocityPwrtY*y;
v_P = collect(v_P,phiDot); v_P = collect(v_P,psiDot); v_P = collect(v_P,thetaDot);

disp(' ');
disp(' Body Frame Components of v_P')
disp(' ');
disp(' Component 1');
disp(v_P(1));
disp(' ')
disp(' Component 2');
disp(v_P(2));
disp(' ')
disp(' Component 3');
disp(v_P(3));

% Time derivatives of partial velocity matrices of point P
% body frame components
partialVelocityPwrtYDot = sym(zeros(3,6));

partialVelocityPwrtYDot(:,1:3) = ...
    -(transpose(omegaBPrimeHatTilde)*RBrelF*rQwrtOPrimeTilde) - ...
    (rPwrtQPrimeTilde*transpose(omegaBPrimeHatTilde)*RBrelF);

partialVelocityPwrtYDot = simplify(partialVelocityPwrtYDot);

% Acceleration of P - body frame components
a_PPart01 = partialVelocityPwrtY*yDot;
a_PPart01 = collect(a_PPart01,phiDDot);

```

```

a_PPart01 = collect(a_PPart01,psiDDot);
a_PPart01 = collect(a_PPart01,thetaDDot);

a_PPart02 = partialVelocityPwrtYDot*y;

a_P      = a_PPart01 + a_PPart02;
a_P      = simplify(a_P);

a_PPart03 = omegaBPrimeTilde*v_P;
a_PPart03 = expand(a_PPart03);

a_P      = a_P + a_PPart03;
a_P      = simplify(a_P);

expr1     = sym(sin(2*theta));
expr2     = 2*sinTheta*cosTheta;
a_P       = subs(a_P,expr1,expr2);

```

```

disp(' ');
disp(' Body Frame Components of a_P')
disp(' ');
disp(' Component 1');
disp(a_P(1));
disp(' ')
disp(' Component 2');
disp(a_P(2));
disp(' ')
disp(' Component 3');
disp(a_P(3));

```

Body Frame Components of v_P
Component 1
 $(-len_l*cos(psi) - len_a*cos(theta))*phiDot + (-len_a*cos(theta))*psiDot$

Component 2
 $(len_l*cos(theta)*sin(psi))*phiDot$

Component 3
 $len_a*thetaDot + (-len_l*sin(psi)*sin(theta))*phiDot$

Body Frame Components of a_P
Component 1
 $-len_l*sin(psi)*phiDot^2 + 2*len_a*thetaDot*sin(theta)*phiDot - len_l*phiDDot*cos(psi) - len_a*phiDDot*cos(theta) - len_a*psiDot*cos(theta) + 2*len_a*psiDot*thetaDot*sin(theta)$

Component 2
 $len_l*phiDDot*cos(theta)*sin(psi) - len_a*phiDot^2*cos(theta)^2 - len_a*psiDot^2*cos(theta)^2 - len_l*phiDot^2*cos(psi)*cos(theta) - 2*len_a*phiDot*psiDot*cos(theta)^2 - len_a*thetaDot^2$

Component 3
 $len_a*thetaDDot - len_l*phiDDot*sin(psi)*sin(theta) + len_l*phiDot^2*cos(psi)*sin(theta) + len_a*phiDot^2*cos(theta)*sin(theta) + len_a*psiDot^2*cos(theta)*sin(theta) + 2*len_a*phiDot*psiDot*cos(theta)*sin(theta)$

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Addendum B: Functions used in Example 3

Function: calculateTransformationMatrixFromAngleX

```
function [transformationMatrixX] = ...
    calculateTransformationMatrixFromAngleX(thetaX)

%
% This function computes the coordinate transformation matrix associated
% with a positive X rotation.
%
% Input:
% thetaX is the angle in radians
%
% Output:
% transformationMatrixX

% calculate the sine and cosine of the angle theta
sinThetaX = sin(thetaX);
cosThetaX = cos(thetaX);

% Elements for an X rotation
transformationMatrixX = [1,      0,      0; ...
                        0, cosThetaX, sinThetaX; ...
                        0, -sinThetaX, cosThetaX];

end
```

Function: calculateTransformationMatrixFromAngleZ

```
function [transformationMatrixZ] = ...
    calculateTransformationMatrixFromAngleZ(thetaZ)

%
% This function computes the coordinate transformation matrix associated
% with a positive Z rotation.
%
% Input:
% thetaZ is the angle in radians
%
% Output:
% transformationMatrixZ

% calculate the sine and cosine of the angle thetaZ
sinThetaZ = sin(thetaZ);
cosThetaZ = cos(thetaZ);

% Elements for a Z rotation
transformationMatrixZ = [ cosThetaZ, sinThetaZ, 0; ...
                        -sinThetaZ, cosThetaZ, 0; ...
                        0,      0, 1];

end
```

Function: buildTildeMatrix

```
function [tilde_matrix] = buildTildeMatrix(vector)

% This function creates a skew symmetric matrix using the components of
% "vector" for the purpose of performing cross products.
%
% Input: 3x1 matrix "vector"
%
% Output: 3x3 skew symmetric matrix "tilde_matrix"
%
    tilde_matrix = zeros(3);

    tilde_matrix(1,2) = -vector(3);
    tilde_matrix(2,1) = vector(3);
    tilde_matrix(1,3) = vector(2);
    tilde_matrix(3,1) = -vector(2);
    tilde_matrix(2,3) = -vector(1);
    tilde_matrix(3,2) = vector(1);

end
```